# Rank One Imaginary Perturbation for Hermitian Random Matrices in the Case of Band Matrices

M.Shcherbina

Institute for Low Temperature Physics, Kharkiv, Ukraine

based on the joint paper with

T.Shcherbina
University of Wisconsin-Madison, USA

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# General model

H is an $n \times n$ matrix ( hermitian or real symmetric) $\Gamma$ is a rank one matrix of the form

$$\Gamma x = (x, w)w, \quad w \in \mathbb{C}^n, \|w\| = 1$$

We consider Hermitian cases with H being GUE, Wishart (MP), weighted adjacency matrix of $G(n, p_n)$, or band matrices (Wegner model)

$$H_{\text{eff}} = H + i\gamma\Gamma, \quad \gamma > 0$$

## Observation

Since Gaussian mean field type models have a symmetry with respect to the replacement of lines and columns we assume without loss of generality that

$$\Gamma = E_{11} = \text{diag}\{1, 0, \ldots, 0\}$$

# Results for real perturbation

There are a lot of interesting works for the case

$$H_{\text{eff}} = H + \gamma\Gamma, \quad \gamma \in \mathbb{R}$$

In this situation the eigenvalues are real, the main part of the spectrum does not change but some "outlier" can be separated as $\gamma$ grows and we observe so called BBP transition

- J. Baik, G. Ben Arous, and S. Peche (2005) Ann of Prob 33 , no. 5,
- S. Peche (2006) PTRF 134 , no. 1,
- D.Feral, S.Peche (2006) CMP 272, n.1.
- ...

The results for this case were discussed in the yesterday's talk of Ke Wang.

# Results for imaginary perturbation

The model of this form was proposed in physics to study statistics of resonances in quantum chaotic scattering, phase shifts, time delays, and so on. In particular, the case of GUE with finite rank perturbation was studied by Fyodorov with co-authors:

- Y.V. Fyodorov and H.J. Sommers, (1996) JETPL 63 , no. 12, 1026–1030.
- Y.V. Fyodorov and H.J. Sommers (1997) JMP , no. 4, 1918–1981.
- Y. V. Fyodorov and B. A. Khoruzhenko, (1999) PhysRevLet 83, no. 1, 65.
- Y.V. Fyodorov and H.J. Sommers, JPhys A: Math, General 36 (2003), no. 12, 3303.
- Y. V. Fyodorov and B. Mehlig, PhysRev E 66 (2002), no. 4.

The dynamics of a rank-one perturbation of a Hermitian matrix was studied in the "fresh" preprint

- G.Dubach and L.Erdos (2021) ArXiv: 21.08.13694v1

see also

- G. Akemann, J. Baik, and P. Di Francesco, The Oxford Handbook of Random Matrix Theory, Oxford University Press, 2011

and references therein

# The main object

## Observation

It is straightforward to check that for $\gamma > 0$ the eigenvalues of $H_{\rm eff}$ has the form

$$\lambda_j(\gamma) = \lambda_j(H) + \zeta_j(\gamma), \quad \Im\zeta_j > 0$$

Moreover, since and eigenvectors $\{\Psi_j\}$ of H (e.g. for GUE) are uniformly distributed over sphere, it is naturally to expect that

$$\zeta_j(\gamma) \sim i\gamma(E_{11}\Psi_j, \Psi_j) \sim in^{-1}y_j$$

Hence it appears a planar density of eigenvalues which will be concentrated in the strip $\Im z \sim n^{-1}$, and in the formulas below one should consider the density at the points of the form $z = E + iy/n$ $(E \in \sigma(H))$.

This means, in particular, that we are studying the local regime.

# General method for densities on the plane

## Logarithmic potential approach (by Girko)

If we define $X(z) = (H_{eff} - z)(H_{eff} - z)^*$, then for any test function $h(z, \bar{z})$ we have that the linear eigenvalue statistics $\mathcal{N}_n[h]$ satisfies the relation

$$E\{\mathcal{N}_n[h]\} = \sum_j \frac{1}{4\pi} \int dxdy \Delta h(z, \bar{z}) E\{\log |z_j - z|^2\}$$
$$= \frac{1}{4\pi} \int dxdy h(z, \bar{z}) \Delta E\{\log \det X(z)\}$$
$$\rho(z) = \frac{1}{\pi n} \frac{\partial^2}{\partial z \partial \bar{z}} E\{\log \det X(z)\}$$

Here and below $E\{\}$ means the averaging with respect to H

# Averaging of logarithm

## Averaging of logarithm (by Fyodorov and Sommers)

Technically, instead studying of $E\{\log \det X(z)\}$ it is convenient to introduce the generating function

$$\mathcal{Z}(\varepsilon, z, z_b) = E\Big\{ \frac{\det(X(z) + \kappa^2)}{\det(X(z_b) + \kappa^2)} \Big\}$$

with $\kappa$ being a regularisation parameter.
Then we use the formula

$$\rho(z) = -\frac{1}{\pi n} \lim_{\kappa \to 0} \frac{\partial}{\partial \bar{z}} \lim_{z_b \to z} \frac{\partial}{\partial z_b} \mathcal{Z}(\kappa, z, z_b)$$

# Representation of $\mathcal{Z}(z, z_b)$ as a super-symmetric integral

Let us first linearise $\det(X(z) + \kappa^2)$. It is easy to see, that is we introduce a matrix

$$\tilde{X}(z) = \begin{pmatrix} -\kappa & i(H_{\text{eff}} - z) \\ i(H_{\text{eff}} - z)^* & -\kappa \end{pmatrix}$$

then

$$\det \tilde{X}(z) = \kappa^{-n} \det((H_{\text{eff}} - z)(H_{\text{eff}} - z)^* + \kappa^2)$$

Now introduce complex variables $\Phi = (\phi_1, \dots, \phi_{2n})$, $\bar{\Phi} = (\bar{\phi}_1, \dots, \bar{\phi}_{2n})$, and Grassmann (anticommuting) variables $\Psi = (\psi_1, \dots, \psi_{2n})$, $\bar{\Psi} = (\bar{\psi}_1, \dots, \bar{\psi}_{2n})$

## Intergal representation for $\mathcal{Z}(z, z_b)$

$$\frac{\det(X(z) + \kappa^2)}{\det(X(z_b) + \kappa^2)} = \pi^{-n} \int d\Psi d\bar{\Psi} d\Phi d\bar{\Phi}\; e^{(\tilde{X}(z)\Psi, \bar{\Psi}) + (\tilde{X}(z)\Phi, \bar{\Phi})}$$

$$\mathcal{Z}(z, z_b) = \pi^{-n} \int d\Psi d\bar{\Psi} d\Phi d\bar{\Phi}\; E\{e^{(\tilde{X}(z)\Psi, \bar{\Psi}) + (\tilde{X}(z)\Phi, \bar{\Phi})}\}$$

# Final representation for $\mathcal{Z}(z, z_b)$ (for GUE)

$$\mathcal{Z}(z, z_b) = n^4 \int_{|u_1|=1} \int_{|u_2|=1} du_1 du_2 \int_{-\infty}^{\infty} da_1 da_2$$
$$\exp\{n(\phi(u_1, z_\kappa) + \phi(u_2, z_\kappa) - \phi(a_1, z_{\kappa_b}) - \phi(a_2, z_{\kappa_b}))\}$$
$$\times F(u_1, u_2, a_1, a_2, U, S) dU dS$$

where U is a unitary $2 \times 2$ matrix ( $U \in U_j \in \mathring{U}(2)$) and S is a hyperbolic $2 \times 2$ matrix ($S \in \mathring{U}(1, 1)$)

$$z_\kappa = E + in^{-1}\sqrt{y^2 + \kappa^2}, \quad \phi(u, z) = \frac{u^2}{2} - izu - \log u$$

F is a rather complicated function of $u_1, u_2, a_1, a_2, U, S$ which does not contain n in the main order.

The analysis of $\mathcal{Z}(z, z_b)$ is a standard but rather involved problem of the saddle point method, since there are 4 saddle points and the factor $n^4$ before the integral makes it necessary to take into account all terms of the forth order in the expansion near the saddle points.

# Density for GUE

## Fyodorov and Sommers (96)

Recall that $z = E + iy/n$. We set a scaling $\tilde{y} = y/\rho_H(E)$

$$\rho(z) = -\frac{d}{d\tilde{y}}\left(e^{-\tilde{y}\tau}\frac{\sinh\tilde{y}}{\tilde{y}}\right) \tag{1}$$

where $\tau = (\pi\rho_H(E))^{-1}\left(\pi\gamma + (\pi\gamma)^{-1}\right)$ and

$$\rho_H(E) = \rho_{sc}(E) = \frac{1}{2\pi}\sqrt{4 - E^2}, \quad E \in (-2, 2)$$

## Conjecture (Fyodorov, private communication)

For all Hermitian matrices with a local behaviour of GUE type the density $\rho(z)$ with $z = E + iy/n\rho_H(E)$ is defined by formula (1).

We checked this for MP (Wishart) model (TS (21) work in preparation)

# Our model: 1d Wegner type band matrix (RBBM)

H is $N \times N$ hermitian block matrix which has n blocks of the size $W \times W$ $(N = nW)$ in the main diagonal. Only 3 block diagonals are non zero.

$$
H = \begin{pmatrix}
\square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & \square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & \square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & \square & \square & \square & \square & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & \square & \square & \square & \square & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & \square & \square & \square & \square & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & \square & \square & \square & \square & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & \square & \square & \square & \square & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & \square & \square & \square & \square \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & \square & \square & \square & \square
\end{pmatrix}
$$

where $\square$s mean independent normal variables with the variance $(1 - 2\alpha)/W$ and $*$s mean independent normal variables with the variance $\alpha/W$ with some $0 < \alpha < \frac{1}{4}$. Crossover in the local behaviour is expected for $W^2 \sim N \Leftrightarrow W \sim n$

# Variance matrix

In other words for Wegner type band matrix we have

$$\langle \mathrm{H}_{\mathrm{j}_1\mathrm{k}_1,\alpha_1\gamma_1}\mathrm{H}_{\mathrm{j}_2\mathrm{k}_2,\alpha_2\gamma_2}\rangle = \delta_{\mathrm{j}_1\mathrm{k}_2}\delta_{\mathrm{j}_2\mathrm{k}_1}\delta_{\alpha_1\gamma_2}\delta_{\gamma_1\alpha_2}\mathrm{J}_{\mathrm{j}_1\mathrm{k}_1}$$

with

$$\mathrm{J} = 1/\mathrm{W} + \alpha\Delta/\mathrm{W},$$

where $\mathrm{W} \gg 1$ and $\Delta$ is the discrete Laplacian on $\Lambda$.

# "Anderson transition" for block band random matrices (conjectures)

**Conjecture (in the bulk of the spectrum):**

| | | |
|---|---|---|
| $d = 1$ : | $W \gg n$ | Local GUE statistics, delocalization |
| | $W \ll n$ | Local Poisson statistics, localization |

# Integral representations for $\mathcal{Z}(z, z_b)$

There is a $70 \times 70$ matrix kernel $\mathcal{K}(X_1, X_2)$ (containing $z, z_b$ as parameters) such that

$$\mathcal{Z}(z, z_b) = C_{N,W} \int g_\delta(X_1)\mathcal{K}(X_1, X_2) \dots \mathcal{K}(X_{n-1}, X_n) f_0(X_n) \prod dX_i,$$
$$= C_{N,W}(\mathcal{K}^{n-1} f, g)$$

where

$$X_j = (x_j, y_j, U_j, S_j, ), \qquad x_j, y_j \in \mathbb{R}^2 \quad U_j \in \mathring{U}(2),\ S_j \in \mathring{U}(1,1)$$

dX means an integration over the Haar measure of X,

To be compared with

## Integral representations for $\mathcal{R}_2$ -the second correlation function

$$\mathcal{R}_2 = C_{N,W} \int g_2(X_1)\mathcal{K}_2(X_1, X_2) \dots \mathcal{K}_2(X_{N-1}, X_N) f_2(X_N) \prod dX_i,$$
$$= C_{N,W}(\mathcal{K}_2^{n-1} f_2, g_2)$$

# "Standard" sigma-model definition

$$\mathcal{R}^{+-}_{n,\beta}(E,\varepsilon,\xi) = C_{E,\xi}\int \exp\Big\{\frac{\beta}{4}\sum \mathrm{Str}Q_jQ_{j-1} - \frac{c_0}{2|\Lambda|}\sum \mathrm{Str}Q_j\Lambda_{\xi,\varepsilon}\Big\}dQ,$$

$$C_{E,\xi} = e^{E(\xi_1+\xi_2-\xi_1'-\xi_2')/2\rho_{sc}(E)},\quad \Lambda_{\xi,\varepsilon} = \mathrm{diag}\{\varepsilon - i\xi_1, -\varepsilon - i\xi_2, \varepsilon - i\xi_1'\varepsilon - i\xi_2'\}$$

Here $Q_j$ $j\in\Lambda$ are $4\times 4$ supermatrices

$$Q_j = \begin{pmatrix} U_j^* & 0\\ 0 & S_j^{-1}\end{pmatrix}\begin{pmatrix} (I+2\hat\rho_j\hat\tau_j)L & 2\hat\tau_j\\ 2\hat\rho_j & -(I-2\hat\rho_j\hat\tau_j)L\end{pmatrix}\begin{pmatrix} U_j & 0\\ 0 & S_j\end{pmatrix},$$

$$dQ = \prod dQ_j,\quad dQ_j = (1-2n_{j,1}n_{j,2})\,d\rho_{j,1}d\tau_{j,1}\,d\rho_{j,2}d\tau_{j,2}\,dU_j\,dS_j$$

with $U_j\in\mathring{U}(2)$, $S_j\in\mathring{U}(1,1)$ (recall that $S\in\mathring{U}(1,1)\Leftrightarrow S^*LS=L$),
$n_{j,1}=\rho_{j,1}\tau_{j,1},\quad n_{j,2}=\rho_{j,2}\tau_{j,2},$

$$\hat\rho_j = \mathrm{diag}\{\rho_{j1},\rho_{j2}\},\quad \hat\tau_j = \mathrm{diag}\{\tau_{j1},\rho_{j2}\},\quad L=\mathrm{diag}\{1,-1\}$$

Here $\rho_{j,l}$, $\tau_{j,l}$, $l=1,2$ are anticommuting Grassmann variables, and

$$\mathrm{Str}\begin{pmatrix} A & \sigma\\ \eta & B\end{pmatrix} = \mathrm{Tr}A - \mathrm{Tr}B,$$

# Why do we need sigma-model?

## Representation of $\mathcal{R}_2^\sigma$

$$\mathcal{R}_2^\sigma = \mathrm{C}_{n,\beta}\int \mathrm{g}_2^\sigma(\mathrm{X}_1)\mathcal{K}_2^\sigma(\mathrm{X}_1,\mathrm{X}_2)\ldots\mathcal{K}_2^\sigma(\mathrm{X}_{N-1},\mathrm{X}_N)\mathrm{f}_2^\sigma(\mathrm{X}_N)\prod d\mathrm{X}_i,$$
$$= \mathrm{C}_{n,\beta}((\mathcal{K}_2^\sigma)^{n-1}\mathrm{f}_2^\sigma, \mathrm{g}_2^\sigma)$$
$$\mathrm{X}_j = (\mathrm{U}_j, \mathrm{S}_j), \quad \mathrm{U}_j \in \mathring{\mathrm{U}}(2),\ \mathrm{S}_j \in \mathring{\mathrm{U}}(1,1),$$

and $\mathcal{K}_2^\sigma(\mathrm{X}_1,\mathrm{X}_2)$ is $4\times 4$ matrix kernel.

On the other hand, physicists believe that $\mathcal{R}_2^\sigma$ inherits all important properties of $\mathcal{R}_2$.

# "Anderson transition" for sigma-model (conjectures)

**Conjecture (in the bulk of the spectrum):**

| | | |
|---|---|---|
| $d = 1$ : | $\beta \gg n$ | GUE form |
| | $\beta \ll n$ | Poisson form |

# Limit $W \to \infty$

$$J = 1/W + \tilde\beta\Delta/W^2, \quad \tilde\beta > 0 \tag{2}$$

and consider the limit $W \to \infty$ for fixed $\tilde\beta$ and n

## Theorem [MS,TS:18] (JSP)

Given $\mathcal{R}^{+-}_{Wn\tilde\beta}$ with any dimension d, any fixed $\tilde\beta$, $|\Lambda|$, $\varepsilon > 0$, and $\xi = (\xi_1, \bar\xi_2, \xi_1', \bar\xi_2') \in \mathbb{C}^4$ $(|\Im\xi_j| < \varepsilon \cdot \rho(E)/2)$ we have, as $W \to \infty$:

$$\mathcal{R}^{+-}_{Wn\beta}(E,\varepsilon,\xi) \to \mathcal{R}^{+-}_{n,\beta}(E,\varepsilon,\xi), \quad \frac{\partial^2\mathcal{R}^{+-}_{Wn\beta}}{\partial\xi_1'\partial\xi_2'}(E,\varepsilon,\xi) \to \frac{\partial^2\mathcal{R}^{+-}_{n,\beta}}{\partial\xi_1'\partial\xi_2'}(E,\varepsilon,\xi),$$

$\beta = (2\pi\rho(E))^2\tilde\beta$, $U_j \in \mathring{U}(2)$, $S_j \in \mathring{U}(1,1)$

# Result for $\mathcal{R}^{+-}_{n,\beta}$ for $\beta \gg n$ (GUE statistics)

## Theorem 1 [MS,TS:18] (JSP)

For the sigma-model in the regime $C\beta/\log^3\beta > n$

$$|\mathcal{R}^{+-}_{n,\beta} - (\tilde{F}^{2(n-1)}\tilde{f}, \tilde{g})| \to 0, \text{ where}$$

$$\hat{F}_0 = F_0 \begin{pmatrix} 1 & F_1 & F_2 & F_1F_2 \\ 0 & 1 & 0 & F_2 \\ 0 & 0 & 1 & F_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; F_0 \sim e^{\varphi(U,S)/n}, \quad F_{1,2} \sim \varphi_{1,2}(U,S)/n$$

$\varphi(U,S) = -\mathrm{Tr}U^*LUD_1 - \mathrm{Tr}S^{-1}LSD_2,$

$\varphi_{1,2}(U,S)$ are defined similarly to $\varphi(U,S)$

$\tilde{f} = (e_4 - e_1), \quad \tilde{g} = (e_1 - e_4)$

## Corollary

The second order correlation function of RBBM with $\alpha = \beta/W$ in the limit $W \to \infty$ and then $\beta, n \to \infty$, $(\beta \gg n)$ coincides with that for GUE.

# Our "deformed" sigma-model

$$\mathcal{Z}_{n\beta}(\kappa, z_1, z_2) = e^{E(x_1+x_2)} \int \exp\Big\{\frac{\tilde\beta}{4}\sum \mathrm{Str} Q_j Q_{j-1} - \frac{c_0}{2n}\sum \mathrm{Str} Q_j \Lambda_{\kappa,y}\Big\}$$
$$\times\, \mathrm{Sdet}^{-1}\Big(Q_1 - \frac{iE}{2\pi\rho(E)}\mathcal{L} + \frac{i\gamma}{\pi\rho(E)}\Sigma\Big) dQ,$$
$$= (\mathcal{K}^{n-1}_{z,z_b}\tilde f, \tilde g),$$

where $Q_j$ $j \in \Lambda$ are the same as in the "standard" sigma-model

$$\Lambda_{\kappa,y,y_b} = \mathrm{diag}\{L_1, L_2\}, \quad \mathcal{L} = \mathrm{diag}\{I, -I\}, \quad \Sigma = \mathrm{diag}\{\hat\sigma, \hat\sigma\},$$
$$L_1 = \begin{pmatrix} \kappa & iy \\ -iy & -\kappa \end{pmatrix}, \quad L_2 = \begin{pmatrix} \kappa & iy_b \\ -iy_b & -\kappa \end{pmatrix} \quad \hat\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

$$\mathrm{Sdet}\begin{pmatrix} A & \sigma \\ \eta & B \end{pmatrix} = \frac{\det(A - \sigma B^{-1}\rho)}{\det B}$$

# Limit as $\mathrm{W} \to \infty$ for $\mathcal{Z}_{\mathrm{Wn}\tilde{\beta}}$

## Theorem 2 [MS,TS:21]

Given $\mathcal{Z}_{\mathrm{Wn}\tilde{\beta}}$ with any dimension d, any fixed $\tilde{\beta}$, $|\Lambda|$, $\kappa > 0$, and $\mathrm{y}, \mathrm{y_b} \in \mathbb{R}_+$ we have, as $\mathrm{W} \to \infty$:

$$\mathcal{Z}_{\mathrm{Wn}\beta}(\mathrm{E}, \kappa, \mathrm{y}, \mathrm{y_b}) \to \mathcal{Z}_{\mathrm{n},\beta}(\mathrm{E}, \kappa, \mathrm{y}, \mathrm{y_b}),$$
$$\frac{\partial^2 \mathcal{Z}_{\mathrm{Wn}\beta}}{\partial \mathrm{y} \partial \mathrm{y_b}}(\mathrm{E}, \kappa, \mathrm{y}, \mathrm{y_b}) \to \frac{\partial^2 \mathcal{Z}_{\mathrm{n},\beta}}{\partial \mathrm{y} \partial \mathrm{y_b}}(\mathrm{E}, \kappa, \mathrm{y}, \mathrm{y_b}),$$

# Result for $\mathcal{Z}_{n,\beta}$ for $\beta \gg n$ (GUE statistics)

## Theorem 3 [MS,TS:21]

For the sigma-model in the regime $C\beta/\log^3\beta > n$

$$|\mathcal{Z}_{n,\beta} - (\hat{F}^{2(n-1)}\tilde{f}, \tilde{g})| \to 0,$$

where $\hat{F}$ has the same triangular form as in Theorem 1 with

$$F_0 \sim e^{\tilde{\varphi}(U,S)/n}, \quad F_{1,2} = \tilde{\varphi}_{1,2}(U,S)/n$$
$$\tilde{\varphi}(U,S) = -\mathrm{Tr}U^*LUL_1 - \mathrm{Tr}S^{-1}LSL_2,$$

where $L_1$ and $L_2$ were defined in (2), $\tilde{\varphi}_{1,2}(U,S)$ are defined similarly to $\tilde{\varphi}(U,S)$, and end $\tilde{f}, \tilde{g}$, $\tilde{f}, \tilde{g}$ are the same as in Theorem 1.

## Corollary

The density of complex eigenvalues of $H_{\mathrm{eff}}$ for the Wegner matrix with $\alpha = \beta/W$ in the limit $W \to \infty$ and then $\beta, n \to \infty$, ($\beta \gg n$) coincides with density (1) obtained for GUE.