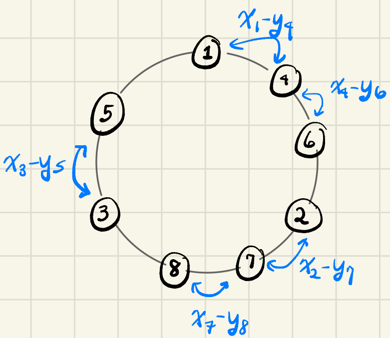
The TASEP on a ring, Schubert polynomials, & evil-avoiding permutations

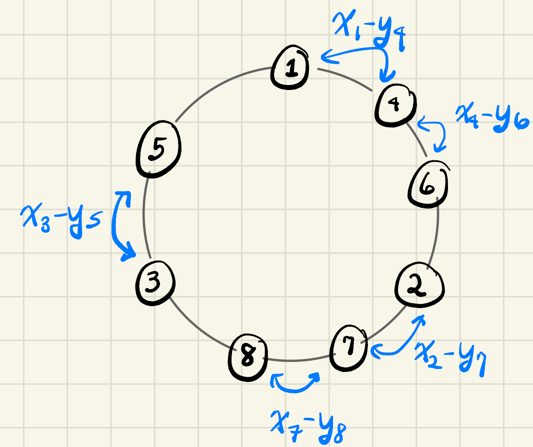


Lauren Williams

joint work w/ Donghyun Kim

arXiv: 2106.13378

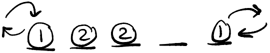
Outline

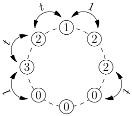


- Exclusion process on a ring
- Stationary dist & Schubert polynomials
- Results for evil-avoiding permutations

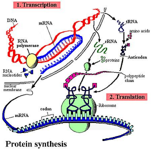
I. The asymmetric simple exclusion process (ASEP)

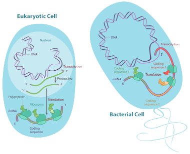
- Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
- Particles hop on a 1D lattice; at most one particle per site. Particles may have different *weights*, which affect their hopping rate.





- Lattice could be a line with open boundaries or a *ring*...
- Cited as a model for traffic flow and for translation in protein synthesis

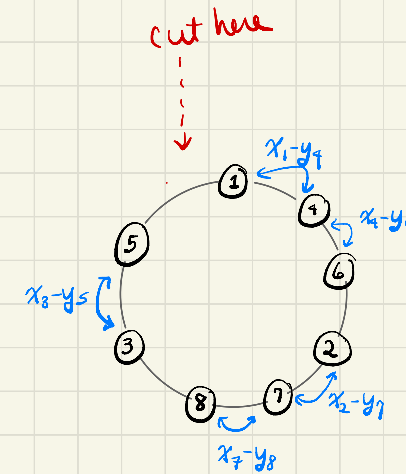




- Over 1000 papers on the exclusion process on the arXiv: Liggett, Derrida, Evans, Hakim, Pasquier, Spohn, Sasamoto, Yau, Borodin, Corwin, Ferrari, Seppalainen, Tracy-Widom, ...

Inhomogeneous TASEP on ring

Fix lattice of n sites on a ring, filled w/ particles of weights $1, 2, \ldots, n$.

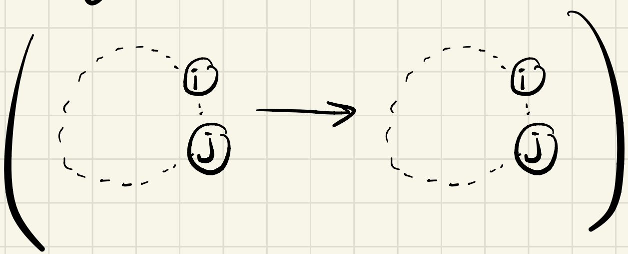


Markov chain (discrete time)

$n!$ states (by cutting open circle, identify state w/ permutation)

Identify state above w/ $(1, 4, 6, 2, 7, 8, 3, 5)$

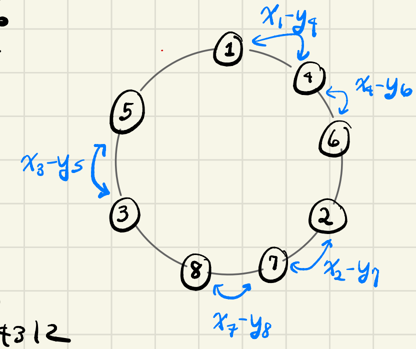
Transitions:

If $i<j$, prob 

$= x_i - y_j$.

If $i>j$, prob ( ) $= 0$.

What are the steady state probabilities?



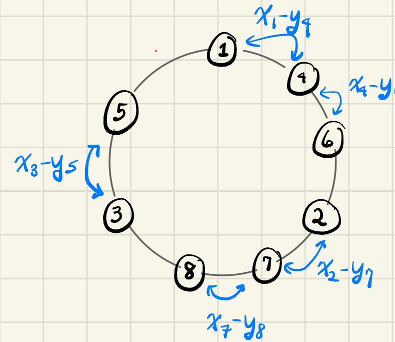
- For $w \in S_n$, let Ψ_w be (unnormalized) steady state prob of being in state w.
- Circular symmetry in model $\Rightarrow$ e.g. $\Psi_{3124} = \Psi_{1243} = \Psi_{2431} = \Psi_{4312}$
- So can restrict attention to $w \in S_n$ with $w_1 = 1$.

State w	Probability ψ_w (we set $y_i = 0$ for all i)
1234	$x_1^3 x_2$
1324	$x_1(x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3)$
1342	$x_1 x_2(x_1^2 + x_1 x_2 + x_2^2)$
1423	$x_1^2 x_2(x_1 + x_2 + x_3)$
1243	$x_1^2(x_1 x_2 + x_1 x_3 + x_2 x_3)$
1432	$(x_1^2 + x_1 x_2 + x_2^2)(x_1 x_2 + x_1 x_3 + x_2 x_3)$

Observation #1: If we factor out the largest monomial from each Ψ_w, all monomials are distinct.

Observation #2

After factoring out this largest monomial, what is left is a Schub poly or product of them



State w	Probability ψ_w (we set $y_i = 0$ for all i)
1234	$x_1^3x_2 = x_1^3x_2$
1324	$x_1\mathfrak{S}_{1432} = x_1(x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3)$
1342	$x_1x_2\mathfrak{S}_{1423} = x_1x_2(x_1^2 + x_1x_2 + x_2^2)$
1423	$x_1^2x_2\mathfrak{S}_{1243} = x_1^2x_2(x_1 + x_2 + x_3)$
1243	$x_1^2\mathfrak{S}_{1342} = x_1^2(x_1x_2 + x_1x_3 + x_2x_3)$
1432	$\mathfrak{S}_{1423}\mathfrak{S}_{1342} = (x_1^2 + x_1x_2 + x_2^2)(x_1x_2 + x_1x_3 + x_2x_3)$

- Here $\mathfrak{S}_w$ is the Schubert polynomial associated to permutation w.

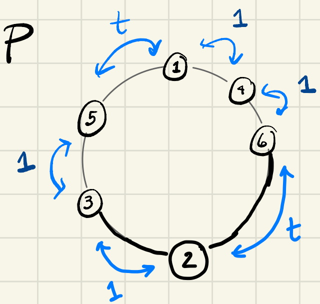
Main questions :

- Find combinatorial formula for steady state prob's

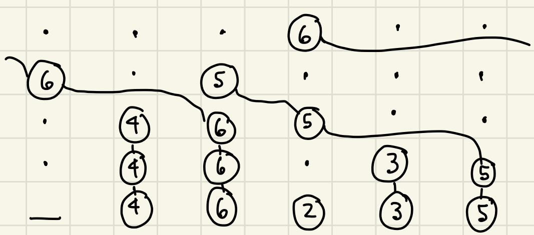
- Figure out those monomial factors
  + relationship to Schubert polynomials

Asymmetric exclusion process on a ring: two versions

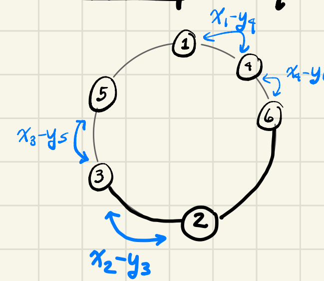
"usual" multispecies ASEP



Formula for steady state prob's in terms of <u>multiline queues</u>
(Martin; Corteel-Mandelshtam-W.)

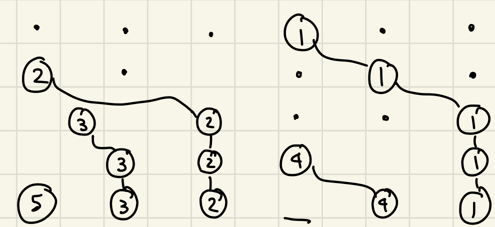


inhomogeneous, <u>totally</u> asymmetric (TASEP)

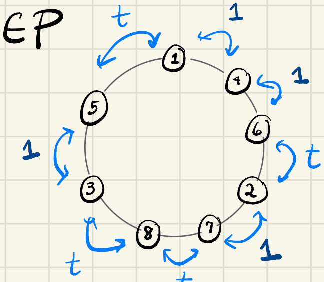


When $y_i=0$, have formula for prob's in terms of multiline queues
(Ayyer-Linusson (conj), Arita-Mallick (proved))

[No formula in general case w/ $y_i \neq 0$]



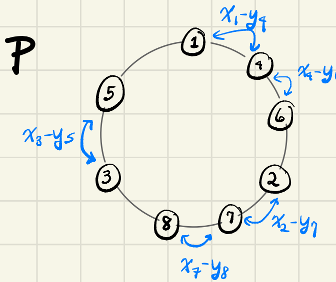
"usual" multispecies ASEP



Connected to Macdonald polynomials

- Partition function equals Macdonald poly $P_\lambda(x_1,\dots,x_n;q,t)$ specialized at $x_i=1$, $q=1$, where λ obtained by sorting particle weights in weakly decreasing order (Cantini-de Gier-Wheeler)
- Each steady state prob is a specialization of a permuted basement Macdonald poly (Corteel-Mandelshtam-W.)
- Can use multiline queues to give formulas for (permuted) Macdonald poly's (CMW)

inhomogeneous TASEP



Connected to (double) Schubert polynomials

- 2012: conj in $y_i=0$ case (Lam-W.)
- 2016: n of the $n!$ states have prob's Ψ_w $\sim$ prod's of double Schubert poly's (Cantini)
- 2021: same as above, but for $\sim \frac{(2+\sqrt{2})^{n-1}}{2}$ out of $n!$ states (Kim-W.)

Conj all other Ψ_w = nontrivial sum of Schuberts.

Rk: When $y_i=0$, $Z_n=\prod_{i=1}^{n} h_{n-i}(x_1,x_2,\dots,x_{i-1},x_i,x_i)$. Has $\prod_{i=0}^{n}\binom{n}{i}$ terms.

(Double) Schubert polynomials

- For $1 \le i < n$, the *divided difference operator* ∂_i acts on polynomials $P(x_1, \ldots, x_n)$ as follows:

$$(\partial_i P)(x_1, \ldots, x_n) = \frac{P(\ldots, x_i, x_{i+1}, \ldots) - P(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

- If $s_{i_1} \ldots s_{i_m}$ is a reduced expression for a permutation $w \in S_n$, then $\partial_{i_1} \ldots \partial_{i_m}$ depends only on w; we denote this operator by ∂_w.
- Let $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_n)$ be two sets of variables;

$$\text{Let } \Delta(\mathrm{x}, \mathrm{y}) = \prod_{i+j \le n} (x_i - y_j).$$

- To each $w \in S_n$ we associate the *double Schubert polynomial* (Lascoux-Schutzenberger)

$$\mathfrak{S}_w(\mathrm{x}, \mathrm{y}) = \partial_{w^{-1}w_0} \Delta(\mathrm{x}, \mathrm{y}),$$

where the *divided difference operator* acts on the x-variables.

(Double) Schubert polynomials, cont.

- Ordinary Schubert polynomials $\mathfrak{S}_w(\mathrm{x})$ obtained from $\mathfrak{S}_w(\mathrm{x}, \mathrm{y})$ when all $y_i = 0$.

- When $w \in S_n$ is a Grassmannian permutation, $\mathfrak{S}_w(\mathrm{x})$ is a Schur polynomial. In particular, both Macdonald polynomials and Schubert polynomials generalize Schur polynomials.

- Geometric significance: Schubert polynomials represent cohomology classes of Schubert varieties in the cohomology ring of the complete flag variety $\mathcal{F}\ell_n$.

- Double Schubert polynomials represent Schubert classes in equivariant cohomology for the Borel group action on $\mathcal{F}\ell_n$.

- Many combinatorial formulas for Schubert polynomials, starting with Billey-Jokusch-Stanley '93, Fomin-Kirillov '96, Kohnert, etc.

Steady State Probabilities when $n=5$

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

$x^{(5,2,0)} = x_1^5 x_2^2 x_3^0$

Most Ψ_w here are product of monomial & some Schub poly's

Probabilities seem to have nice expressions in terms of Schubert polynomials. Often they are products of Schubert polynomials!

Monomial Factor Conjecture (2012 conj from Lam-W)

Def: Let $w = (w_1, \dots, w_n) \in S_n$. Let $r = w^{-1}(i+1)$ and $s = w^{-1}(i)$.

Let $\alpha_i(w) :=$ # integers greater than $i+1$ among $\{ \overset{i+1}{\overset{\shortparallel}{w_r}}, w_{r+1}, w_{r+2}, \dots, \overset{i}{\overset{\shortparallel}{w_s}} \}$.

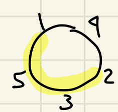
Theorem (Kim-W): Consider inhomog TASEP on S_n (where $y_i = 0 \ \forall i$).

For $w \in S_n$, Let $\eta(w)$ be the largest monomial that can be factored out of Ψ_w.

Then $$\eta(w) = \prod_{i=1}^{n-2} x_i^{\alpha_i(w) + \dots + \alpha_{n-2}(w)}.$$ $$n-1 \longrightarrow n-2 \longrightarrow n-3 \longrightarrow \dots \longrightarrow i$$

If two states $w, w' \in S_n$ have $\eta(w) = \eta(w')$, then w & w' are cyclically equivalent.

Ex: If $w = (1,4,2,3,5)$,



$\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 0$ so

$$\eta(w) = x_1^{\alpha_1+\alpha_2+\alpha_3} x_2^{\alpha_2+\alpha_3} x_3^{\alpha_3} = x_1^4 x_2^2$$

(cf previous page)

Pattern Avoidance

- We say a permutation $w = (w_1, \ldots, w_n)$ *avoids the pattern* 2413 if there are not four positions $1 \leq i < j < k < l \leq n$ such that w_i, w_j, w_k, w_l are in relative order 2413.
- Example: $w = (3, 4, 6, 1, 2, 5)$ contains the pattern 2413 while $w = (1, 2, 6, 4, 3, 5)$ avoids the pattern 2413.

Def: Say permutation w is evil-avoiding if it avoids patterns 2413 and also patterns coming from

$$\begin{array}{cccc} 2 & 4 & 1 & 3 \\ \| & \| & \| & \| \\ e & v & i & l \end{array}$$

these anagrams of evil:

vile $= 4132$

veil $= 4213$

leiv $= 3214$

Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP. WLOG $w_1 = 1$. Then the (unnormalized) probability Ψ_w is proportional to product of k (double) Schubert poly's, where $k = \#$ descents of w^{-1}.

evil = 2413

vile = 4132

veil = 4213

leiv = 3214

	State w	Probability ψ_w (we set $y_i = 0$ for all i)	
	12345	$x^{(6,3,1)}$	
	12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$	
	12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$	
	12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$	
	12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$	
	12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	
	13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$	
	13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$	
	13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$	
	13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$	
pattern 2413	13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$	
	13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$	
	14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$	
	14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$	
3214	14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$	
	14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$	
	14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$	
	14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$	
	15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$	
4132	15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$	
4213	15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$	
	15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$	
	15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$	
	15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	← inverse has 3 descents

Combinatorics of evil-avoiding permutations

evil $= 2413$
vile $= 4132$
veil $= 4213$
leiv $= 3214$

- Let $e(n) = \#$ evil-avoiding perms in S_n.

  Then $e(n) = \dfrac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}}{2}$

  (Sloane Encyclopedia A006012)

- Apparently $e(n)$ also counts
  - rectangular perms on $[n]$, those avoiding 2413, 2431, 4213, 4231
  - (several other objects)

  Bijective proof?

Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP w/ $\mathrm{des}(w^{-1}) = k$. WLOG $w_1 = 1$.
Then there are partitions $\lambda^1 \dots \lambda^k$ and $\mu := \left(\binom{n-1}{2}, \binom{n-2}{2}, \dots, \binom{2}{2}\right) - \sum_{i=1}^{k} \lambda^i$
such that
$$\Psi_w = x^{\mu} \prod_{i=1}^{k} \underbrace{s_{\lambda^i}(x_{n-\lambda_1^i}, x_{n-\lambda_2^i}, \dots)}$$
flagged Schur poly of shape λ^i w/ entries in row j bounded above by $n-\lambda_j^i$

To construct $\lambda^1, \dots, \lambda^k$:

Let $c = (c_1, \dots, c_n)$ be Lehmer code of w^{-1}.

Let $a_0 = 0$, and $a_1 \dots a_k$ be descent positions of $c_1 \dots c_n$.

Define $\lambda^i = (n-a_i)^{a_i} - (\underbrace{0, \dots, 0}_{a_{i-1}}, c_{a_{i-1}+1}, c_{a_{i-1}+2}, \dots c_{a_i})$

Rem: I stated Theorem in case all y_i's $= 0$, but we have version when y_i's $\neq 0$ using double Schub poly's.

Ex: If $w = 14253$, $\mathrm{code}(w^{-1}) = (0,1,2,0,0)$, $a_0 = 0$, $a_1 = 3$,
$\lambda^1 = (5-3)^3 - (0,1,2) = (2,2,2) - (0,1,2) = (2,1,0)$.

descents of w^{-1}

k	$w \in \mathrm{St}(5,k)$	$\Psi(w)$	probability ψ_w	$s(w)$
0	12345	$\varnothing$	$\mathbf{x}^{(6,3,1)}$	(0)
1	12354	$(1,1,1)$	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$	(0)
1	12435	$(2,2,1)$	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$	(0)
1	12453	$(2,2)$	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$	(0)
1	12534	$(1,1)$	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$	(0)
1	13245	$(3,2)$	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$	(0)
1	13425	$(3,1)$	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$	(0)
1	13452	(3)	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$	(0)
1	14235	$(2,1,1)$	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$	(0)
1	14253	$(2,1)$	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$	(0)
1	14523	(2)	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$	(0)
1	15234	(1)	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$	(0)
2	12543	$(2,2),(1,1,1)$	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	$(0,-1)$
2	13254	$(3,2),(1,1,1)$	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$	$(0,0)$
2	13542	$(3),(1,1,1)$	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$	$(0,-1)$
2	14352	$(3),(2,2,1)$	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$	$(0,-1)$
2	14532	$(3),(2,2)$	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$	$(0,-1)$
2	15342	$(3),(1,1)$	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$	$(0,-1)$
2	15423	$(2),(1,1,1)$	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$	$(0,-2)$
3	15432	$(3),(2,2),(1,1,1)$	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	$(0,-1,-2)$

sum of these vectors is constant

$(2,2)+(1,1,1)+(3,0,0)$

$=(6,3,1)$

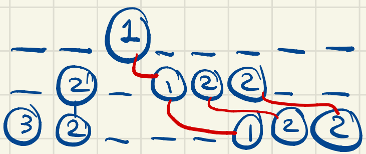
sequence of partitions $\lambda^1 \ldots \lambda^k$

Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP w/ $\mathrm{des}(w^{-1}) = k$. WLOG $w_1 = 1$.
Then there are partitions $\lambda^1 \dots \lambda^k$ and $\mu := \left(\binom{n-1}{2}, \binom{n-2}{2}, \dots, \binom{2}{2}\right) - \sum_{i=1}^{k} \lambda^i$ such that

$$\Psi_w = x^{\mu} \prod_{i=1}^{k} s_{\lambda^i}\left(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \dots\right)$$

Note: If y_i's $= 0$, there is combinatorial formula for stationary dist in terms of multiline queues (Ayyer-Linusson and Arita-Mallick)

In this case & when $k=1$, can use it to prove Thm, by mapping MLQ's to semistandard tableaux.



But to prove theorem for general y_i's & k, need another technique...

To prove our main result connecting steady state prob's $\Psi_w(x_1 \ldots x_n, y_1 \ldots y_n)$ to double Schubert poly's, use theorem of Cantini that adds auxiliary parameters $z_1 \ldots z_n$ into the picture.

Gives recursion for the Ψ_w.

To prove our main result connecting steady state prob's $\Psi_w(x_1 \ldots x_n, y_1 \ldots y_n)$ to double Schubert poly's, use theorem of Cantini that adds auxiliary parameters $z_1 \ldots z_n$ into the picture.

Gives recursion for the Ψ_w.

Theorem (Cantini)

For each state $w \in S_n$ we define the quantity $\psi_w(\mathrm{z}) = \psi_w(z_1, \ldots, z_n)$ via:

$$\psi_{(1,2,\cdots,n)}(\mathrm{z}) = \prod_{1 \le i < j \le n} (x_i - y_j)^{j-i-1} \prod_{i=1}^{n} \Big(\prod_{j=1}^{i-1} (z_i - x_j) \prod_{j=i+1}^{n} (z_i - y_j) \Big),$$

$$\psi_{s_\ell w}(\mathrm{z}) = \pi_\ell(w_\ell, w_{\ell+1}; n) \psi_w(\mathrm{z}) \quad \text{if } w_\ell > w_{\ell+1},$$

where $\pi_\ell(\beta, \alpha; n)$ is the *isobaric divided difference operator* defined by

$$\pi_\ell(\beta, \alpha; n) G(\mathbf{z}) = \frac{(z_\ell - y_\beta)(z_{\ell+1} - x_\alpha)}{x_\alpha - y_\beta} \frac{G(\mathbf{z}) - s_\ell G(\mathbf{z})}{z_\ell - z_{\ell+1}}.$$

Then the leading coefficient $\mathrm{LC}_z(\psi_w(\mathrm{z}))$ w/ respect to z equals the steady state prob ψ_w.

Then need to show that for evil-avoiding ω,

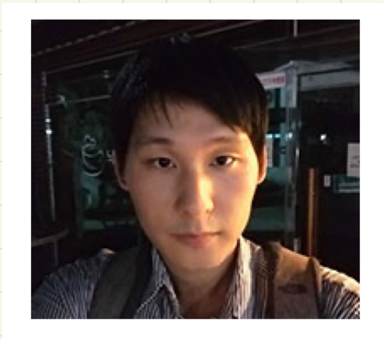
$LC_z(\Psi_\omega(z)) =$ product of Schuberts.

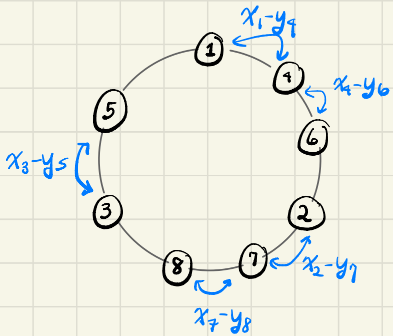
Conj: If w not evil-avoiding, can write Ψ_w as monomial $\cdot$ (pos sum of Schubert poly's)

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

Open: Is there some geometric interpretation of steady state prob's?

Thank you!





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