Localization and delocalization in Erdős-Rényi graphs

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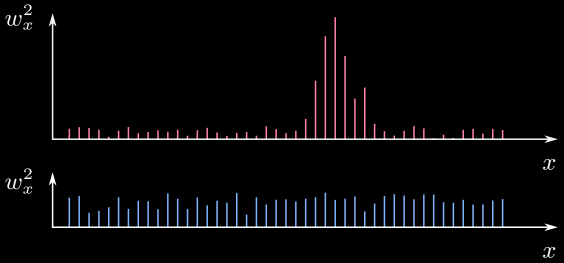
Universality conjecture for disordered quantum systems

A disordered quantum system can be in one of two phases:

(1) Localized (insulator, strong disorder):
Eigenvectors are localized.
Local spectral statistics are Poisson.

(2) Delocalized (metal, weak disorder):
Eigenvectors are delocalized.
Local spectral statistics follow random matrix theory (e.g. GOE).

ℓ^2-normalized eigenvector $\mathbf{w} = (w_x)$:



Conjectured phase diagram for Anderson model

Standard model of quantum disorder: Anderson model

$$-\Delta + \lambda V \quad \text{on } \Lambda \subset \mathbb{Z}^d\,, \qquad V = (V_x)_{x\in\Lambda} \text{ i.i.d. } \mathcal{N}(0,1)\,.$$

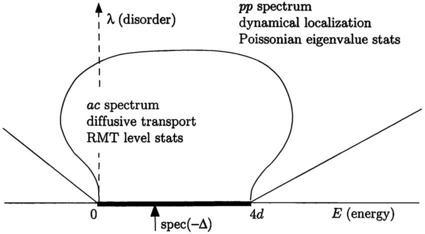


Figure 1.2. The predicted shape of the phase diagram of the Anderson model (1.2) in dimensions $d > 2$ for site potentials given by bounded iid random variables with a distribution similar to (1.5).

(From M. Aizenman, S. Warzel, *Random Operators*, AMS.)

Localized phase very well understood ([Fröhlich, Spencer; 1983], [Aizenman, Molchanov; 1993], [Molchanov; 1981], [Minami; 1996], . . .)

Delocalized phase wide open (extended states conjecture).

Random matrices

- Wigner matrices with light tails are in the delocalized phase [Erdős, Schlein, Yau, Yin; 2009–. . .], [Tao, Vu; 2009–. . .].
- Heavy-tailed Wigner matrices proposed as a simple model that exhibits a phase transition [Cizeau, Bouchaud; 1994], [Tarquini, Biroli, Tarzia; 2016].

  For $1 < \alpha < 2$, any bounded interval lies in the delocalized phase [Bordenave, Guionnet; 2013], [Aggarwal, Lopatto, Yau; 2020].

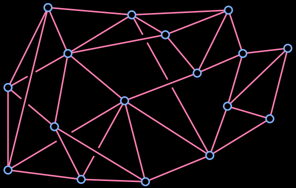
  For $0 < \alpha < 1$, delocalized phase in some neighbourhood of origin [Bordenave, Guionnet; 2017], [Aggarwal, Lopatto, Yau; 2020].

  For $0 < \alpha < 2/3$, partially localized phase far away from origin [Bordenave, Guionnet; 2013].
- This talk: Sparse matrices.

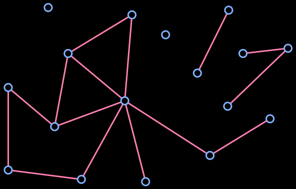
Erdős-Rényi graph and critical regime

Erdős-Rényi graph $\mathbb{G}(N, d/N)$

Critical regime: $d \approx \log N$, below which degrees do not concentrate.



$d \gg \log N$



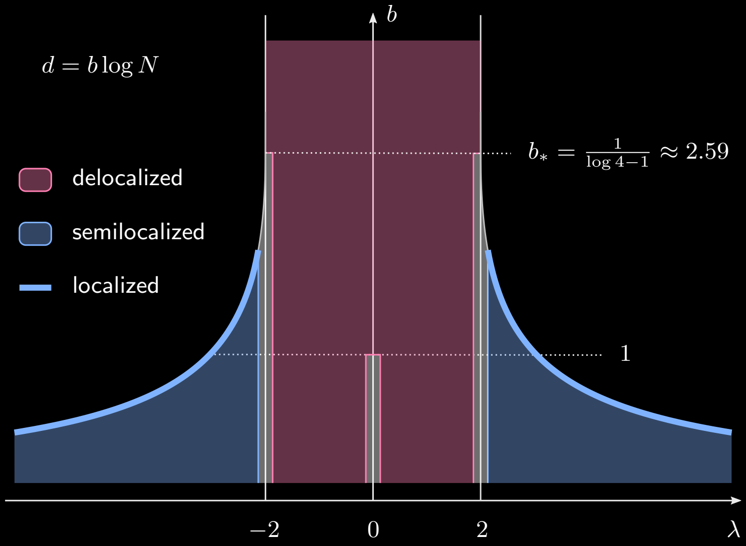
$d \ll \log N$

Supercritical $d \gg \log N$: homogeneous.

Subcritical $d \ll \log N$: inhomogeneous (hubs, leaves, isolated vertices, . . .).

Consider the adjecency matrix $A = (A_{xy}) \in \{0, 1\}^{N \times N}$.

Phase diagram for $H := d^{-1/2} A$

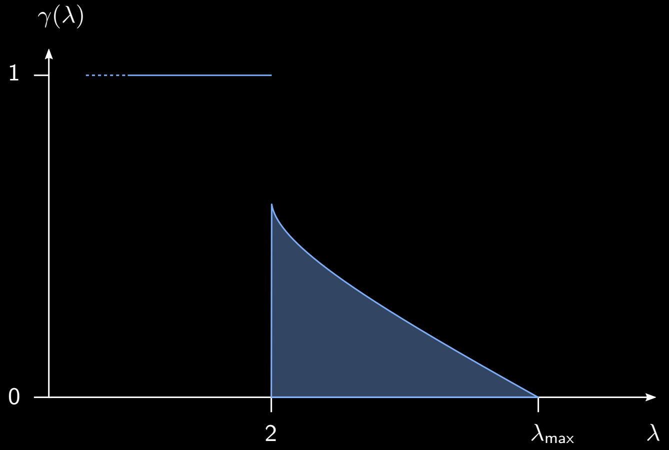


Behaviour of localization exponent

Eigenvalue λ with eigenvector $\mathbf{w}$ has localization exponent $\gamma(\lambda) \in [0,1]$:

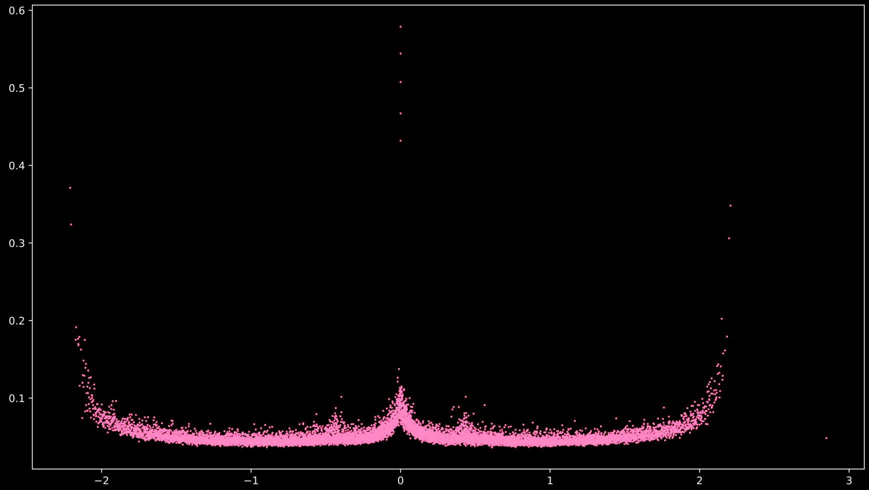
$$\|\mathbf{w}\|_\infty^2 =: N^{-\gamma(\lambda)}\,.$$

Asymptotically allowed region for $\gamma(\lambda)$ (plotted for $b = 1$):



Simulation of eigenvectors

Scatter plot of $(\text{eigenvalue}, \|\text{eigenvector}\|_\infty)$. $(N = 10'000,\ b = 0.6)$



Localized Phase I: Poisson eigenvalue statistics

Theorem [Alt, Ducatez, K; 2021]. Suppose that

$$(\log \log N)^4 \leqslant d \leqslant (1 - o(1))\, b_* \log N\,.$$

There exist deterministic u, σ, τ, θ (which are explicit functions of d and N) such that the rescaled eigenvalue process

$$\Phi := \sum_i \delta_{d\tau(\lambda_i(H) - \sigma)}$$

is asymptotically close to a Poisson point process Ψ on $\mathbb{R}$ on intervals $[-\kappa, \infty)$ containing at most $\mathcal{K} \gg 1$ points.

Corollary. Asymptotic equality in law of $k = O(\mathcal{K})$ largest points.

Intensity of Ψ

The intensity of Ψ is

$$\rho(\mathrm{d}s) := \sum_{\ell \in \mathbb{Z}} u^{\langle du \rangle + \ell} \, g\big(s + \theta(\langle du \rangle + \ell)\big) \, \mathrm{d}s \,,$$

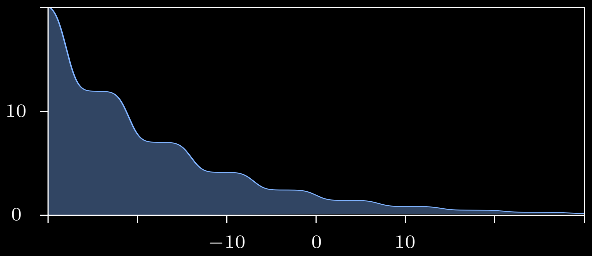
where $\langle \cdot \rangle$ is the periodic representative in $[-1/2, 1/2)$, and $g(s) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2} s^2}$.

Scaling laws

$$u \asymp \tau \asymp \sigma^2 \asymp \theta^2 \asymp \frac{t}{\log(t \vee 2)} \,, \qquad t := \frac{\log N}{d} \,.$$

Distribution of ρ in critical regime $t \asymp 1$:

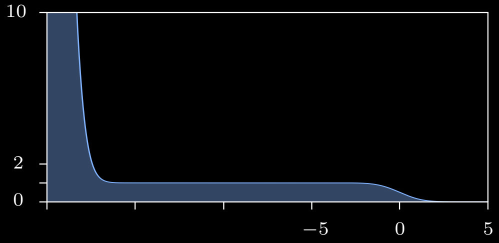
Top eigenvalue not governed by Gumbel law.



Distribution of ρ in subcritical regime $t \gg 1$:

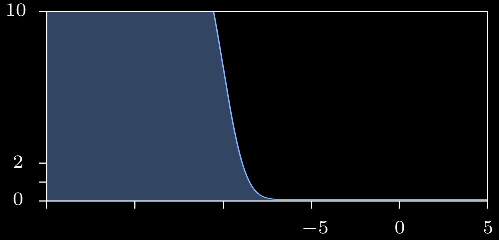
Resonance $\langle du \rangle = 0$:

Top eigenvalue not governed by Gumbel law.



Off-resonance $|\langle du \rangle| \geqslant c$:

Top eigenvalue governed by Gumbel law.



Localized Phase II: eigenvector localization

Let x be a vertex whose normalized degree $\alpha_x := |S_1(x)|/d$ is greater than 2. Define

$$u_1(x) = \frac{\sqrt{\alpha_x}}{\sqrt{\alpha_x - 1}} u_0(x) \,, \qquad u_{i+1}(x) = \frac{1}{\sqrt{\alpha_x - 1}} u_i(x) \quad (i \geqslant 1) \,,$$

and the exponentially decaying radial vector (with $r \gg 1$)

$$\mathbf{v}(x) := \sum_{i=0}^{r} u_i(x) \frac{\mathbf{1}_{S_i(x)}}{\|\mathbf{1}_{S_i(x)}\|} \,.$$

Theorem [Alt, Ducatez, K; 2021]. Let $\mathbf{w} = (w_x)$ be an eigenvector associated with one of the top (or bottom) $\mathcal{K}$ eigenvalues. Then with high probability there exists a vertex x with $\alpha_x > 2$ such that $\|\mathbf{w} - \mathbf{v}(x)\| = o(1)$.

Overview of proof

Basic intuition: one-to-one correspondence between eigenvalues and vertices of large degree.

Main steps of proof:

Step 1. Characterize the fluctuations of an eigenvalue associated with a vertex of large degree.

Step 2. Establish a one-to-one relation between such eigenvalues and the eigenvalues of H near the edge.

Step 1

Consider neighbourhood of vertex in

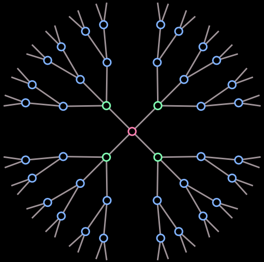
$$\mathcal{U} := \{x : \alpha_x \geqslant 2 + o(1)\}\,.$$

Use the tridiagonal representation of $H := d^{-1/2}A$ around x: write H in the basis $\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \dots$ obtained by orthogonalizing $\mathbf{1}_x, H\mathbf{1}_x, H^2\mathbf{1}_x, \dots$.

Apply transfer matrix (or orthogonal polynomial) analysis.

Problem: Fluctuations of transfer matrices very hard to control precisely, because $\mathbf{h}_i$ is unwieldy.

Toy model: in a rooted regular tree, the degree depends only on the distance to the root.



Exercise: if $\mathbb{G}|_{B_r(x)}$ is a rooted regular tree, then $\mathbf{h}_i = \mathbf{1}_{S_i(x)}$ for $i \leqslant r$.

- Naive attempt: write H in basis $(\mathbf{1}_{S_i(x)})$ instead of $(\mathbf{h}_i)$, to get an almost tridiagonal matrix.

  Problem: off-tridiagonal matrix is too large.

- More refined attempt: If $\mathbb{G}|_{B_r(x)}$ is a tree, the vector $H^i \mathbf{1}_x$ can be decomposed as a sum over simple walks in $\mathbb{N}$ of length i.

jump left / right $\iff$ terms decreasing / increasing distance from root

- Basis $(\mathbf{h}_i)$: all walks
- Basis $(\mathbf{1}_{S_i(x)})$: only steps to the right

Define basis $(\mathbf{f}_i)$ using walks with at most one step to the left.
For instance,

$$\mathbf{f}_3 = \mathbf{1}_{S_3(x)} + \sum_{y \in S_1(x)} (d\alpha_y - F)\mathbf{1}_y \,, \qquad F \in \mathbb{R}\,.$$

Proposition. Let $r \gg 1$ be suitably chosen. Let M be the matrix H in the basis $(\mathbf{f}_i)_{i=0}^r$. Then

$$\|M - Z_{\mathfrak{d}}(\alpha_x, \beta_x)\| \leqslant d^{-1-c},$$

where

$$\alpha_x = \frac{|S_1(x)|}{d}, \qquad \beta_x = \frac{|S_2(x)|}{|S_1(x)|d},$$

and

$$Z_{\mathfrak{d}}(\alpha, \beta) := \begin{pmatrix} 0 & \sqrt{\alpha} & & & & \\ \sqrt{\alpha} & 0 & \sqrt{\beta} & & & \\ & \sqrt{\beta} & 0 & \sqrt{\mathfrak{d}} & & \\ & & \sqrt{\mathfrak{d}} & 0 & \sqrt{\mathfrak{d}} & \\ & & & \sqrt{\mathfrak{d}} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \qquad \mathfrak{d} := 1 + \frac{1}{d}.$$

Remark. $\sqrt{d} Z_{\mathfrak{d}}(\alpha, \beta)$ is the tridiagonalization at the root of the rooted regular tree with degree sequence $\alpha d, \beta d, d+1, d+1, \dots$.

$Z_{\mathfrak{d}}(\alpha_x, \beta_x)$ has a unique eigenvalue $\Lambda_{\mathfrak{d}}(\alpha_x, \beta_x) > 2 + o(1)$, with exponentially decaying eigenvector $(u_i)_{i=0}^r$.

Back to graph $\mathbb{G}$ with

$$\mathbf{y}(x) := \sum_{i=0}^{r} u_i \frac{\mathbf{f}_i}{\|\mathbf{f}_i\|} \,.$$

It is possible to show that

$$\big\|\big(H - \Lambda_{\mathfrak{d}}(\alpha_x, \beta_x)\big)\mathbf{y}(x)\big\| \leqslant d^{-1-c} \,. \tag{1}$$

Step 1 is concluded by analysing the fluctuations of $\Lambda_{\mathfrak{d}}(\alpha_x, \beta_x)$ (of order d^{-1}).

Step 2

Need to ensure:

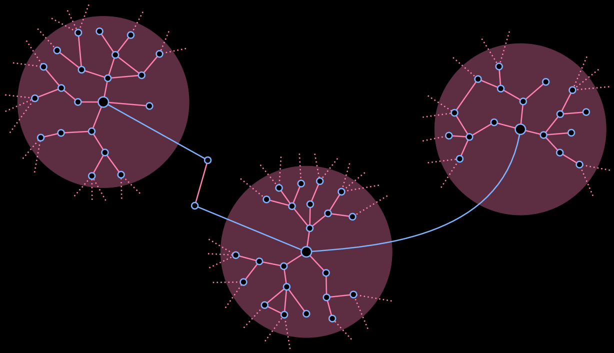
(a) $(\mathbf{y}(x) : x \in \mathcal{U})$ are orthogonal (i.e. $(B_r(x) : x \in \mathcal{U})$ are disjoint).

(b) The high probability bounds hold simultaneously for all $x \in \mathcal{U}$.

(c) The remaining eigenvalues cannot "pollute" the edge of the spectrum.

All of these present significant complications. In fact, (a) and (b) are wrong.

(a) $(B_r(x) : x \in \mathcal{U})$ are disjoint only if either (i) $\mathcal{U}$ is small or (ii) we prune the graph by removing edges to disconnect balls.

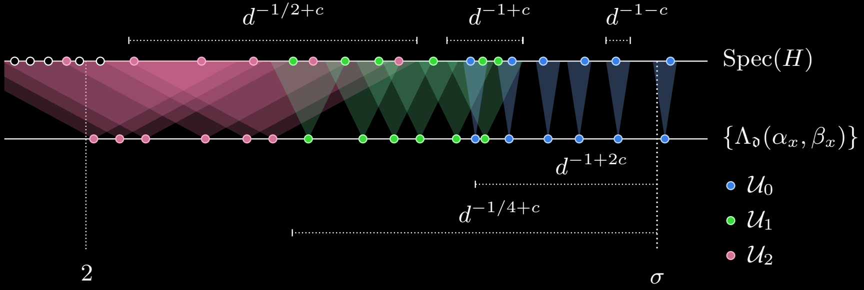


The pruning is potentially deadly, since in general removing even a single edge perturbs an eigenvalue by $O(1/\sqrt{d})$.

We have to prune in places that have a small impact on the extreme eigenvalues: prune only in the neighbourhoods of vertices x whose α_x is far from the top degree.

(b) The estimate (1) is not true simultaneously for all $x \in \mathcal{U}$. Solution: three-scale rigidity argument with the partition $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2$, where $\alpha_x > \alpha_y$ for $x \in \mathcal{U}_i$ and $y \in \mathcal{U}_{i+1}$.

The sets $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$ are increasing in size, but the accuracy of the estimate (1) deteriorates as i increases.



Block diagonal representation

$$O^{-1}HO = \begin{pmatrix} \mathcal{D}_0 & 0 & 0 & E_0^* \\ 0 & \mathcal{D}_1 & 0 & E_1^* \\ 0 & 0 & \mathcal{D}_2 + \mathcal{E}_2 & E_2^* \\ E_0 & E_1 & E_2 & X \end{pmatrix}$$

where

$$\mathcal{D}_i = \mathrm{diag}(\Lambda_{\mathfrak{d}}(\alpha_x, \beta_x) + O(\kappa_i) : x \in \mathcal{U}_i)$$

$$\kappa_i + \|E_i\| = \begin{cases} d^{-1-c} & \text{if } i = 0 \\ d^{-1+c} & \text{if } i = 1 \\ d^{-1/2+c} & \text{if } i = 2 \end{cases} \qquad \longleftarrow \text{ main estimates}$$

$$\|\mathcal{E}_2\| = O(d^{-1/2+c}) \qquad \longleftarrow \text{ pruning}$$

$$\|X\| \leqslant 2 + o(1) \qquad \longleftarrow \text{ (c)}$$

(c) Estimate of $\|X\|$ relies on:

- analysis of nonbacktracking version of A
- approximate Ihara-Bass identities
- local delocalization bounds from a radial Combes-Thomas argument