The Fyodorov-Hiary-Keating conjecture

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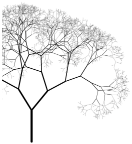
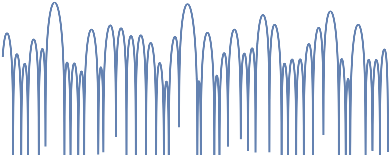




$\log|\zeta(\frac{1}{2}+\mathrm{i}s)|, \quad 10^6 \leq s \leq 10^6+1$

Extreme values for ζ

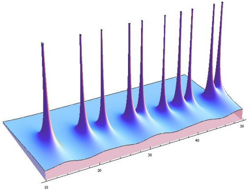
The Riemann zeta function :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^s}}, \text{ if } \mathfrak{Re}(s) > 1,$$

Analytic continuation to $\mathbb{C}$, except at 1. This continuation admits a functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

$|\zeta|^{-1}$:



Understanding the size of the Riemann zeta function on the line $\mathrm{Re}s = \frac{1}{2}$ is of great interest. There are two compelling conjectures concerning the maximal size of ζ.

Conjecture 1

There exists $C > 0$ such that $\max_{0<t<T} \log|\zeta(\frac{1}{2} + \mathrm{i}t)| = (C + \mathrm{o}(1))\frac{\log T}{\log\log T}$.

This would mean that the known upper bounds (under RH) are close to the truth : Chandee-Soundararajan proved $C = (\log 2)/2$ is possible.

There is an alternative conjecture due to Gonek-Farmer-Hughes.

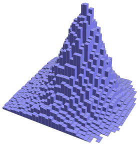
Conjecture 2

We have $\max_{0<t<T} \log|\zeta(\frac{1}{2} + \mathrm{i}t)| = (\sqrt{2} + \mathrm{o}(1))\sqrt{\log T \log\log T}$.

Consistent with $\log|\zeta(\frac{1}{2} + \mathrm{i}t)|$ having Gaussian tail with wide uniformity.

Theorem (Selberg's CLT (1946))

$$\frac{1}{T}\mathrm{meas}\left\{t \leq T : \frac{\log|\zeta\left(\frac{1}{2} + \mathrm{i}t\right)|}{\sqrt{\frac{1}{2}\log\log T}} \leq x\right\} \to \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y$$



Fyodorov-Hiary-Keating conjecture (2012)

For any $y > 0$, as $T \to \infty$ we have

$$\frac{1}{T}\text{meas}\left\{t \le T : \max_{|t-u|\le 1} \left|\zeta\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| > \frac{\log T}{(\log\log T)^{3/4}} e^{y}\right\} \to F(y),$$

where $F(y) \sim Cye^{-2y},\ y \to \infty$.

This exponential decay seems to suggest that Conjecture 1 is the right order, i.e. the tail in Selberg's CLT is exponential instead of Gaussian.

This extrapolation is not correct : A more precise conjecture is : uniformly in $1 < y < \log\log T$,

$$\frac{1}{T}\text{meas}\left\{t \le T : \max_{|t-u|\le 1} \left|\zeta\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| > \frac{\log T}{(\log\log T)^{3/4}} e^{y}\right\} \asymp Cy^{-2y}e^{-\frac{y^2}{\log\log T}},$$

a Gaussian decay supporting Conjecture 2.

Extremes and log-correlation

The independent, Gaussian case. Exercise : X_i's i.i.d. with density $\frac{e^{-x^2}}{\sqrt{\pi}}$ (variance $\frac{1}{2}$).

$$\mathbb{P}\left(\max_{1\leq i\leq N} X_i \leq \lambda\right) = \mathbb{P}(X_1 \leq \lambda)^N = \left(1 - \int_\lambda^\infty \frac{e^{-x^2}}{\sqrt{\pi}}\mathrm{d}x\right)^N \sim \exp\left(-N\frac{e^{-\lambda^2}}{\lambda\sqrt{\pi}}\right)$$

In particular, for i.i.d. Y_i's $\mathscr{N}(0, \frac{1}{2}\log N)$, we obtain

$$\max_{1\leq i\leq N} Y_i = \log N - \frac{1}{4}\log\log N + Z_N$$

with $\mathbb{P}(Z_N \leq \lambda) \to \exp(-e^{-2\lambda}/(2\sqrt{\pi}))$.

So the factor $3/4$ in the FHK conjecture suggests some form of (positive) correlations between the ζ values around maxima.

Beyond independence : Logarithmically-correlated fields. Metric spaces $V_1 \subset V_2 \subset \dots$ with distance d, stochastic process X^N on V_N satisfying

$$\mathbb{E}(X_v^N X_{v'}^N) = -\log\left(d(v,v') + \frac{1}{N}\right) + \text{bounded function}$$

Slow decay of correlations.

Superposition of ind. fields on scales : $\log u = \int_0^\infty \left(e^{-\frac{1}{2t}} - e^{-\frac{u^2}{2t}}\right)\frac{\mathrm{d}t}{t}$.

From [B, 2009], $\log|\zeta(1/2 + \mathrm{i}(t+h))|$ and $\log|\det(e^{\mathrm{i}h} - U_N)|$ are log-correlated as h varies :

(i) for ζ this follows from Selberg's original method

(ii) for U_N this relies on the seminal moment identities of Diaconis and Shashahani (distributional sense by Hughes-Keating-O'Connell).

Universality for the maximum of Gaussian log-correlated fields by Ding, Roy, Zeitouni.

Example 1 : the Branching Brownian motion.

Branching rule : After a random time with exponential distribution, a Brownian motion splits into two independent ones. And so on.

$$d(v,v') = e^{t-(v\wedge v')}.$$

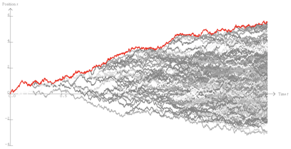


Image : M. Roberts

McKean (1975) connected it to the Fisher-Kolmogorov-Petrovsky-Piskunov reaction-diffusion equation,

$$\partial_t u = \tfrac{1}{2}\partial_{xx}u + u^2 - u, \text{ with step initial condition :}$$
$$u(t,x) = \mathbb{P}(\max_v X_v(t) < x)$$

If e^t ind. Gaussians of variance $t/2$, the maximum $\approx t - \frac{1}{4}\log t$. But :

Theorem (Bramson, 1978)

The maximum $\approx t - \frac{3}{4}\log t$.

Bramson's barrier method.

Let $Z = \#\{v : X_v(t) > t - \frac{3}{4}\log t + b_t\}$, $b_t \to \infty$ slowly.

We have $\mathbb{E}\, Z \to \infty$, first sign that the branching structure matters for the subleading order. The divergence of the expectation comes from atypical events that inflate the expectation.

Look for A_v such that $\widetilde{Z} = \#\{v : X_v(t) > t - \frac{3}{4}\log t + b_t, A_v\}$, satisfies $\mathbb{E}\,\widetilde{Z} \to 0$ but $\mathbb{P}(\cap_v A_v) \to 1$. A pertinent choice :

$$A_v = \{X_v(s) < s + M, s \leq t\},\ M = (\log t)^2 \text{ for example.}$$

Implementation requires the classical Ballot theorem :

Theorem

Conditioned on $B_t = p < M$, the probability that a Brownian motion remains below M up to time t is of order

$$\frac{M(M-p)}{t}.$$

Example 2 : the 2d discrete Gaussian free field.

On a $N \times N$ square of $\mathbb{Z}^2$, density

$$\frac{1}{Z} e^{-\frac{1}{8}\sum_{v\sim v'}(X_v - X_{v'})^2}$$

with zero boundary condition.

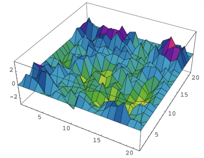


Image : S. Sheffield

$$\mathbb{E}(X_N(v)X_N(v')) = \mathbb{E}_v\left[\sum_{k=1}^{\text{exit time}} \mathbb{1}_{S_k=v'}\right] \sim C_1 - C_2 \log\left(d(v,v') + \frac{1}{N}\right).$$

Theorem (Bramson, Ding and Zeitouni, 2013)

$$C_N \max_v X_N(v) = \log N - \frac{3}{4}\log\log N + Z_N$$

with Z_N converging in distribution. The tail is $\asymp \lambda e^{-c\lambda}$ up to $\lambda < \sqrt{\log N}$.

For the proof, help from a branching structure behind X_N obtained by averaging the field on boxes of size $2^{k_0-k} \times 2^{k_0-k}$, $k \le k_0 = \log N$.

Back to Fyodorov-Hiary-Keating

Fyodorov-Hiary-Keating conjecture. Consider a $N \times N$ Haar-distributed random unitary matrix U_N, and t be uniform on $[T, 2T]$.

$$\max_{\theta\in[0,2\pi]} \log|\det(e^{i\theta} - U_N)| = \log N - \frac{3}{4}\log\log N + X_{\mathrm{U}} + o_{\mathbb{P}}(1),$$

$$\max_{|t-u|\leq 1} \log\left|\zeta\left(\tfrac{1}{2} + iu\right)\right| = \log\log T - \frac{3}{4}\log\log\log T + X_\zeta + o_{\mathbb{P}}(1),$$

as $N, T \to \infty$.

X_{U}, X_ζ have the same tail distribution

$$\mathbb{P}(X_\zeta > y) \sim Cye^{-2y}, \ y \to \infty$$

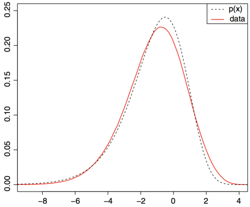


FIG. 1 (color online). Numerical computation (solid red line) compared to theoretical prediction (13) (dashed black line) for $p(x)$.

Numerics : Fyodorov, Hiary, Keating. $T = 10^{28}$!

Why this correspondence ?

Assume the Riemann hypothesis, and let $\frac{1}{2} \pm \mathrm{i} t_n$ be the ζ zeros.

$$w_n = \frac{t_n}{2\pi} \log \frac{t_n}{2\pi}$$

Theorem (Montgomery, 1972)

If f is a Schwartz function with Fourier transform supported on $(-1, 1)$, then

$$\frac{1}{x} \sum_{1 \leq j,k \leq x, j \neq k} f(w_j - w_k) \underset{x \to \infty}{\longrightarrow} \int_{-\infty}^{\infty} \mathrm{d}y \; f(y) \left(1 - \left(\frac{\sin \pi y}{\pi y}\right)^2\right).$$

All orders correlations coincide : Rudnick, Sarnak (1996), for restricted Fourier support.

Fyodorov-Hiary-Keating conjecture, for $U(N)$.

$$\max_{\theta\in[0,2\pi]}\log|\det(e^{\mathrm{i}\theta}-U_N)|=\log N-\frac{3}{4}\log\log N+X_{\mathrm{U}}+\mathrm{o}_{\mathbb{P}}(1).$$

Method : Computation of moments of moments (made rigorous by Bailey-Keating in the integer case)

$$\mathbb{E}\left[\left(\int_0^{2\pi}|\det(e^{\mathrm{i}\theta}-U)|^{2q}\mathrm{d}\theta\right)^{2p}\right].$$

They essentially coincide with the Fyodorov-Bouchaud gaussian model, for which maxima were predicted.

Theorem (Chhaibi, Najnudel, Madaule, 2016)

The random variable $\max_{\theta\in[0,2\pi]}\log|\det(e^{\mathrm{i}\theta}-U_N)|-(\log N-\frac{3}{4}\log\log N)$ is tight (for any circular β-ensemble).

Related results for $\beta=2$:

(i) Arguin, B. and Belius obtained the $\log N$ (2015)

(ii) Paquette and Zeitouni proved the second order (2016)

Rely on identification of increasingly pertinent branching structures.

Theorem (ABR, 2020 (upper bound) and 2021 (lower bound))

There is $C > 0$ such that for any $y, T > 1$, we have

$$\frac{1}{T}\mathrm{meas}\left\{t \leq T : \max_{|t-u|\leq 1} \left|\zeta\left(\tfrac{1}{2} + \mathrm{i}u\right)\right| > \frac{\log T}{(\log\log T)^{3/4}} e^{y}\right\} \leq C y e^{-2y},$$

For any $\varepsilon > 0$ there exists $y, T_0 > 0$ such that for any $T > T_0$

$$\frac{1}{T}\mathrm{meas}\left\{t \leq T : \max_{|t-u|\leq 1} \left|\zeta\left(\tfrac{1}{2} + \mathrm{i}u\right)\right| > \frac{\log T}{(\log\log T)^{3/4}} e^{-y}\right\} \geq 1 - \varepsilon,$$

Convergence in distribution along subsequences.

The upper tail and uniformity in y is similar to [BDZ] on the 2d GFF. It probably also holds for random matrices.

The method of proof is flexible, and relies on a multiscale analysis with twisted moments as key number theoretic-input.

Related results.

About this maximum

- Najnudel (2016) : leading order, under RH for lower bound
- Arguin, Belius, B., Soundararajan, Radziwill (2016) : same result, no RH
- Harper (2019) : upper bound up to the second order, with error $\mathrm{O}(\log\log\log T)$

Other directions

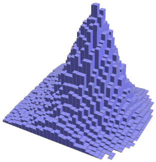
- Arguin, Belius, Harper (2015) : second order for a random model for ζ
- Najnudel (2016) : leading order, under RH, for $\operatorname{Im}\log\zeta$
- Arguin, Ouimet, Radziwill (2019) : first order in intervals of size $(\log T)^{\theta}$
- Arguin, Dubach, Hartung (2021) : transition "from 3/4 to 1/4" for a random model as the size of the interval is $(\log T)^{\theta}$, $\theta \sim (\log\log T)^{-\alpha}$

Proof

First step : Gaussian.

Theorem (Selberg, 1946)

Let ω be uniform on $[0,1]$. Then
$\frac{\log|\zeta\left(\frac{1}{2}+\mathrm{i}\omega T\right)|}{\sqrt{\frac{1}{2}\log\log T}} \to \mathcal{N}(0,1)$ as $T\to\infty$.



Selberg's proof proceeds in two steps :

1. Cut the tail in the Euler product

$$\frac{1}{T}\int_0^T \left|\log\zeta(1/2+\mathrm{i}s) - \sum_{p\leq t}\frac{p^{-\mathrm{i}s}}{\sqrt{p}}\right|^2 \mathrm{d}s < C.$$

2. Then quantify the fact that (U_p's independent on the unit circle)

$$\mathbb{E}\left[\prod_{p\in I}(p^{\mathrm{i}\omega T})^{\alpha_p}\overline{p^{\mathrm{i}\omega T}}^{\beta_p}\right] \approx \mathbb{E}\left[\prod_{p\in I}U_p^{\alpha_p}\overline{U_p}^{\beta_p}\right]$$

Second step : Branching. Let w be random uniform on $[0,1]$ and

$$Y_\ell(h) = \sum_{e^{\ell-1}<\log p<e^\ell} \frac{\mathrm{Re} p^{-\mathrm{i}(\omega T+h)}}{\sqrt{p}}, \quad 1 \leq \ell \leq \log\log T$$

$$S_k(h) = \sum_{\ell=1}^{k} Y_\ell(h)$$

From the prime number theorem, $\mathbb{E}\left[|Y_\ell(h)|^2\right] = \frac{1}{2}$ with good precision. Moreover, log-correlation comes from

$$\mathbb{E}\left[Y_\ell(h_1)Y_\ell(h_2)\right] \approx \frac{1}{2} \text{ if } |h_1-h_2| \ll e^{-\ell},\ 0 \text{ if } |h_1-h_2| \gg e^{-\ell}.$$

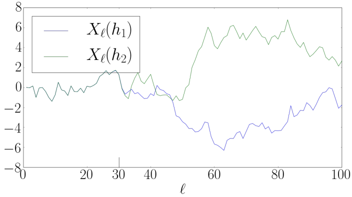


Figure – Illustration of the processes $S_k(h_1)$ and $S_k(h_2)$.

Basic heuristics : Let $n = \log\log T$.Then S_n achieving a high value $\approx n$ requires all $Y_\ell, \ell \leq n$ to be unusually large. These increments need to line up and the partial sums lie in a corridor : $S_k \approx k,\ k \leq n$.

Analytic number theory barrier. To find h such that $S_k(h) \approx k$, we need to identify the moments of the random walk

$$\mathbb{E}\left[(S_k)^{2q}\right] = \mathbb{E}\left[\mathscr{N}(0, k/2)^{2q}\right] + \mathrm{O}\left(\frac{\exp(2qe^k)}{T}\right)$$

up to $q \approx k$.

The error term allows to do this for $k < \log\log T - C \log\log\log T$. This poor control on last increments is a number theory barrier ; one cannot work directly with primes.

This problem is avoided through lower barrier estimates, obtained thanks to twisted moments of ζ.

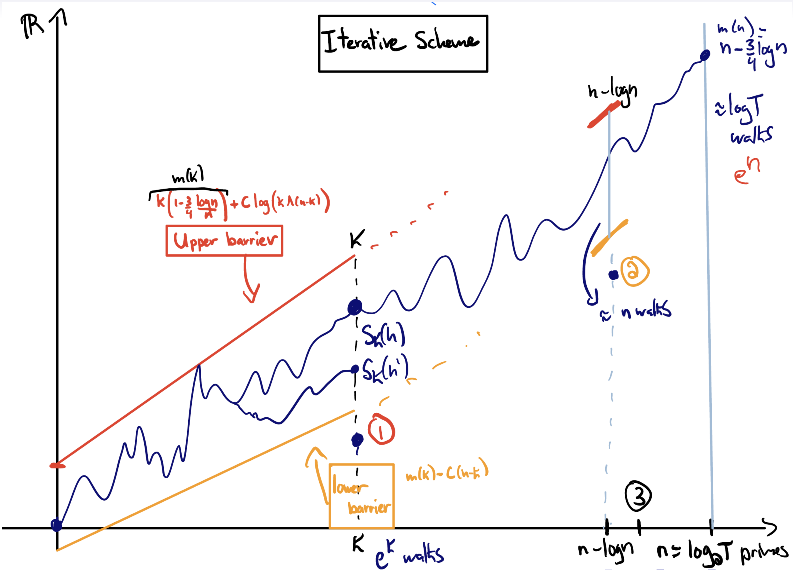


Image : L.-P. Arguin

Iteration for the upper bound.

First, discretize : the maximum over $\log T$ points h is enough (Poisson summation formula).

Let G_k be the set of h such that the walk keeps in the corridor up to time k, and $H = \{h : |\zeta| > \frac{e^n}{n^{3/4}} e^y\}$ The key estimate is (approximately)

$$\mathbb{P}(\exists h \in H \cap G_{n_\ell} \cap G^c_{n_{\ell+1}}) \leq \frac{y e^{-2y}}{(n - n_\ell)^2}, \; n_\ell = n - \log_{\ell+1} n.$$

Iterations then show that high points need to be in the corridor. But for $n - n_\ell = \mathrm{O}(1)$ this is unlikely by twisted moments for ζ.

These twisted moments are also used in the proof of the above inequality.

Decoupling between primes : in a primitive version, the **twisted fourth moment** states that

$$\mathbb{E}\Big[|(\zeta\mathcal{M}^{(k)})|^4\cdot|\mathcal{Q}^{(k)}|^2\Big]\ll\mathbb{E}\Big[|(\zeta\mathcal{M}^{(k)})|^4\Big]\cdot\mathbb{E}\Big[|\mathcal{Q}^{(k)}|^2\Big],$$

where $\mathcal{M}^{(k)}(s)$ is a proper approximation of $\zeta^{-1}(s)=\sum\frac{\mu(n)}{n^s}$ which takes into account all primes up to $\exp(e^k)$, and $\mathcal{Q}^{(k)}$ is any Dirichlet series with non-trivial summands supported on multiples of primes smaller than $\exp(e^k)$.

In our case, pick $k=n_{\ell+1}$ encode the event "upper barrier up to $n_{\ell+1}$, lower barrier up to n_ℓ" into $\mathcal{Q}^{(n_\ell+1)}$. Then the final gap cannot be too large because of the above decoupling.

It is remarkable that

(i) no higher moments than 4 are needed to obtain tightness ;

(ii) twisted moment are accessible with current number theory technology only up to fourth order.

Additional ideas, with inspiration from a simple, alternate proof of Selberg's CLT by Radziwill and Soundararajan.

Shift from the critical axis for the lower bound to have better approximations for ζ. For us this is harmless because ($\varepsilon > 0$, $V > 1$ and $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-1/2-\varepsilon}$)

$$\mathbb{P}\left(\max_{|t-u|\leq 1} |\zeta\left(1/2 + \mathrm{i}t\right)| > V\right) \geq \mathbb{P}\left(\max_{|t-u|\leq \frac{1}{4}} |\zeta\left(\sigma + \mathrm{i}t\right)| > 2V\right) + \mathrm{o}(1).$$

Avoid large multiplicities for the approximation of ζ^{-1} and $\mathcal{Q}^{(k)}$,as they hurt in large moments. This can be achieved in accordance with the Erdős-Kac theorem, which states that N typically has $\log_2 N$ prime factors (and $\sqrt{\log\log N}$ normal fluctuations) :

$$\mathcal{M}_\ell(h) = \sum_{\substack{p|m \implies p\in(T_{\ell-1},T_\ell] \\ \Omega_\ell(m)\leq (n_\ell - n_{\ell-1})^{10^5}}} \frac{\mu(m)}{m^{\sigma+\mathrm{i}\tau+\mathrm{i}h}},$$

and $\mathcal{M}^{(k)} = \prod_\ell \mathcal{M}_\ell$.

At the level of extremes, the universality class of log-correlated fields includes non-Gaussian models such as $|\det(z - U_N)|$ and $|\zeta|$.

The proofs rely on underlying branching structures.

The analysis of high modes, or the process on the last edges of the tree, poses challenges to random matrix theory and analytic number theory.

For ζ, the key inputs (in a multiscale analysis involving lower barriers) are twisted moments.

This multiscale analysis is expected to apply to general classes of log-correlated fields with poor control on high modes.