

Stability and chaos  
in last passage percolation

ALAN HAMMOND, U.C. Berkeley

— joint work with

SHIRSHENDU GANGULY, U.C. Berkeley

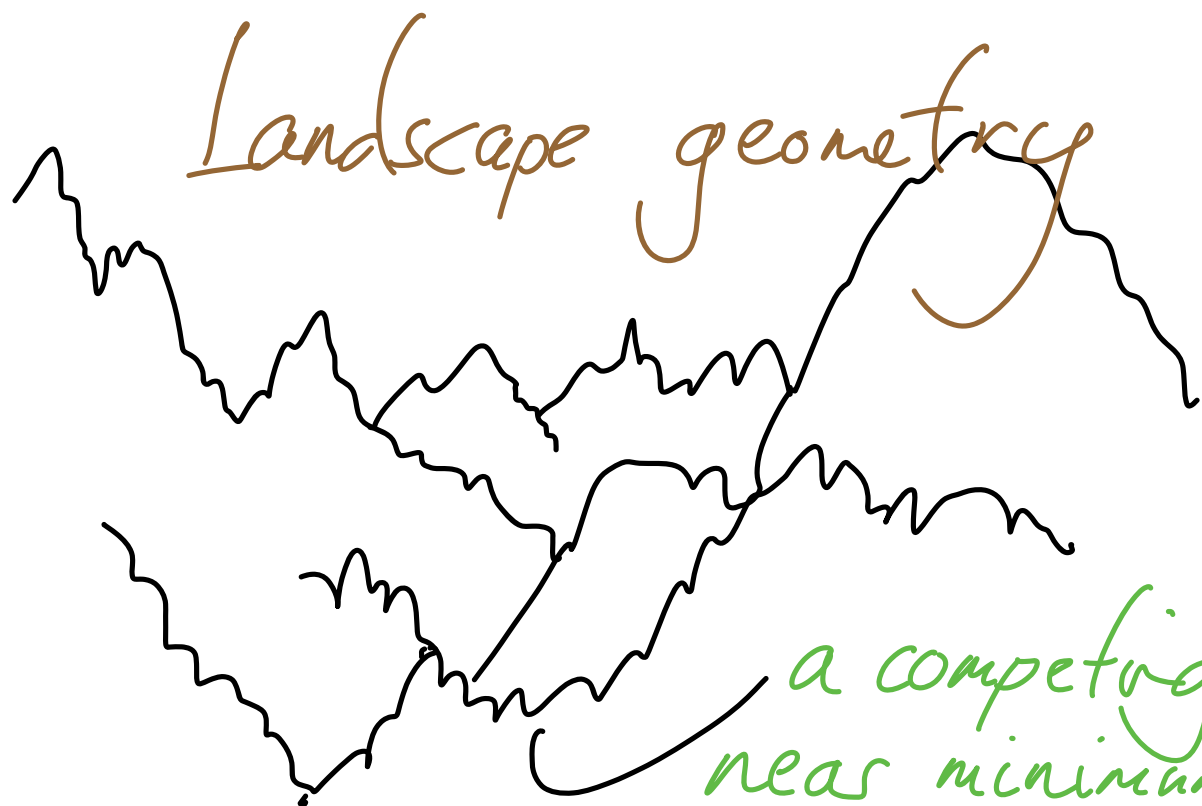
MSRI Workshop

23rd Sept 2021

In complex discrete systems  
in statistical mechanics,  
a probability measure  $\mu$   
is often written

$$\mu(\{x\}) = e^{-H(x)}, \quad x \in X$$

The Hamiltonian  $H: X \rightarrow [0, \infty)$   
may be viewed as an energy  
specified over the landscape  $X$ .



Landscape geometry

a competing near minimum

the ground state  
 $x \in X$  minimizes  $H$

Three natural

questions

about an

energy

landscape

...

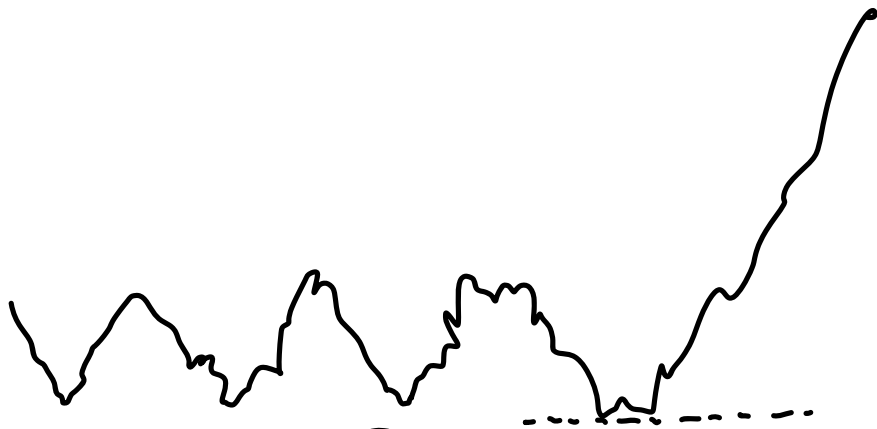
Q1. Is the  
ground state energy  
super-concentrated,  
with sublinear growth  
in variance, slower than

$$\text{Var} \left( \sum_{i=1}^N \mathbb{I}_{\text{HEADS}_i} \right) ?$$

$\approx N$  CLASSICAL CT

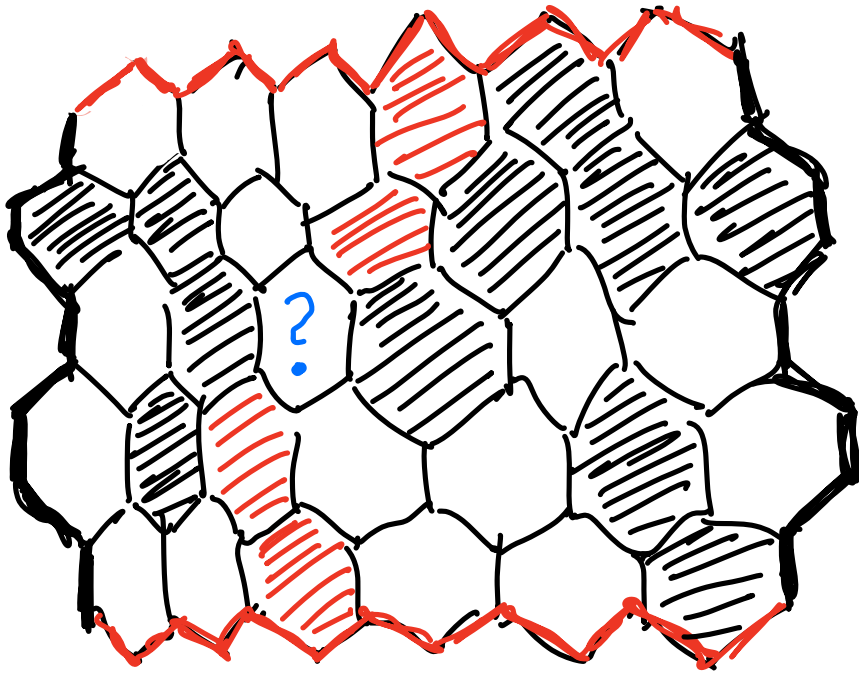
Q2. Are there

MULTIPLE VALLEYS!






Several valleys  
that rival the ground state

Q3. Does chaos reign?



HEADS  OR  TAILS

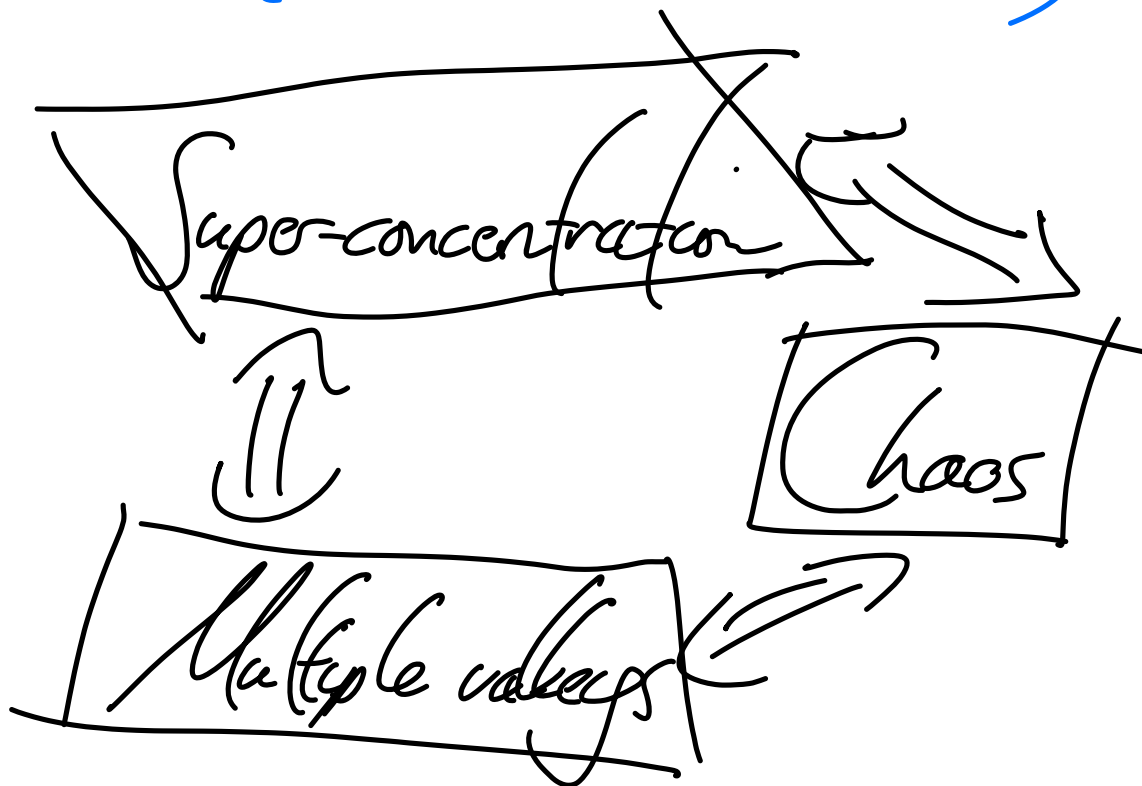
Flipping  decides the  
outcome  / 

Q3 Does chaos reign?  
Is the ground state  
sensitive to small  
perturbations in the  
noise that specifies  
the Hamiltonian  $H$ .



Sourav Chatterjee's 2014 monograph

For a class of models of  
GAUSSIAN disorder,



GOAL: understand  
the order of NOISE  
perturbation that heralds

the onset of chaos

in a dynamical  
form of  
last passage percolation.

The three aspects  
at play in

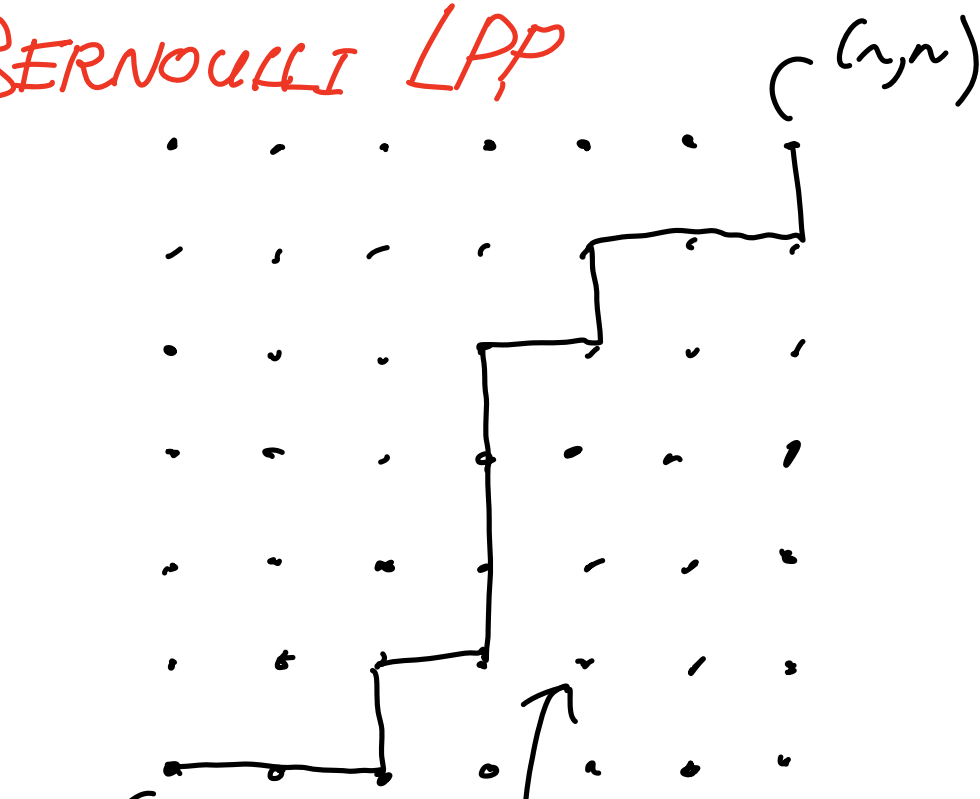
Bernoulli.

last

passage

percolation

# BERNOULLI LPP



$(0,0)$  / zero/one bits  
are assigned HEADS/TAILS

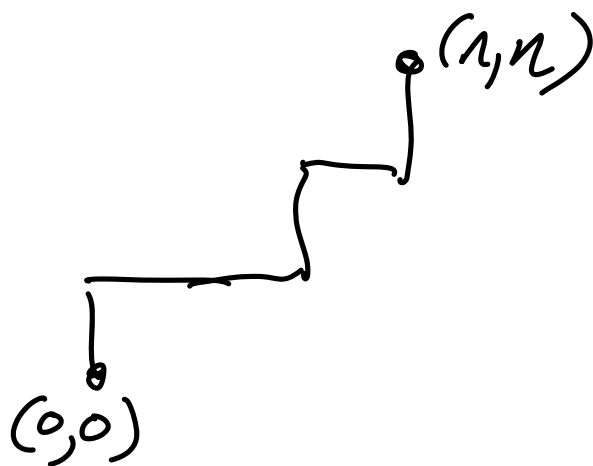
any UPRIGHT path  $P$   
has energy  $E(P)$   
= sum of values along  $P$ .

The geodesic  $\Gamma_n$   
is the path

$$(0,0) \rightarrow (n,n)$$

of maximum energy  $j$

$$\text{set } \mu_n = E(\Gamma_n).$$



# Super-concentration

$$M_n = an + \underbrace{W_n}_{\text{a tight, non-degenerate, sequence of RVs}} n^{1/3}$$

$a \in (1, 2)$

(a tight, non-degenerate, sequence of RVs)

So:

$$\text{Var}(M_n) = \Theta(n^{2/3}) \ll n$$

super-concentration

The energy landscape of LPP.

Consider the correlation

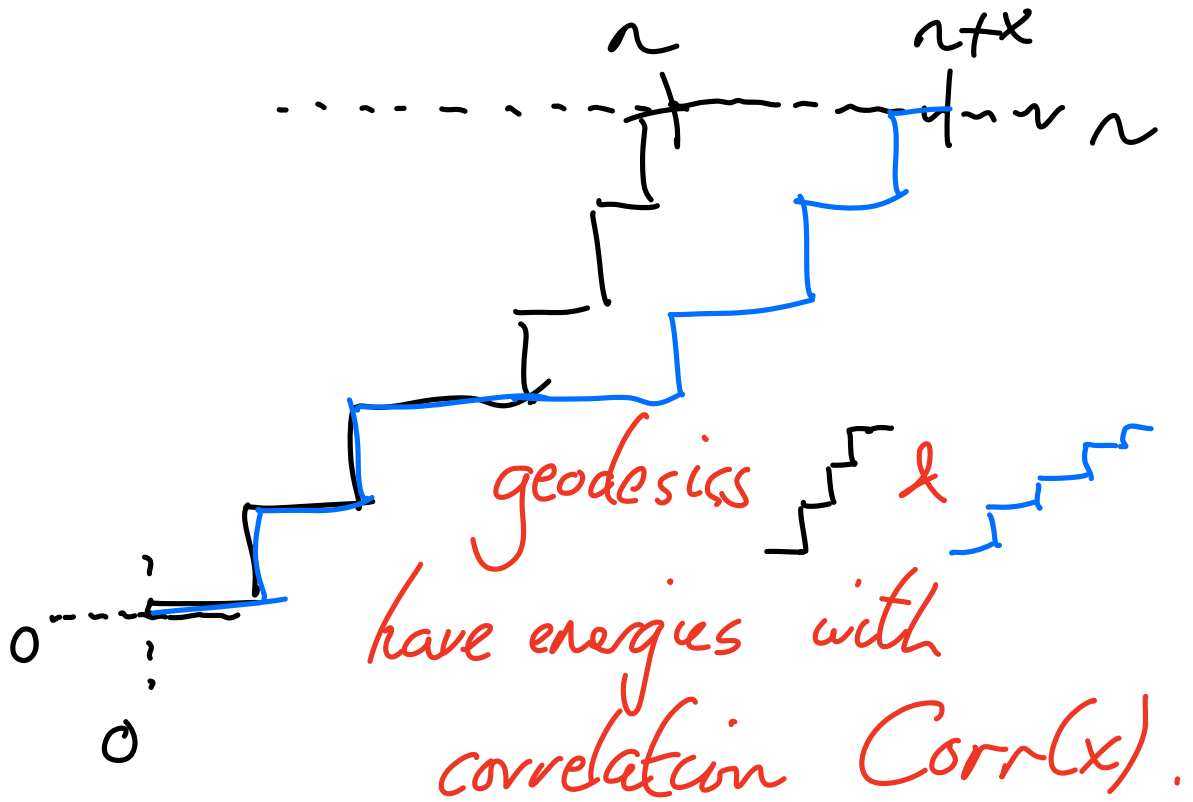
$\text{Corr}(x)$  between

geodesic energies

for the routes

$(0,0) \rightarrow (n,n)$

&  $(0,0) \rightarrow (n+x,n)$ .

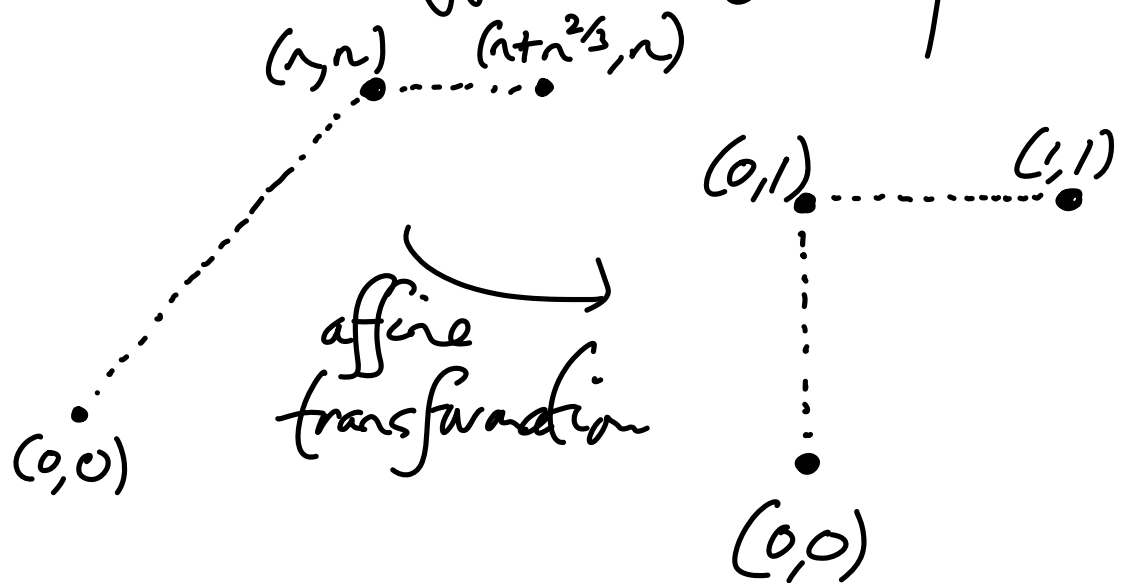


Q: what scale of  $x$  leads to  $\text{Corr}(x) = 50\%$ ?

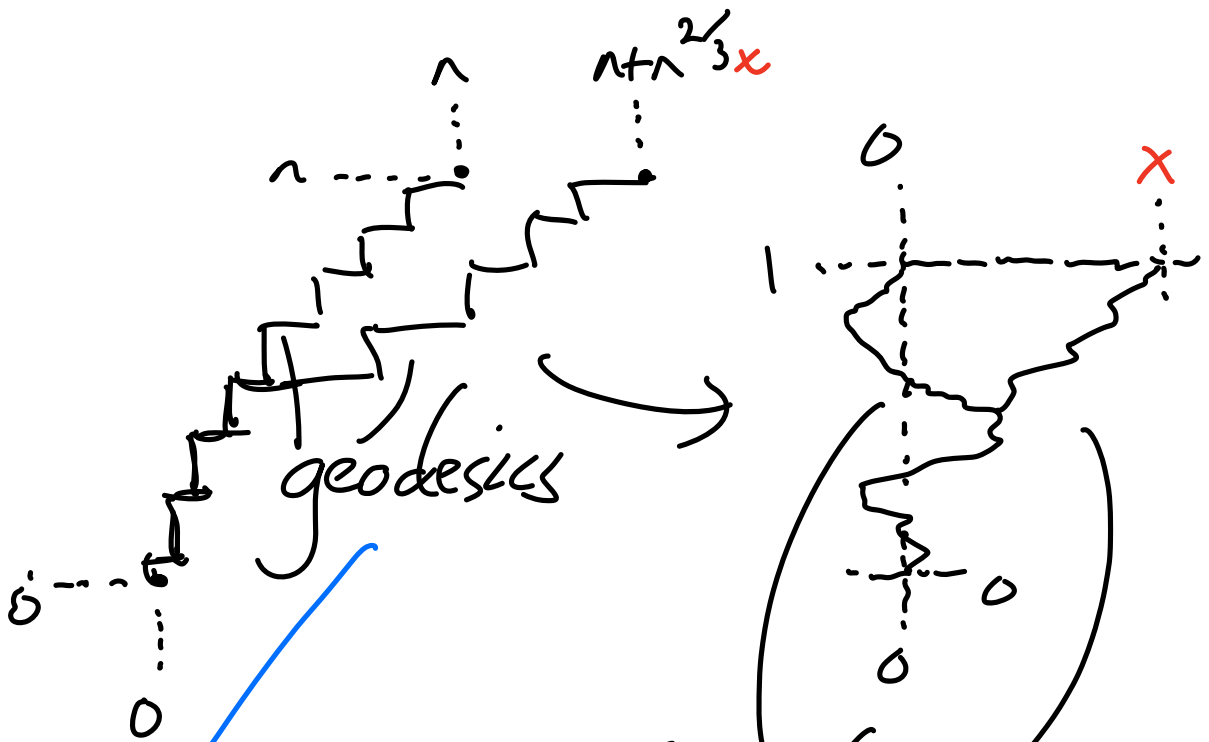
A:  $x = \Theta\left(n^{\frac{2}{3}}\right)$ .



The ANSWER suggests a SCALED PICTURE



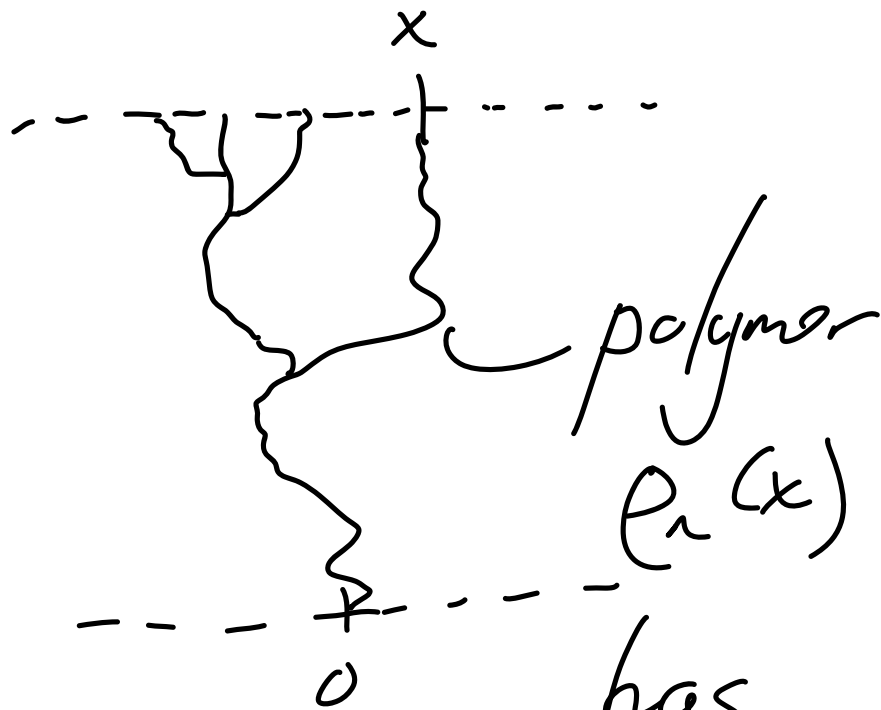
in which  
geodesics may  
be depicted in a  
scaled form ...



scaled geodesics,  
or polymers

have energy  $\underbrace{an + W(x)}_{\text{random}} n^{1/3}$

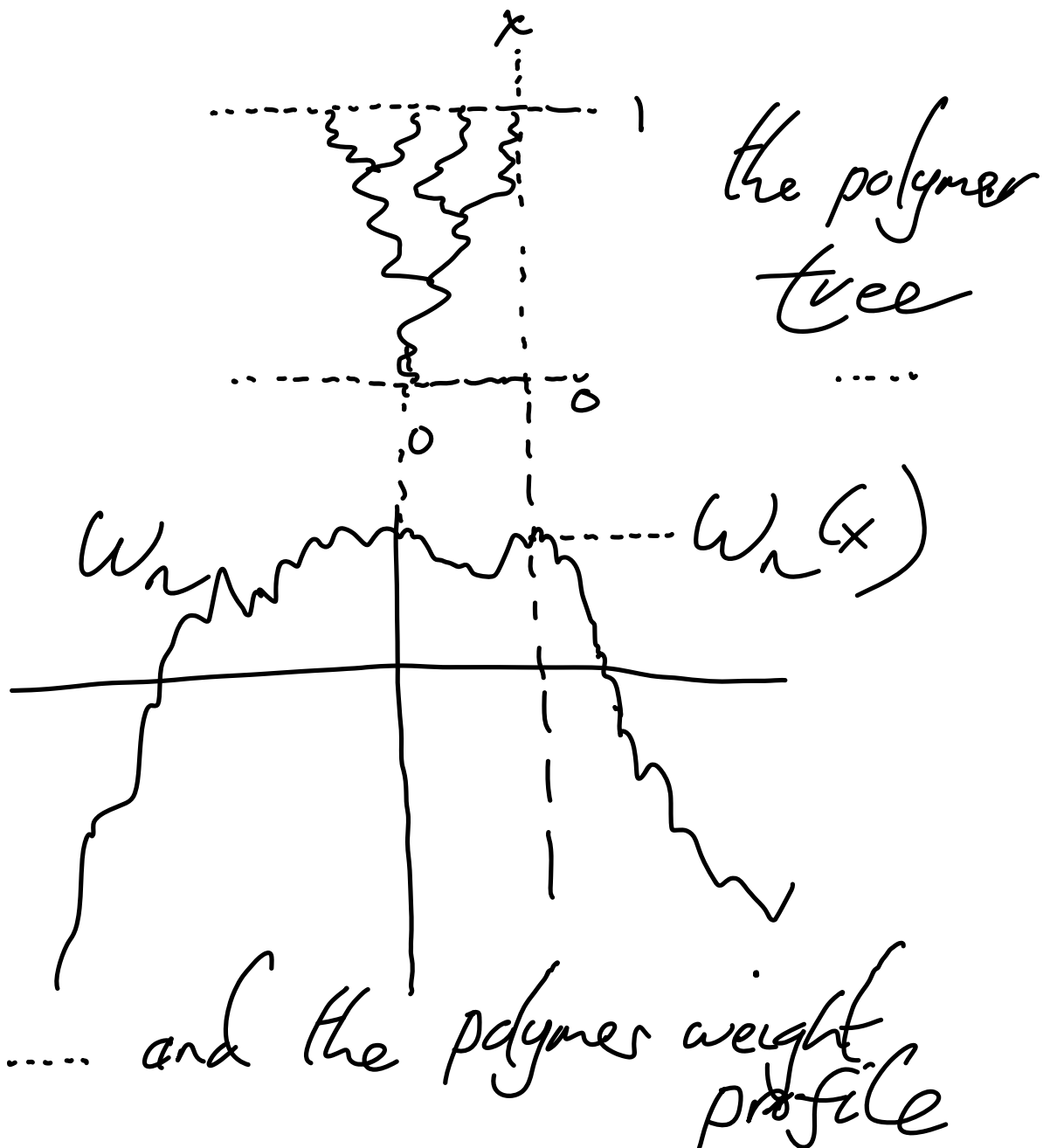
but unit-order scaled energy



has  
scaled  
energy,  
or weight,  
 $U_n(x)$

The scaled picture:

## POLYMERS & THEIR WEIGHT PROFILE



Taking a high  $n$  limit ...

$$\omega_n \xrightarrow{(d)} \omega$$

the parabolic  $A_{\text{rig}_2}$  process

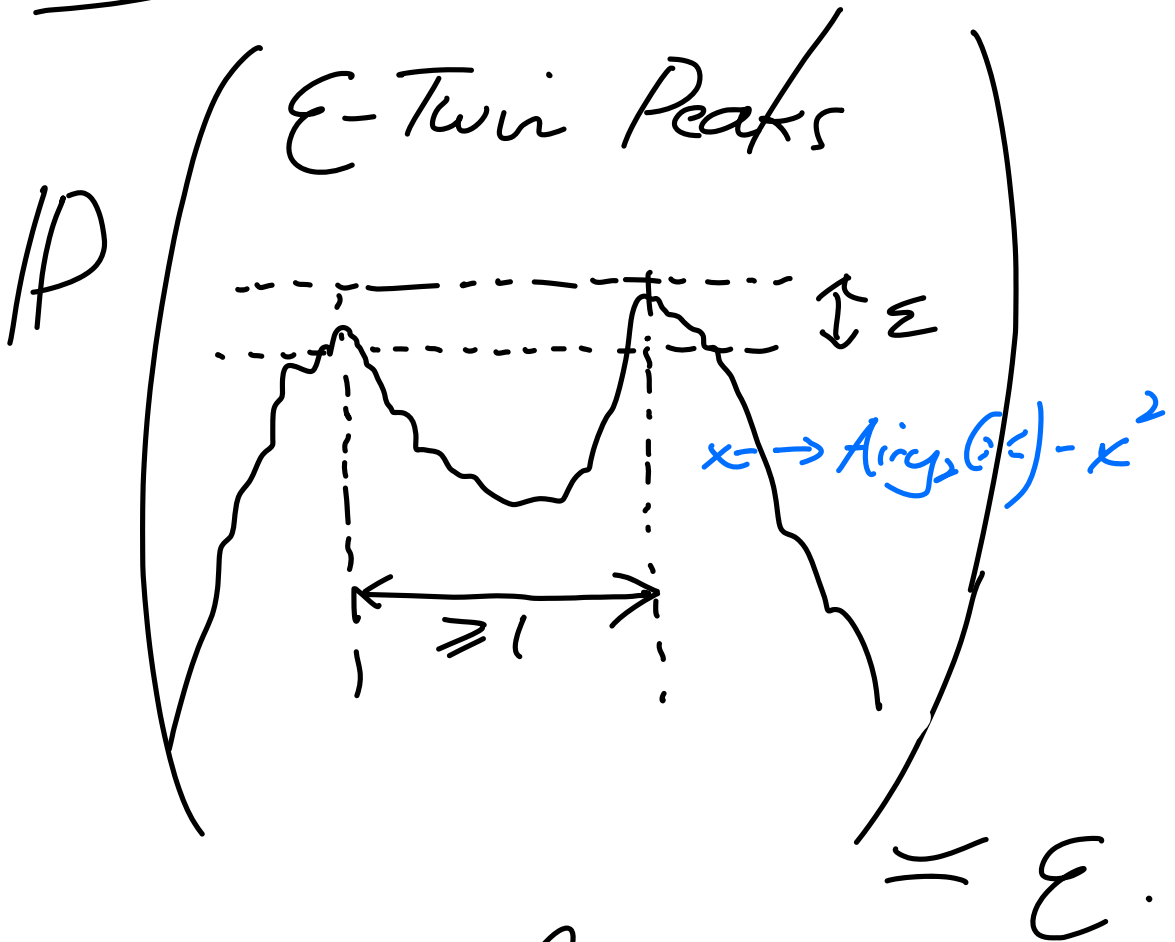


Q concerning the  
SCALED energy landscape:

What is the PROBABILITY  
of NEAR TWIN PEAKS?

(of a walk to within  $\epsilon$   
of the energy maximum  
at distance at least one?

A.



Proof idea next ....

The profile 

closely resembles

Brownian motion

on wide-order

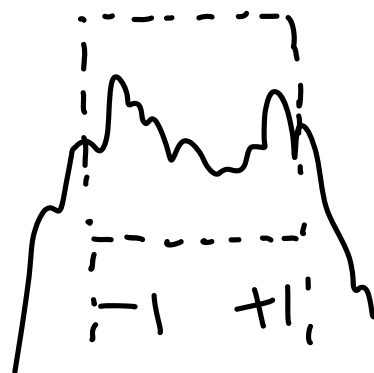
scales — so the

estimate is inherited

from PM.

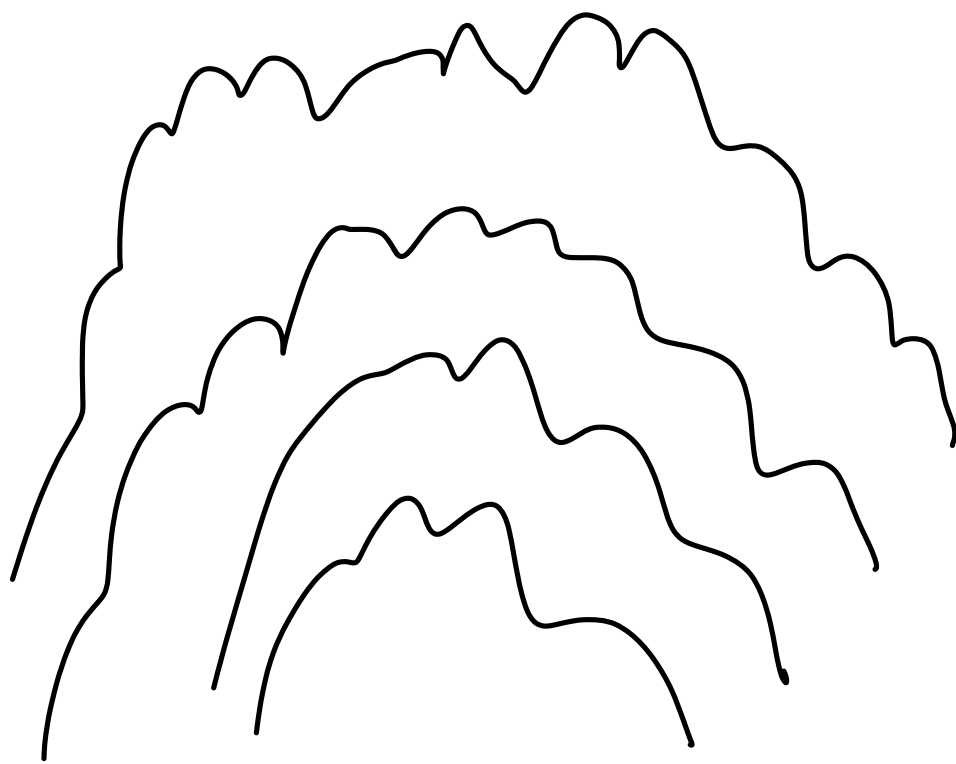


Q: Why Brownian?



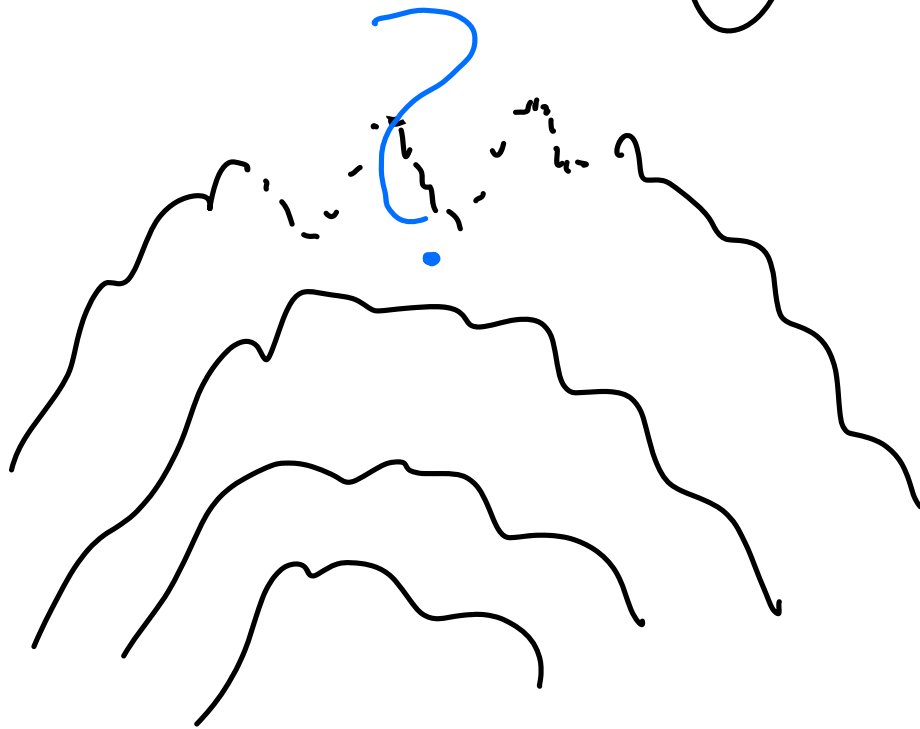
A: embed the  
parabolic  $A_{\text{sig}^2}$  process  
as the uppermost curve  
in an ENSEMBLE  
of random continuous  
curves —

— the parabolic  
Airy line ensemble



via the RSK correspondence

The ensemble satisfies

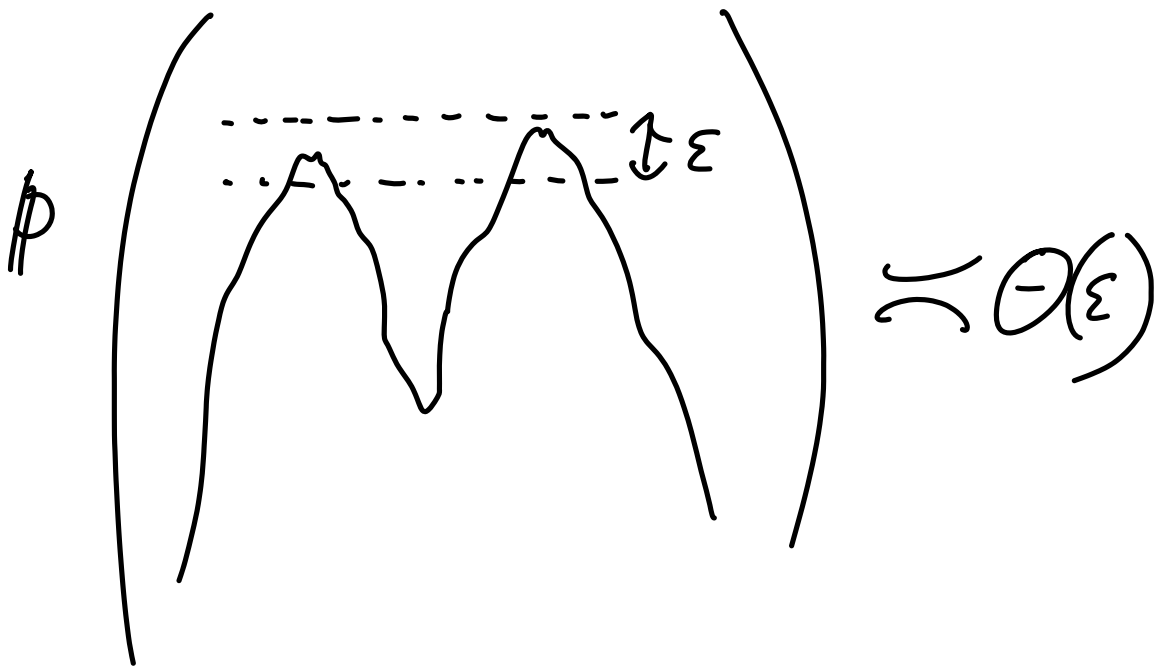


the BROWNIAN GIBBS

property

— a powerful tool for proving  
the Brownianity of the  $Airy_2$  process.

Twin peaks' probability



Calvert, Hegde, H.

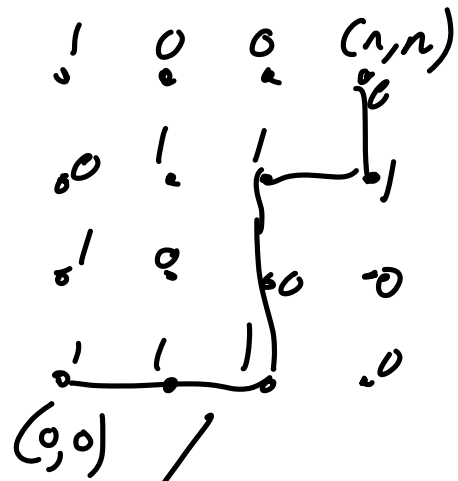
— 2019

What of chaos?

0	0	0	0	0	Dynamical Bernoulli LPP
0	0	0	0	0	
0	0	0	0	0	
0	0	0	0	0	

bits are independently  
updated at rate one  
Passion times

At time zero,  
a copy of static  
Bernoulli LPP

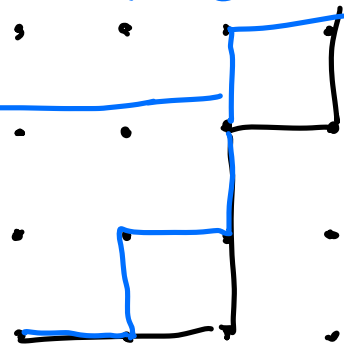


geodesic  $(0,0) \rightarrow (n,n)$

$\Gamma_n(0)$  has energy  $M_n(0)$

Equally, at any later time  $t > 0$ :

geodesic  $\Gamma_n(t)$   
has energy  $M_n(t)$ .

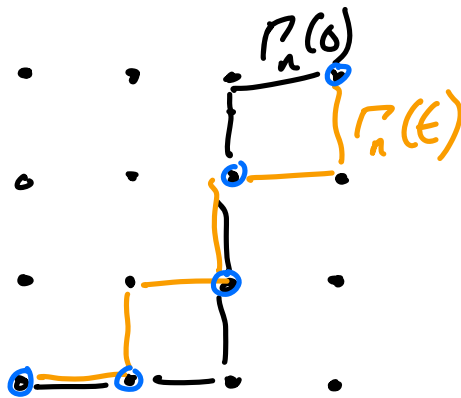


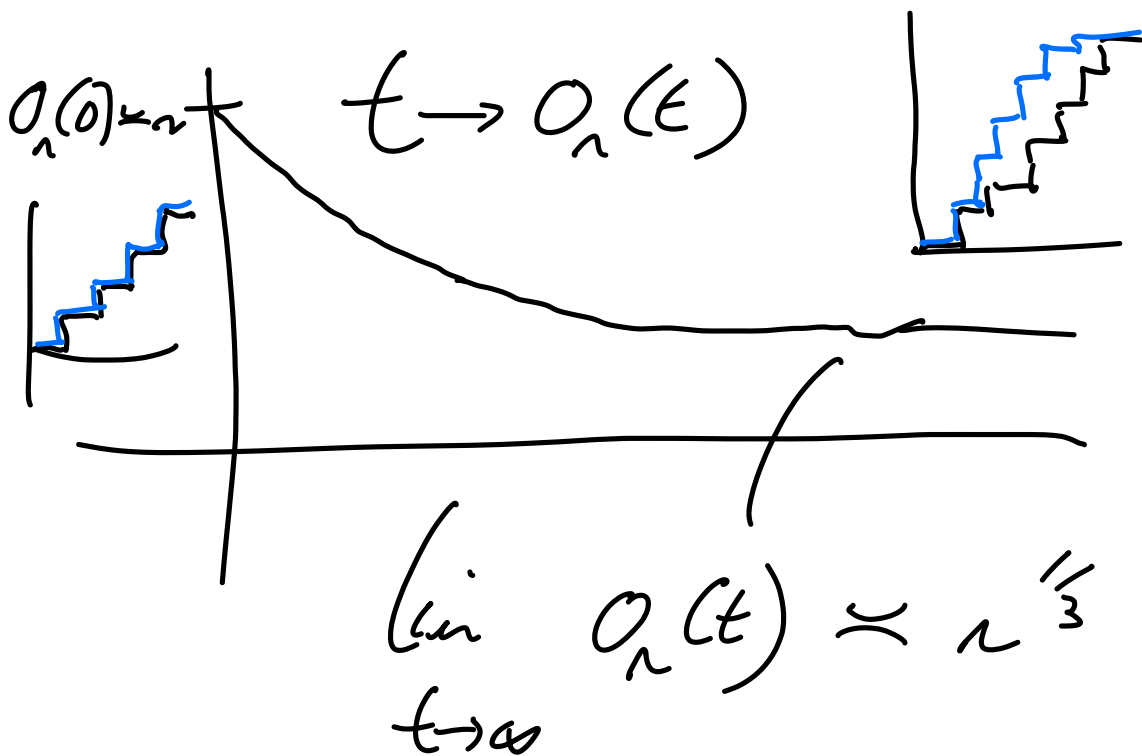
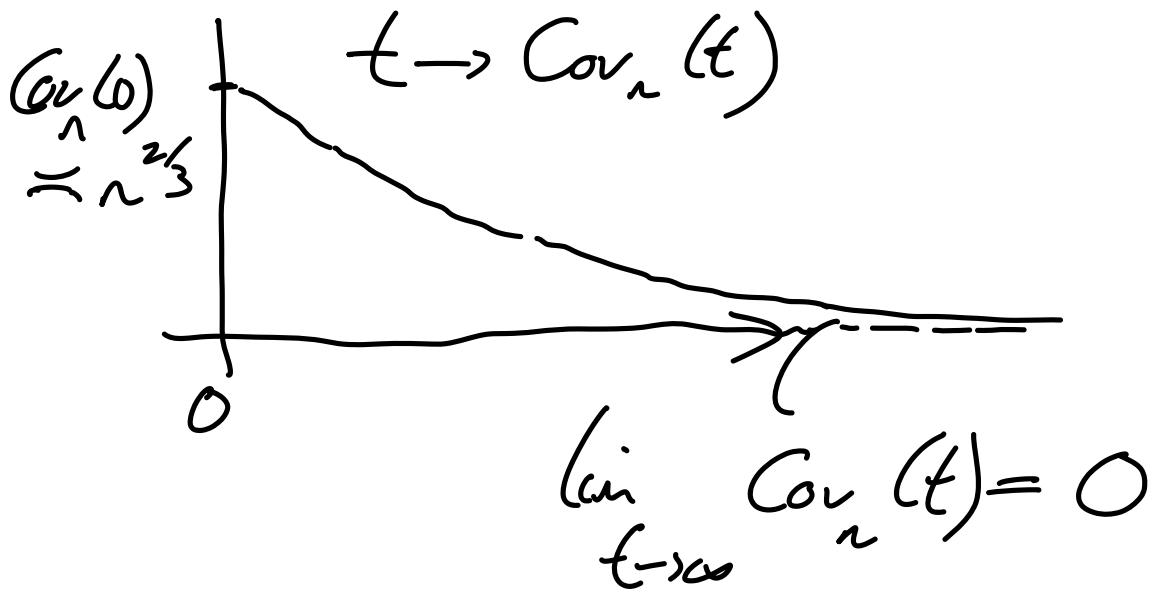
How to measure the fragility  
of these objects to the  
passage of time?

Energy:  $Cov_n(t) = Cov(m_n(0), m_n(t))$

Geodesic overlap:

$$O_n(t) = \# \text{ vertices in } \Gamma_n(0) \cap \Gamma_n(t)$$







Theorem [Ganguly - H.]

For dynamical BROWNIAN  
LPP, the TRANSITION from  
stability to chaos, measured  
by geodesic overlap, occurs  
on the scale  $t = n^{-1/3}$ .

Indeed, ....

There exists  $\delta > 0$  such that

$$P(O_n(t) \geq \delta n) \geq 1 - o(1)$$

when  $t \ll n^{-1/3}$  STABILITY

&

$$P(O_n(t) = o(n)) \geq 1 - o(1)$$

when  $t \gg n^{-1/3}$ .

CHAOS

Signposts to the proof

To show the transition  
stability  $\rightarrow$  chaos,

we may try to prove

Subcritical  $t \ll \tau^{-1/3}$       Supercritical  
 $t \gg \tau^{-1/3}$

Energy  
change

NOT MUCH:  
 $o(\tau^{1/3})$

A LOT:  
 $\Theta(\tau^{1/3})$

Geodesic  
overlap

is work with S.G.  
HIGH  $\checkmark$       LOW  $\checkmark$   
 $\Theta(\tau)$        $o(\tau)$

# Three-step game plan

Subcritical      Supercritical

Energy change

NOT MUCH

A LOT

Geodesic overlap

HIGH

LOW

substantial component of work with S.G.

Chatterjee's monograph / harmonic analysis



Step I:

supercritical  $t \gg n^{-1/3}$

geodesic overlap is LOW.

Chatterjee's theory of  
super-concentration and chaos

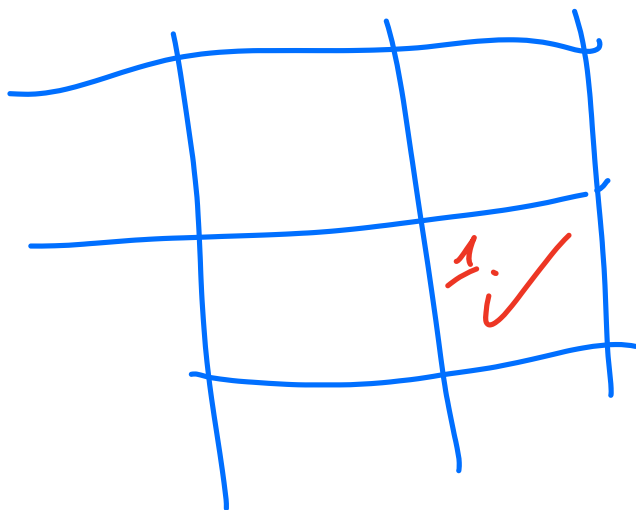
concerns Gaussian polymer models.

A beautiful formula: geodesic overlap  
between times 0 & t.

$$\text{Var } M_n = \int_0^\infty e^{-t} \mathbb{E} O_n(t) dt$$

(geodesic energy  $(0,0) \rightarrow (n,n)$ )

Since  $\text{Var}(\mu_n) = n^{-2/3}$ ,  
 $\# O_n(t)$  must fall  
precipitously from order  $n$   
at  $t$  advances through  
scale  $n^{-1/3}$ .



Step II

2.	
	1. ✓

Subcritical

stability of  
geodesic energy

— via harmonic  
analysis of LPP.

We will argue that, for  $t \gg 0$ ,

and with

$$\mu_n^t = \text{geodesic energy} \\ (0,0) \rightarrow (n,n)$$

in *dynamic* Bernoulli LPP,

$$\mathbb{E} \left( \mu_n^t - \mu_n^0 \right)^2 \leq \Theta(1) t n.$$

Take  $t = \tau n^{-1/3}$ , so that,

for  $\tau \ll 1$  subcritical scaled time,

$$\mu_n^t - \mu_n^0 = o(\mu_n^0).$$

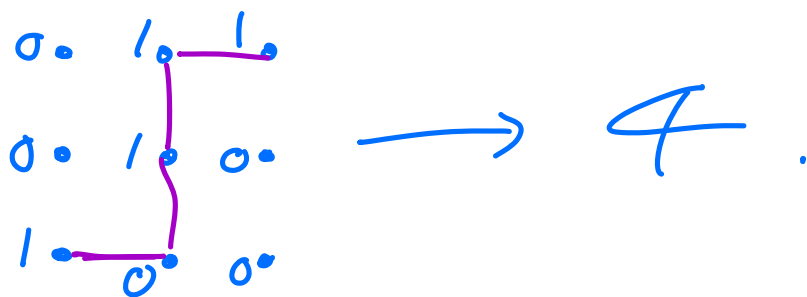


Key technique: discrete  
Fourier analysis.

Set  $\Lambda_n = \{0, 1, \dots, n\}^2$ .

Geodesic energy  $\mathcal{M}_n$   
is a map

$$\mathcal{M}_n: \{0, 1\}^{\Lambda_n} \rightarrow \mathcal{M}$$



To each subset  $S \subseteq \Lambda_n$ ,

define

$$\chi_S : \{0, 1\}^{\Lambda_n} \rightarrow \{-1, 1\}^{\Lambda_n},$$

$$\chi_S(\omega) = \prod_{x \in S} (2\omega(x) - 1),$$

$$\chi_\emptyset = 1.$$

The collection  $\{\chi_S : S \subseteq \Lambda_n\}$  is an orthonormal basis for the  $L^2$ -space of functions mapping  $\{0, 1\}^{\Lambda_n}$  to  $\mathbb{R}$ .

As such, we may decompose

$$\mu_n(\omega) = \sum_{S \subseteq \Lambda_n} \alpha(S) \chi_S(\omega),$$

where  $\alpha(\emptyset) = \mathbb{E} \mu_n$ .

Parseval's formula:

$$\text{Var}(\mu_n) = \sum_{S \neq \emptyset} \kappa(S)^2.$$

This identity permits us to introduce the **SPECTRAL SAMPLE**, a random variable  $f$  under a law — call it  $\mathbb{Q}$  — that is canonically associated to the function  $\mu_n: \{0,1\}^{\Lambda_n} \rightarrow \mathbb{R}$ :

$$Q(f=S) = \frac{\kappa(S)^2}{\text{Var}(\mu_n)}, \quad S \subseteq \Lambda_n.$$

Proposition [fundamental role of the spectral sample in studying dynamics]

Let  $t \geq 0$ . Then

$$\text{Cov}(\mu_n^0, \mu_n^t) = \sum_{S \neq \emptyset} \kappa(S)^2 e^{-t|S|}$$

and

$$\text{Cov}(\mu_n^0, \mu_n^t) = \mathbb{E}_Q [e^{-t|S|}].$$

The mean value of the spectral sample.

We will argue that

$$\mathbb{E}_Q |S| = \Theta(n)^{1/3}$$

(the mean value under  $Q$ ).

Alongside the proposition  
(and Jensen's  $\leq$ ),

we learn that ....

$$\mathbb{E} (\mu_n^t - \mu_n^0)^2 \leq 2 \text{Var}(\mu_n) \cdot \left(1 - e^{-\Theta(1) t n^{1/3}}\right).$$

When  $t = o(1) n^{1/3}$ ,

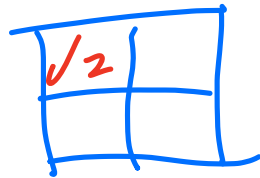
$$\begin{aligned} \text{RHS} &= \text{Var}(\mu_n) \Theta(1) t n^{1/3} \\ &= \Theta(1) t n. \end{aligned}$$

So ...

Subcritical energetic stability

$$\mu_n^\varepsilon - \mu_n^0 = o(\mu_n^0)$$

reduces to



deriving

$$E_Q |\mathcal{S}| = \Theta(1) n^{1/3}.$$

( three steps to this  
formula ...



First: for  $v \in \Lambda_n$ ,  
write  $\mu_n[v]$  for the value  
of  $\mu_n$  when the  $w$ -value  
at  $v$  is flipped.

Then:

$$\mathbb{E}_{\mathcal{Q}} |S| = \frac{1}{4 \text{Var}(\mu_n)} \sum_{v \in \Lambda_n} \mathbb{E} (\mu_n - \mu_n[v])^2.$$

Proof.

$$\mathbb{E}_Q |S| = \sum_{V \in \Lambda_n} Q(V \in S)$$

$$\stackrel{\text{def. of } f}{=} \frac{1}{\text{Var}(\mu_n)} \sum_{V \in \Lambda_n} \sum_{S: V \in S} \alpha(S)^2.$$

To prove the sought formula,  
it thus suffices to show that

$$\begin{aligned} & \mathbb{E} (\mu_n - \mu_n[V])^2 \\ &= 4 \sum_{S: V \in S} \alpha(S)^2 \quad (*) \end{aligned}$$

To see this, note that

$$\mu_n = \sum_S \alpha(s) \chi_S$$

&

$$\mu_n[v] = \sum_S \alpha(s) \chi_S[v]$$

(the value of  $\chi_S$  when the bit at  $v$  is flipped.)

So ...

$$\mathbb{E} (M_n - M_n[\nu])^2 = A - B,$$

$$\text{where } A = 2 \sum_1 \alpha(s)^2$$

$$\& B = 2 \sum_1 \alpha(s)^2 \chi_s \chi_{s^c}$$

$$= -2 \sum_{S: v \in S} \alpha(s)^2 + 2 \sum_{S: v \notin S} \alpha(s)^2$$

$$\Rightarrow A - B$$

$$= 4 \sum_{S: v \in S} \alpha(s)^2$$

so that (x) is obtained, as we sought.

We have confirmed:

$$\mathbb{E}_Q |S| = \frac{1}{4 \cdot \text{Var } \mu_n} \sum_{v \in \Lambda_n} \mathbb{E}(\mu_n - \mu_n[v])^2$$

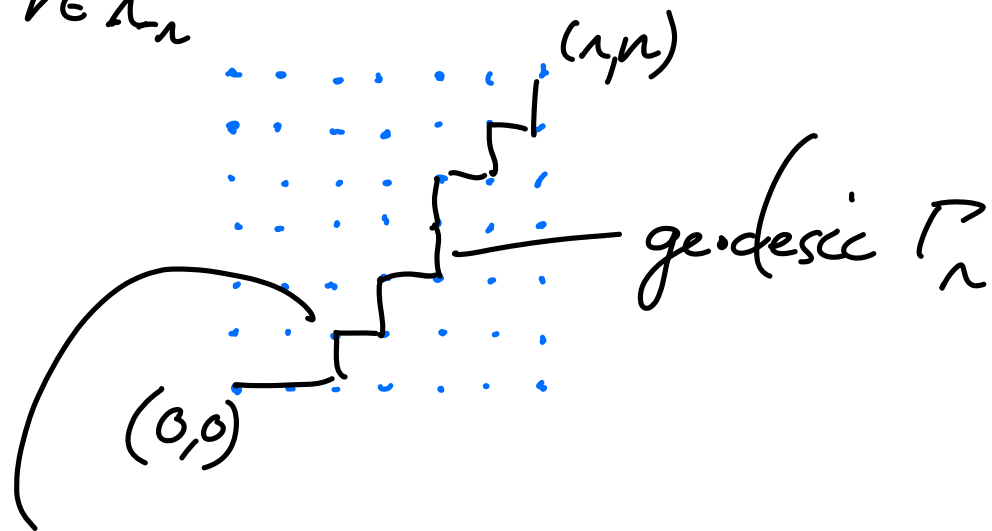
is of order  $n^{2/3}$

— this is LPP  
geodesic energy  
fluctuation

will  
show this  
to be  
 $\Theta(1)n$

To argue that:

$$\sum_{v \in \Lambda_n} \mathbb{E} (\mu_n - \mu_n[v])^2 = \Theta(1)n :$$



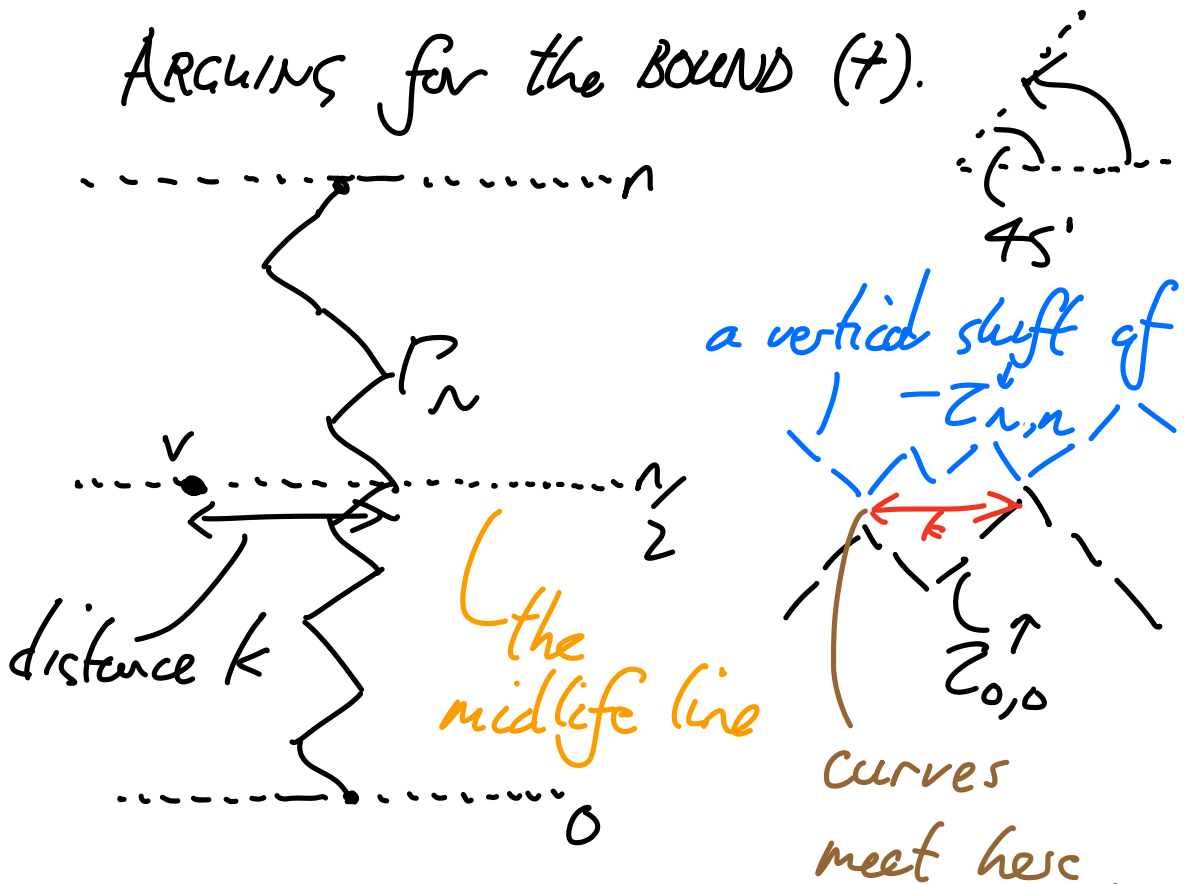
order  $n$  elements of  $\Gamma_n$

have  $|\mu_n - \mu_n[v]| = 1$ .

Wish to argue that only a comparable order of off-geodesic vertices  $v$  have  $\mu_n - \mu_n[v] \neq 0$ .



ARGUING for the BOUND (7).



Let  $Z_{0,0}^{\uparrow}$ : midlife line  $\rightarrow \mathcal{N}$  with probability  $\Theta(1)k^{-3/2}$ .

denote geodesic energy from  $(0,0)$  to points on the midlife line.

Let  $Z_{n,n}^{\downarrow}$ : midlife line  $\rightarrow \mathcal{N}$  denote geodesic energy from points on the midlife line to  $(n,n)$ .



In summary, we have argued for  
subcritical energetic stability

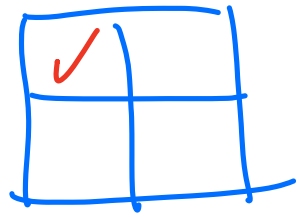
$$\mu_n^E - \mu_n^0 = o(\mu_n^0)$$

	Subcrit. free	Supercrit.
Energy change	✓ 2. NOT MUCH	A LOT
Geod. overlap	HIGH	LOW ✓ 1.

major technical challenge

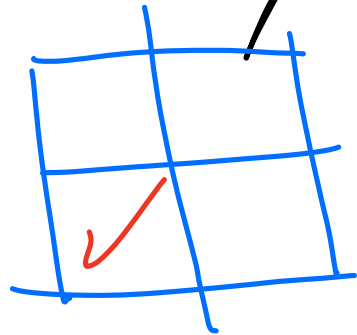
— establish this!

Harnessing  
subcritical energetic  
stability



to prove

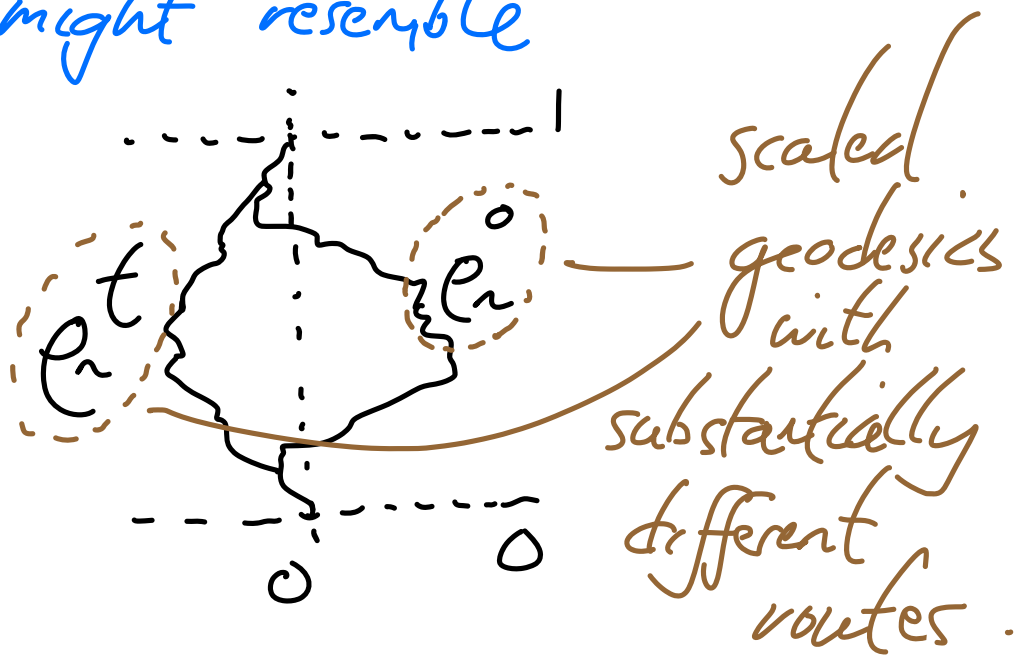
high subcritical overlap



—  
the proxy.

Suppose low overlap between  
geodesics  $\Gamma_n^o$  and  $\Gamma_n^t$   
for a subcritical time  $t \ll n^{-1/3}$ .

In the SCALED picture,  
a TYPICAL realization  
might resemble

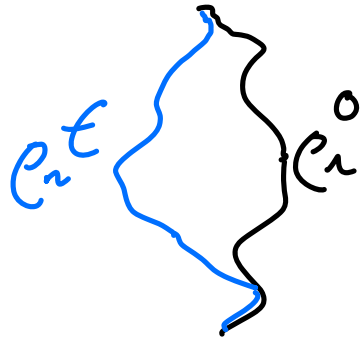


AIM: substantially transport  
the scaled geodesic  $\rho_n^\epsilon$   
at time  $t$  to a PROXY  
 $\rho_n^{\epsilon \rightarrow 0}$  — so that  
the proxy mimics the  
route of  $\rho_n^\epsilon$  and, in its  
time-zero weight, mimics  
the time- $t$  weight  
of  $\rho_n^\epsilon$ .

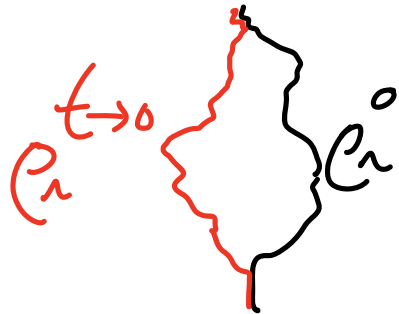


the proxy is formed of  
interpolating time  $\rightarrow 0$  local geodesics.

So the TYPICAL low overlap scenario



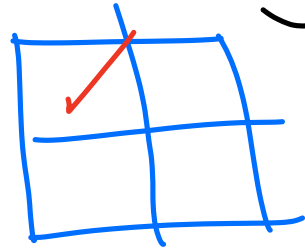
typically forces



a proxy whose route is  
substantially different  
to  $C_n^o$ 's.

But, by  
subcritical weight stability,

$\rho_n^t$  and  $\rho_n^0$



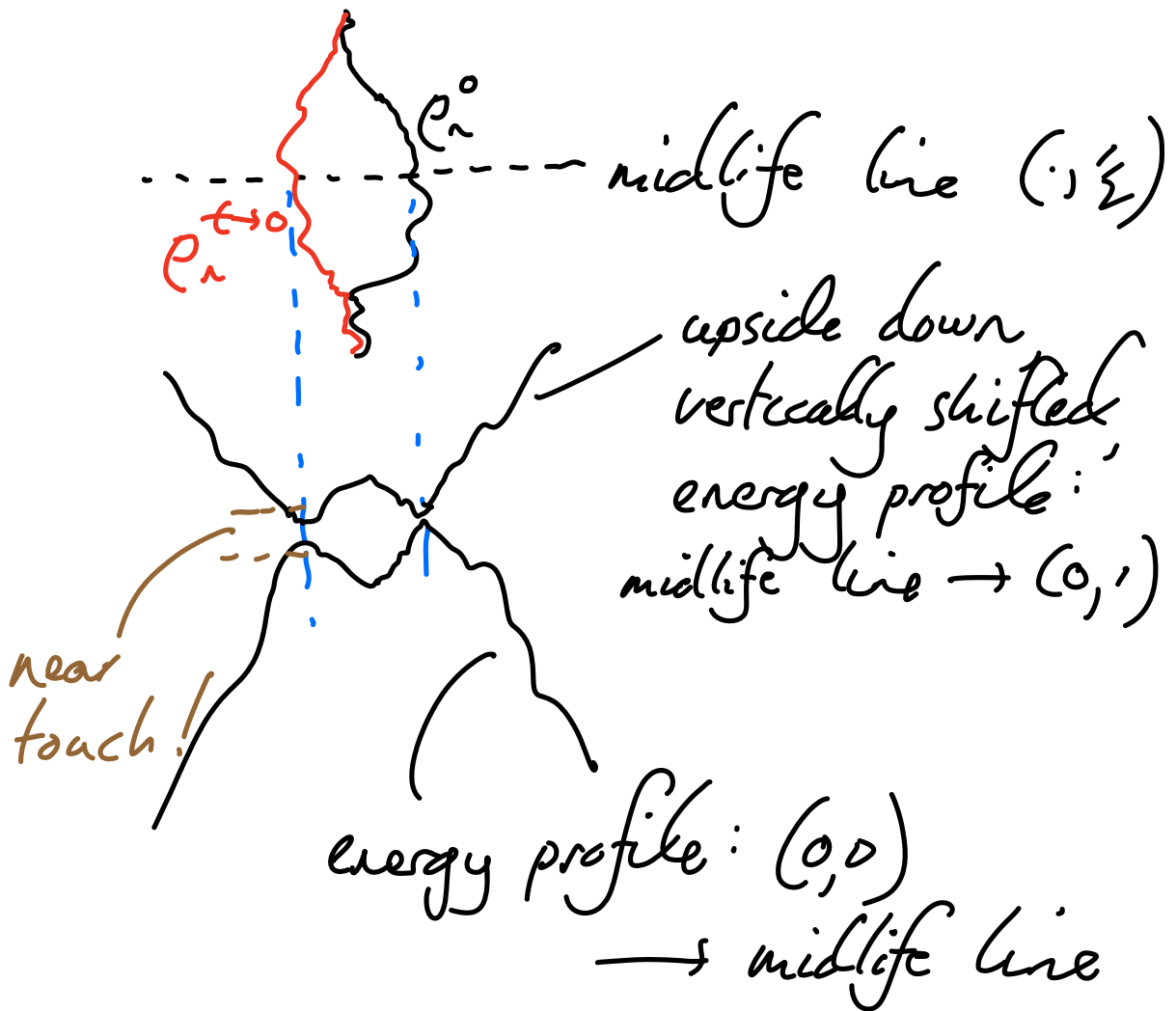
have very similar weight

— with  $o(1)$  difference.

Thus, so do  $\rho_n^{t \rightarrow 0}$  and  $\rho_n^0$ .

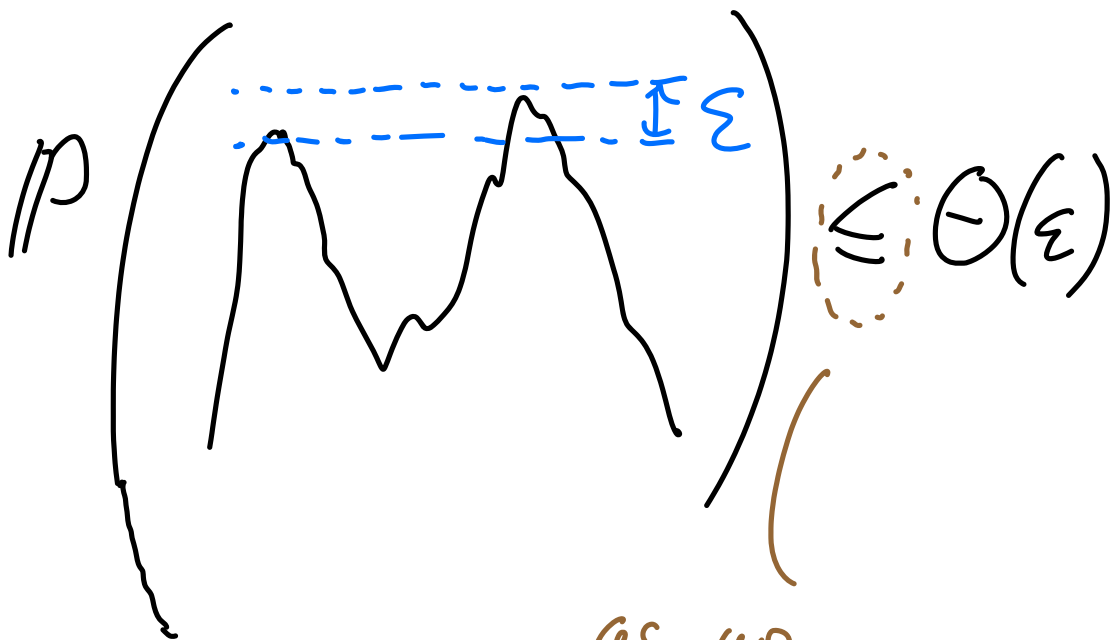
So ....

The proxy  $\rho_n^{t \rightarrow 0}$  is  
 close to  $\rho_n^0$  in its  
 time-zero weight.





This near-touch event is essentially the same as the 'two peaks' event:



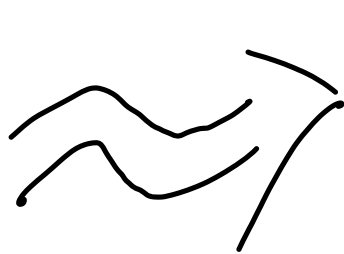
as we discussed earlier

In summary, the PROXY  
transports the geometric  
scenario TYPICALLY forced  
by LOW OVERLAP between  
 $\rho_n^0$  &  $\rho_n^t$ , for  $t \ll n^{-1/3}$ ,  
to a provably rare  
TWIN PEAKS'

scenario at time zero.

Thues,

	Subcrit. time	Supercrit. time
Energy change	LOW 2. ✓	HIGH
Geod. overlap	3. ✓ A LOT	✓ 1. A LITTLE



the MAIN theorem

— transition in overlap to CHAOS.