Spectral gap in regular graphs and hypergraphs

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Joint work with Gerandy Brito, Kameron Harris, Yizhe Zhu

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Graphs

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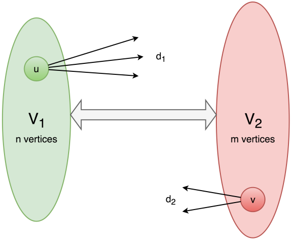

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- Data science / Machine Learning (matrix completion, coding, community detection, recommender systems, pattern recognition, bioinformatics)

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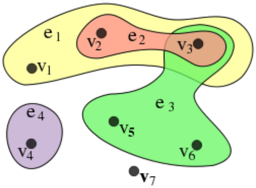

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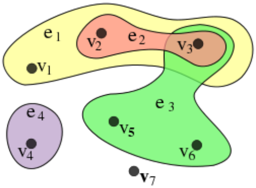

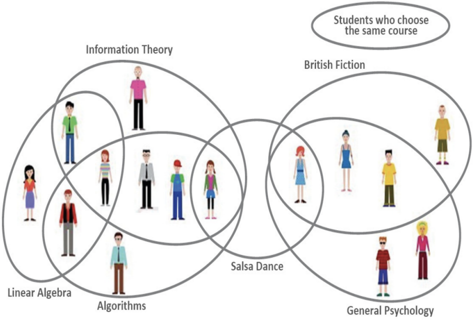

Examples include co-authorship, social networks, protein interactions.

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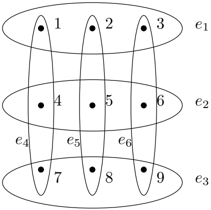

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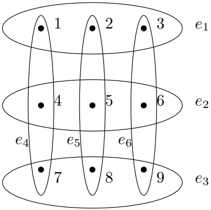

  - Any hypergraph has a natural *adjacency matrix*; any uniform hypegraph has a natural *adjacency tensor*. More later.

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  - Regular: uniform distribution among all possible $(d, k)-$hypergraphs.

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- In particular, the spectra of random graphs/hypergraphs is connected to *expansion*, and as such can (also) be used in the study of satisfiability, recommender systems, pattern recognition, matrix completion, etc. AND recently, neural networks.
- Best way to understand expanders is via the *spectral gap*.

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- If edges are not independent (regular, bipartite biregular) entries are *weakly dependent*. Interesting, slightly harder, but more rigid and other benefits (connectivity).
- For both regular and biregular bipartite, global shape of spectra well-understood since the '80s and '90s (Kesten-McKay, Godsil-Mohar); outliers are more recent.

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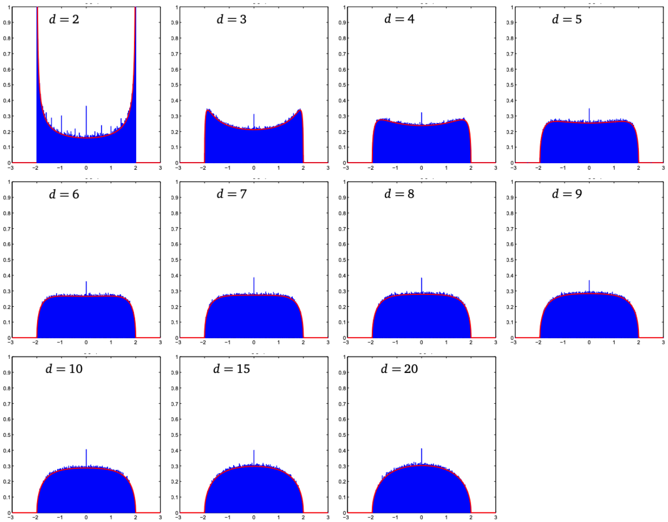
- As $d \to \infty$, f_d converges to the semicircle density (**Wigner**).

From Kesten-McKay to Semicircle



Courtesy of Yufei Zhao.

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- Huang, Yau ('21) : fluctuations at the edge are polynomially small, eigenvalue rigidity, eigenvector delocalization.

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- Remaining to be settled: d slowly growing, $d \gg n^{2/3}$.

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- A BBG graph (m, n, d_1, d_2) has its largest (Perron-Frobenius) eigenvalues $|\lambda_{1,2}| = \sqrt{d_1 d_2}$, with fixed eigenvectors.

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- [BDH'21]: for fixed d_1, d_2, $|\lambda_{3,4}| = \sqrt{d_1 - 1} + \sqrt{d_2 - 1} + o(1)$ as $m, n \to \infty$.

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- Problem is still quite open.

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- Both tensor and adjacency matrix connected to expansion, but matrix is easier to analyze (D., Zhu '20)

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- Algebraic constructions: Martinez-Stark-Terras ('01), Li ('04), Sarveniazi ('07)

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- Connects to expansion, mixing lemma, non-backtracking spectral norm, etc.

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  - For regular graphs, largest eigenvalue is deterministic $\lambda_1 = d$, eigenvector is all-ones; we want $|\lambda_2|$

(Some) RM methods for largest eigenvalue

- Bounding largest eigenvalues general idea:
  - If A positive definite find upper bound on $\lambda_1 = ||A||$ by $|\mathrm{Tr}(\tilde{A}^m)|^{1/m}$ (Gelfand lemma) as m grows large
  - Generally this has to be done with a lot more care than moment method proofs (which have m finite) but it can be done (see, e.g., Sinai and Soshnikov '98, Soshnikov '00)
  - For regular graphs, largest eigenvalue is deterministic $\lambda_1 = d$, eigenvector is all-ones; we want $|\lambda_2|$
  - Can try to work with $\tilde{A} = A - \frac{d}{n}J$.

Illustration on regular graphs

- Bounding second eigenvalues general idea:
  - In case $\lambda_1 = d$, $\lambda_2 = O(\sqrt{d})$, would like to be able to prove

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  - Also, in this case, $\lambda_2 = d$.
  - Same is true for the bipartite biregular case. We need to change the matrix.

Nonbacktracking matrix / Hashimoto operator

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- Uses its connection to Ihara-Bass formula

Non-backtracking (Hashimoto) matrix for RBBGs

- Idea: Examine instead the "non-backtracking" (aka Hashimoto) matrix B (of size $2|E| \times 2|E|$) whose rows/columns indexed by ordered edges, and $B_{ef} = 1$ iff $\sigma(e) = f$ and $\sigma(f) \neq e$. Non-symmetric.

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- One may choose the ordering of edges to write $B = \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}$.
- Largest eigenvalues $\lambda_{1,2|E|} = \pm\sqrt{(d_1-1)(d_2-1)}$, with eigenvector $\mathbf{1}_\alpha := (\mathbf{1}_{|E|}, \pm\alpha\mathbf{1}_{|E|})$ where $\alpha = \frac{\sqrt{d_1-1}}{\sqrt{d_2-1}}$.

Non-backtracking (Hashimoto) matrix

- Can relate the eigenvalues of B to those of the adjacency matrix A via the Ihara-Bass formula

$$\det(B - \lambda I) = (\lambda^2 - 1)^{|E|-n} \det((D - I) - \lambda A + \lambda^2 I) ,$$

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- Spectral gap for B yields spectral gap for A (for BBG, also noticed by Kempton ('16)). Hard part: show spectral gap for B.

NB matrix, classical and modern

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- Since '15, not only used for regular graphs but also for non-homogeneous Erdős-Rényi (Benaych-Georges, Bordenave, Knowles, '17, Alt, Ducatez, Knowles '19, '21), directed graphs (Coste '17), etc.

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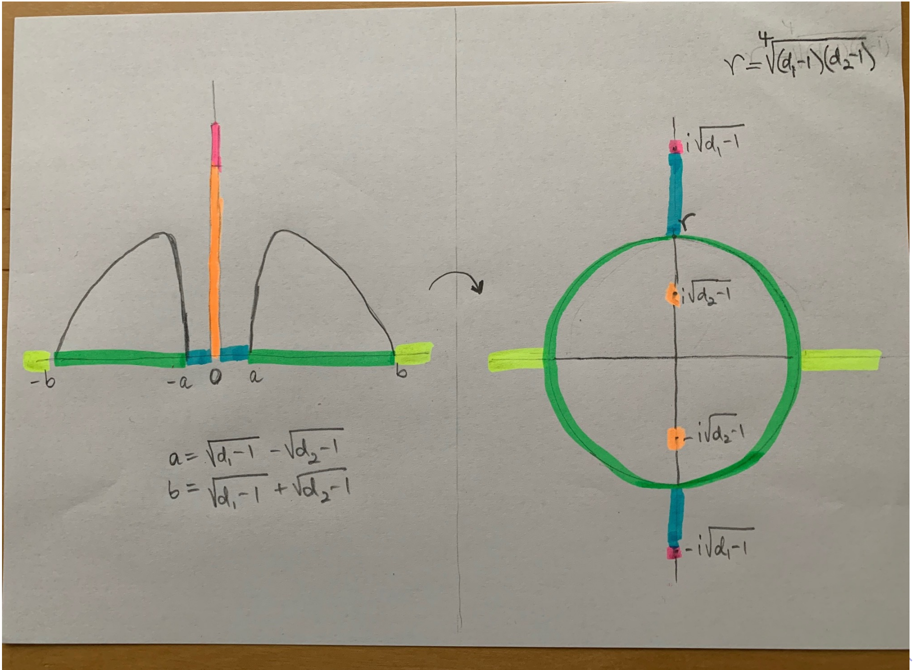
- Evals bigger than right edge of support for A map to real evals of B bigger than $\sqrt[4]{(d_1 - 1)(d_2 - 1)}$.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$-a$
0
a
b
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- Can show $|\lambda_3(B)| \leq \sqrt[4]{(d_1-1)(d_2-1)} + o(1)$; it follows that $\lambda_{3,4}(A) \leq \sqrt{d_1-1} + \sqrt{d_2-1} + o(1)$ BUT also

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- In particular, the rectangular (random!) $0-1$ matrix X with every column summing to d_1 and every row summing to d_2 is whp full-rank if $d_1 \neq d_2$.
- Harder to say what happens when $d_1, d_2 \to \infty$.

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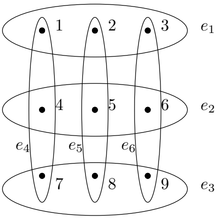

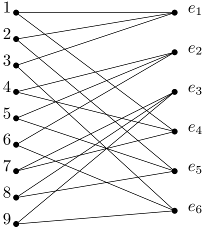

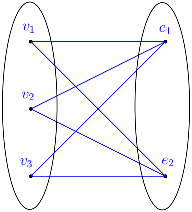

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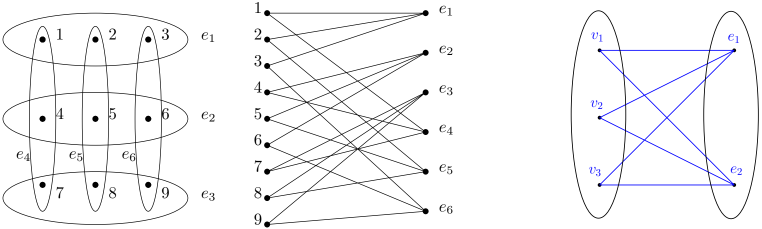

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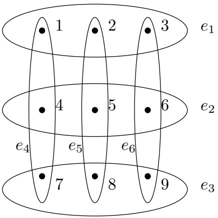

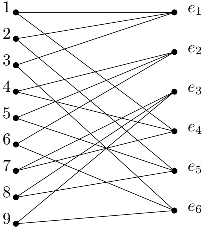

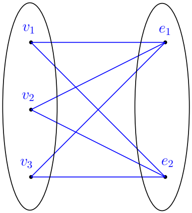

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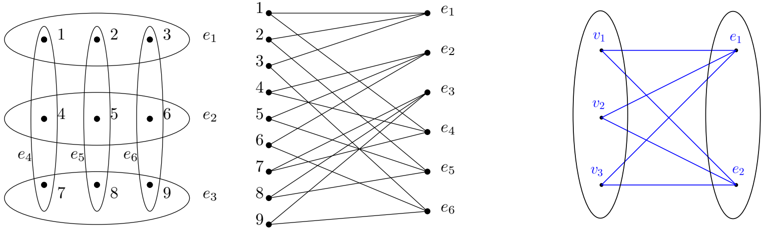

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- Apply the results for RBBGs from D.-Johnson ('14) and Brito-D.-Harris ('20).

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- Keen interest in applications