Eigenvectors of Toeplitz matrices under small random perturbations

Ofer Zeitouni

Joint with Anirban Basak and Martin Vogel

August 2021

An empirical fact 

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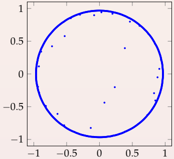

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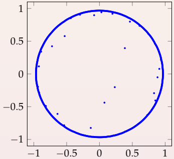

Goes back to Trefethen et als - pseudo-spectrum.

Regularization by noise

   

Set $\gamma > 1/2$.

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Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues L_N^A. Then L_N^A converges weakly to the uniform measure on the unit circle in the complex plane.

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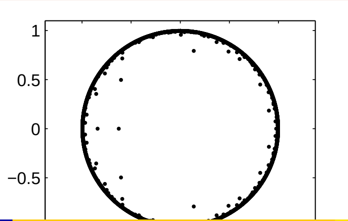

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General criterion - Guionnet, Wood, Z.

More general models? 

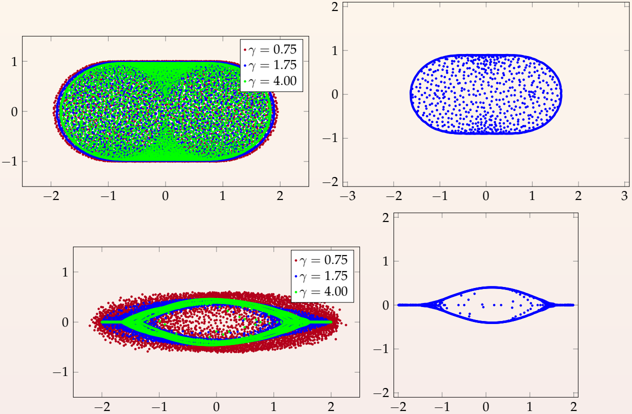

Figure: The eigenvalues of $D_N + J_N + N^{-\gamma} G_N$, with $N = 4000$ and various γ. Top: $D_N(i,i) = -1 + 2i/N$. Bottom: D_N i.i.d. uniform on $[-2, 2]$. On left, actual matrix. On the right, $U_N(D_N + J_N)U_N^*$.

More general models  

Theorem (Basak, Paquette, Z. '17, '18)

$T_N = \sum_{i=-k_-}^{k} a_i J_N^i$ (Toeplitz, finite symbol, $J_N^{-1} := J_N^T$.) General noise model. Then,

$$L_N \to \textit{Law of} \sum_{i=-k_-}^{k} a_i U^i$$

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If upper triangular (i.e. $k_- = 0$), then extends to twisted Toeplitz $T_N(i,j) = a_i(j/N)$, $i = 1, \ldots, k$, a_i continuous:

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Confirms simulations and predictions (based on pseudo-spectrum) of Trefethen et als. Also studied by Sjöstrand and Vogel (2016-2020), more on their approach later

Proof ingredients

Theorem (Replacement principle - after GWZ)

A_N - deterministic, bounded operator norm. Δ_N and G_N - independent random matrices. Assume

(a) *G_N and Δ_N are independent. $\|\Delta_N\| < N^{-\gamma_0}$ whp and G_N noise matrix as before.*

(b) *For Lebesgue a.e. $z \in B_{\mathbb{C}}(0, R_0)$, the empirical distribution of the singular values of $A_N - zI_N$ converges weakly to the law induced by $|X - z|$, where $X \sim \mu$ and $\mathrm{supp}\mu \subset B_{\mathbb{C}}(0, R_0/2)$.*

(c) *For Lebesgue a.e. every $z \in B_{\mathbb{C}}(0, R_0)$,*

$$\mathcal{L}_{L_N^{A+\Delta}}(z) \to \mathcal{L}_\mu(z), \qquad \text{as } N \to \infty, \text{ in probability.} \tag{1}$$

Then, for any $\gamma > \frac{1}{2}$, for Lebesgue a.e. every $z \in B_{\mathbb{C}}(0, R_0)$,

$$\mathcal{L}_{L_N^{A+N^{-\gamma}G}}(z) \to \mathcal{L}_\mu(z), \qquad \text{as } N \to \infty, \text{ in probability.} \tag{2}$$

Proof ingredient II

Theorem

Let T_N be any $N \times N$ banded Toeplitz matrix with a symbol $\mathbf{a}$. Then, there exists a random matrix Δ_N with

$$P(\|\Delta_N\| \geq N^{-\gamma_0}) = o(1), \tag{3}$$

for some $\gamma_0 > 0$, so that $L_N^{T+\Delta}$ converges weakly, in probability, to $\nu_{\mathbf{a}}$.

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This works for Toeplitz with banded symbol, but not for twisted Toeplitz! Main issue - Toeplitz determinant of un-perturbed matrix requires work, e.g. Widom's theorem.

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Let $\{e_i\}$ be eigenvectors of A^*A, $\{f_i\}$ of AA^*, with

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Fix $M > 0$ integer (may depend on N) - these will be eventually the *small* singular values, ie all singular values of A except for smallest M are above a strictly positive threshold α. Let $\{\delta_i\}$ be standard basis of $\mathbb{C}^M$.

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$$R_+ = \sum_{i=1}^{M} \delta_i \circ e_i^*, \quad R_- = \sum_{i=1}^{M} f_i \circ \delta_i^*,$$

$$\mathcal{P} = \begin{pmatrix} A & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M \quad \text{bijection!}$$

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We have

$$\mathcal{P}^{-1} = \mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$$

with

$$E = \sum_{M+1}^{N} \frac{1}{t_i} e_i \circ f_i, \quad E_+ = \sum_{1}^{M} e_i \circ \delta_i^*,$$

$$E_- = \sum_{1}^{M} \delta_i \circ f_i^*, \quad E_{-+} = -\sum_{1}^{M} t_j \delta_j \circ \delta_j^*,$$

and the norm estimates

$$\|E\| \leq \frac{1}{\alpha}, \quad \|E_\pm\| = 1, \quad \|E_{-+}\| \leq \alpha, \quad |\det \mathcal{P}|^2 = \prod_{M+1}^{N} t_i^2.$$

Noisy Grushin problem

$$A^\delta = A + \delta G, \quad 0 \le \delta \ll 1.$$

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Applying $\mathcal{E} = \mathcal{P}^{-1}$ from the right:

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The Schur complement formula applied to $\mathcal{P}^{\delta}$ and $\mathcal{E}^{\delta}$ shows that $\det \mathcal{P}^{\delta} = \det A^{\delta} \cdot \det(-R_+(A^{\delta})^{-1} R_-)$, while $E^{\delta}_+ = -(A^{\delta})^{-1} R_- E^{\delta}_{-+}$ and hence $I = R_+ E^{\delta}_+ = -R_+(A^{\delta})^{-1} R_- E^{\delta}_{-+}$.

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But $\|E_{-+}^{\delta}\| \leq 2\alpha$, thus,

$$\log|\det A^{\delta}| \leq \log|\det \mathcal{P}| + M|\log 2\alpha| + 2\alpha^{-1}\delta N\|G\|.$$

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$$= \left|\Re \int_0^\delta \mathrm{Tr}\left(\begin{pmatrix} E^\tau & E^\tau_+ \\ E^\tau_- & E^\tau_{-+} \end{pmatrix} \cdot \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}\right) d\tau\right| \le 2\alpha^{-1}\delta N \|G\|.$$

$$\text{So, } \left|\frac{1}{N}\log|\det \mathcal{P}^\delta| - \frac{1}{N}\log|\det \mathcal{P}|\right| \le 2\alpha^{-1}\delta\|G\|.$$

But $\|E^\delta_{-+}\| \le 2\alpha$, thus,

$$\log|\det A^\delta| \le \log|\det \mathcal{P}| + M|\log 2\alpha| + 2\alpha^{-1}\delta N\|G\|.$$

Complementary lower bound requires just a bit more work.

Noisy Grushin -ct'd

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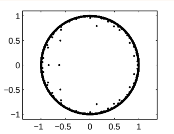
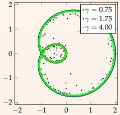
Complementary lower bound requires just a bit more work.
Since $\det \mathcal{P}$ is like erasing the small singular values of A, this gives a version of the deterministic equivalence lemma for general noise (Vogel-Z. '20)

Quick remarks on outliers 

$J_N + N^{-\gamma} G_N$ $\qquad$ $J_N + J_N^2 + N^{-\gamma} G_N$





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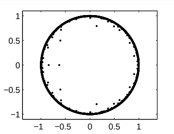

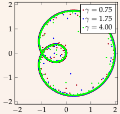

Outliers are random. What is structure of outliers?

Quick remarks on outliers

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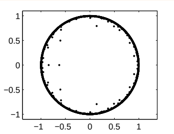

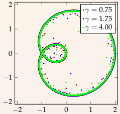

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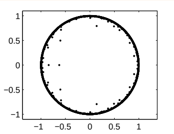

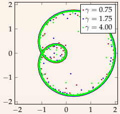

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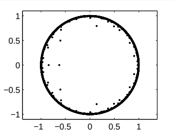

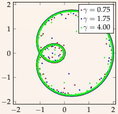

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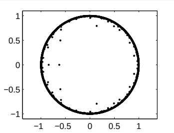

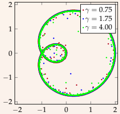

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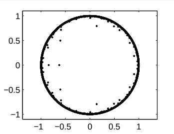

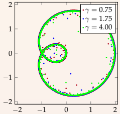

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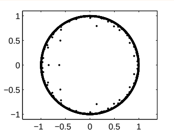

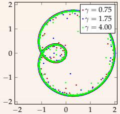

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Generalizes to general Toeplitz. Proof uses study of determinant.

Example

Develop the determinant of $zI - J_N - N^{-\gamma} G_N$:

$$z^N - N^{-\gamma} \sum_{k=0}^{N-1} \sum_{i,j:i+j=k+2} G_{i,j} z^k + \text{remainder.}$$

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For general Toeplitz matrices, decompose the determinant to factors of this form!

Eigenvectors  

What are the eigenvectors of perturbed Toeplitz matrices?

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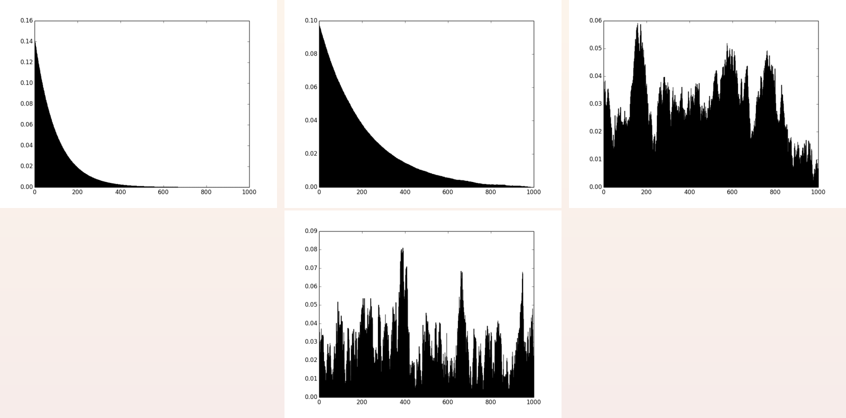

Figure: Eigenvectors for $\gamma = 2, 1.5, 0.9, 0.75$, $T_N = J_N$, $N = 1000$

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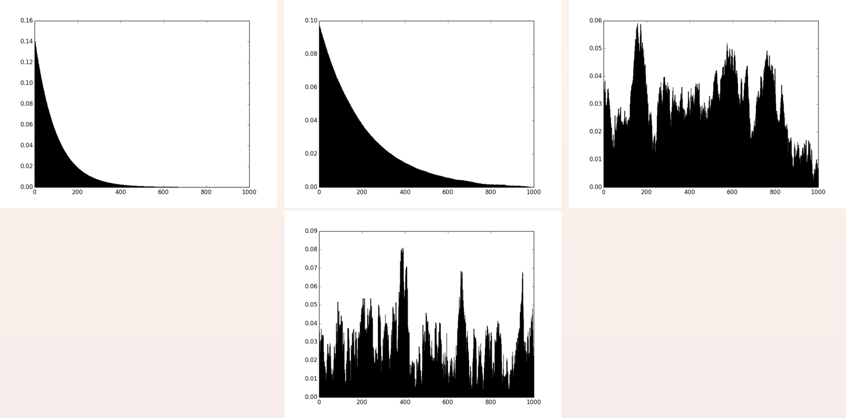

Figure: Eigenvectors for $\gamma = 2, 1.5, 0.9, 0.75$, $T_N = J_N$, $N = 1000$

Phase transitions?

Eigenvectors (w/Anirban Basak, Martin Vogel)


Back to bijective Grushin problem, introduceed by Sjöstrand-Vogel. Fix M,

$$\mathcal{P} = \begin{pmatrix} A & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M, \; R_+ = \sum_{i=1}^{M} \delta_i \circ e_i^*, \quad R_- = \sum_{i=1}^{M} f_i \circ \delta_i^*.$$

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As soon as $I + \delta QE$ is invertible, $\mathcal{P}^\delta$ is invertible, and

$$E_{-+}^\delta = E_{-+} - E_-(I + \delta QE)^{-1} \delta QE_+, \; E_+^\delta = E_+ - E(I + \delta QE)^{-1} \delta QE_+.$$

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Eigenvectors

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Also, $M = 1$ for $\gamma > 1$ and $M = N^{2(1-\gamma)}$ for $\gamma < 1$. Consider $\gamma > 1$ first

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We take $A_N = J_N - zI_N$ and Q Gaussian iid, $\delta = N^{-\gamma}$, where z is eigenvalue of $J_N + \delta Q$.
Important fact: $|z| = 1 - c_{\gamma}(\log N)/N$, with $c_{\gamma} = \gamma - 1$; set $v = [1, z, z^2, \ldots, z^{N-1}]^T/\sqrt{(N/\log N)}$, of norm $O(1)$ (pseudomode).

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$\|A_N v\|_2 = o(1/N)$, so $t_1 \ll 1/N$ while $t_i \sim i/N$ for $i \geq 2$.

Eigenvectors

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E^δ_+ is a bijection from the kernel of E^δ_{-+} to the kernel of A^δ, with inverse given by the unperturbed operator R_+.

Easiest case: $\gamma > 3/2$, $M = 1$. Then $\|\delta QE\|_\infty \sim N^{-(\gamma-3/2)} \ll 1$, so kernel of E^δ_{-+} is essentially 1, so kernel of A^δ is essentially pseudomode.

Eigenvectors

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But $\delta^p \|(EQ)^p E_+\|_\infty = o(1)$, so same conclusion as for $\gamma > 3/2$.

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Also, $M = 1$ for $\gamma > 1$ and $M = N^{2(1-\gamma)}$ for $\gamma < 1$. Consider $\gamma > 1$ first

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We take $A_N = J_N - zI_N$ and Q Gaussian iid, $\delta = N^{-\gamma}$, where z is eigenvalue of $J_N + \delta Q$. Also, $M = 1$ for $\gamma > 1$ and $M = N^{2(1-\gamma)}$ for $\gamma < 1$. Consider $\gamma > 1$ first

Important fact: $|z| = 1 - c_\gamma (\log N)/N$, with $c_\gamma = \gamma - 1$; set $v = [1, z, z^2, \ldots, z^{N-1}]^T/\sqrt{(N/\log N)}$, of norm $O(1)$ (pseudomode).

$\|A_N v\|_2 = o(1/N)$, so $t_1 \ll 1/N$ while $t_i \sim i/N$ for $i \geq 2$.

Easiest case: $\gamma > 3/2$, $M = 1$. Then $\|\delta QE\|_\infty \sim N^{-(\gamma-3/2)} \ll 1$, so kernel of E^{δ}_{-+} is essentially 1, so kernel of A^{δ} is essentially pseudomode.

$\gamma \in (1, 3/2]$. Here $\|\delta QE\|_\infty$ is not small, but whp $I + \delta QE$ is invertible, inverse norm bounded by polynomial in N. Expand:

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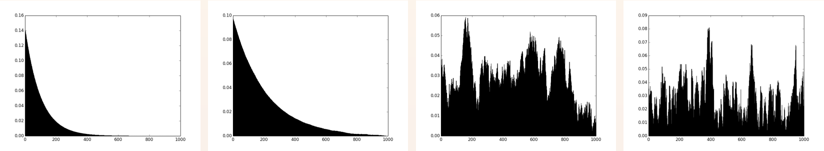

Figure: Eigenvectors for $\gamma = 2, 1.5, 0.9, 0.75$, $T_N = J_N$, $N = 1000$

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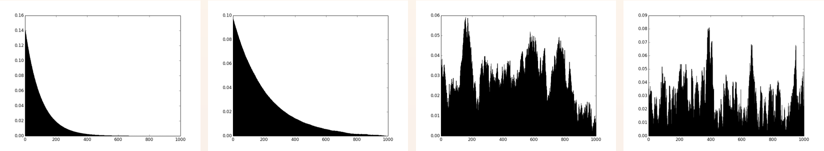

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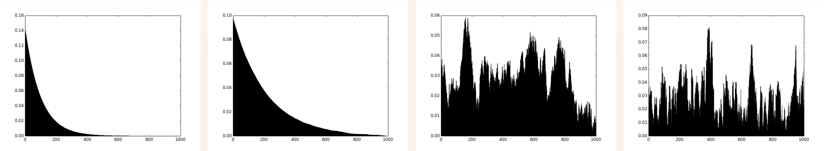

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Major cheat: norm estimates stated were for deterministic z, not the random eigenvalue!
Solution uses a net of deterministic z's, and a good probabilistic estimate on norm.

Eigenvectors

We slightly shift notation:

$$P_N = \begin{pmatrix} a_0 & a_{-1} & \dots & a_{-N_-} & \dots \\ a_1 & a_0 & a_{-1} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{N_+} & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & a_{N_+} & \dots & a_0 \end{pmatrix}.$$

$$P^Q_{N,\gamma} = P_N + N^{-\gamma} Q_N,$$

(i) The entries of Q are jointly independent and have zero mean.

(ii) For any $h \in \mathbb{N}$ there exists an absolute constant $\mathfrak{C}_h < \infty$ such that

$$\max_{i,j=1}^{N} E[|Q_{i,j}|^{2h}] \leq \mathfrak{C}_h.$$

(We also impose an anti-concentration assumption on the entries of Q.)

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For a point $z \in \mathbb{C}$, let $d(z)$ be the winding number of $p(\cdot)$ around z. Let

$$\Omega(\varepsilon, C, N) := \{z \in \mathbb{C} : C^{-1} \log N/N < \operatorname{dist}(z, \mathcal{G}_{p,\varepsilon}) < C \log N/N, d(z) \neq 0\}$$

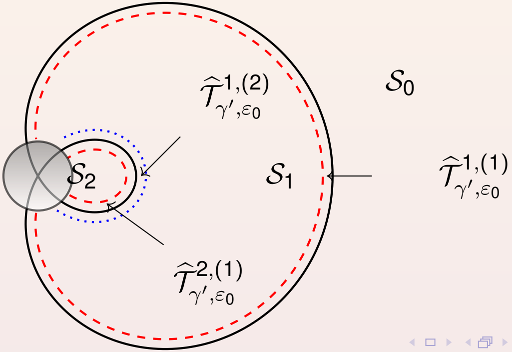
Let $\mathcal{N}_{\Omega(\varepsilon,C,N),N,\gamma} := |\{\lambda_i^N \in \Omega(\varepsilon, C, N)\}|$ denote the number of eigenvalues of $P_{N,\gamma}^Q$ that lie in $\Omega(\varepsilon, C, N)$.

Location of eigenvalues

Theorem (BVZ21)

Fix $\mu > 0$ and $\gamma > 1$. Then there exist $0 < \varepsilon, C < \infty$ (depending on γ, μ and p only) so that

$$P(\mathcal{N}_{\Omega(\varepsilon,C,N),N,\gamma} < (1-\mu)N) \to_{N\to\infty} 0.$$



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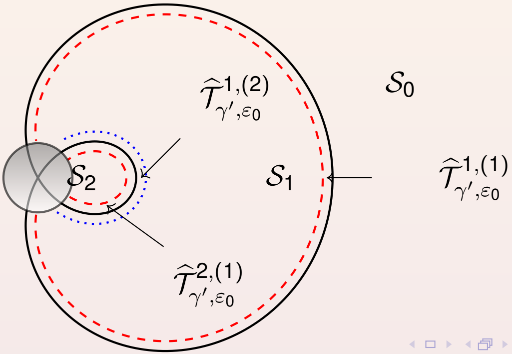

Eigenvectors

Theorem (BVZ21, $\gamma > 1$)

1. The following occurs with probability approaching one as $N \to \infty$. For each $\hat{z} \in \Omega(\varepsilon, C, N)$ which is an eigenvalue of $P^Q_{N,\gamma}$, let $v = v(\hat{z})$ denote the corresponding (right) eigenvector, normalized so that $\|v\|_2 = 1$. Then there exists a vector w, linear combination of the d smallest eigenvectors of $|d|$ eigenvectors of $(P_N - \hat{z}I)^*(P_N - \hat{z}I)$, with $\|w\|_2 = 1$ such that $\|v - w\|_2 = o(1)$ and a constant $c_\gamma > 0$, so that for any $\ell \in [N]$,

$$\begin{aligned} \|w\|_{\ell^2([\ell,N])} &\leq \varepsilon^{-c\ell \log N/N}/c, \quad \text{if } d > 0, \\ \|w\|_{\ell^2([1,N-\ell])} &\leq \varepsilon^{-c\ell \log N/N}/c, \quad \text{if } d < 0. \end{aligned}$$

Eigenvectors

Theorem (BVZ21, $\gamma > 1$)

Fix $z_0 = z_0(N) \in \Omega(\varepsilon, C, N)$ deterministic, C_0, $\widetilde{C}_0$ large, and $\eta > 0$ small. Then, there exist constants $c_1 = c_1(\eta, C_0, \widetilde{C}_0)$ and $c_0 = c_0(\gamma) \in (0,1)$, with $c_0 \to 1$ as $\gamma \to 1$ and $c_0 \to 0$ as $\gamma \to \infty$, so that, with probability at least $1-\eta$, for every $\hat{z} = \lambda_i^N \in D(z_0, C_0 \log N/N)$, any $0 < \ell \le \ell' \le \widetilde{C}_0 N/\log N$ satisfying $\ell' - \ell > N^{c_0}$ and all large N,

$$\begin{aligned} \|w\|^2_{\ell^2([\ell,\ell'])} &\ge c_1(\ell'-\ell)\log N/N, && \text{if } d > 0, \\ \|w\|^2_{\ell^2([N-\ell',N-\ell])} &\ge c_1(\ell'-\ell)\log N/N, && \text{if } d < 0. \end{aligned}$$

Further, for any $0 < c' \le \widetilde{C}_0$,

$$\begin{aligned} \|v\|^2_{\ell^2([1,c'N/\log N])} &\ge c'c_1/2, && \text{if } d > 0, \\ \|v\|^2_{\ell^2([N-c'N/\log N,N])} &\ge c'c_1/2, && \text{if } d < 0. \end{aligned}$$

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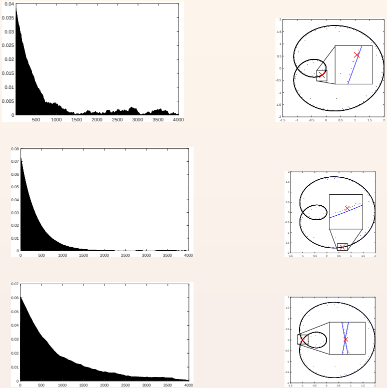

Figure: $N = 4000$, $\gamma = 1.2$, symbol $\zeta + \zeta^2$. The bottom row is not covered by the theorem, because the chosen eigenvalue is at vanishing distance from $\mathcal{B}_1$.

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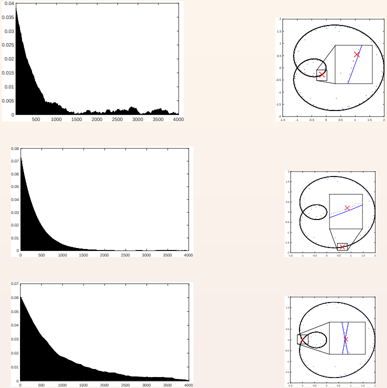

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Localization at scale $N/\log N$. The w's can in turn be approximated by *pseudomodes* ψ, with $\|(P_N - \widehat{z}I)\psi\| \to_{N\to\infty} 0$.

Speculations on $\gamma < 1$

$E^{\delta}_{-+} = E_{-+} - E_{-}(I + \delta QE)^{-1}\delta QE_{+}$, $E = \sum_{i=M+1}^{N} \frac{1}{t_i} e_i \circ f_i^*$, $E_{+} = \sum_{i=1}^{M} e_i \circ \delta_i^*$, $E^{\delta}_{+} = E_{+} - E(I + \delta QE)^{-1}\delta QE_{+}$.

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The following speculations work in the case $A_N = J_N$, general case work in progress.

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- The ith singular value of $zI - T$ is bounded below by i/N. The norm of δQE is bounded above by $N^{-\gamma+1/2+1}/M$, while that of δQE_+ is bounded above by $N^{-\gamma}M^{1/2}$.

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- By resolvent expansion, the norm of $(\delta QE)^2$ is small. We chose M so that $\|E_{-}\delta QE_{+}\| < M/N$, i.e. $\sqrt{M}N^{-\gamma} < M/N$, i.e. $M = N^{2(1-\gamma)}$. Now, the kernel of E^{δ}_{-+} is given by the kernel of

$$K = \begin{pmatrix} t_1 & \cdots & 0 & 0 \\ 0 & t_2 & 0 & \cdots \\ 0 & \cdots & t_j & 0 \\ 0 & \cdots & \cdots & t_M \end{pmatrix} - E_{-}\delta QE_{+}$$

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- $E_-\delta QE_+$ is a noise matrix of dimension M and entries $N^{-\gamma}$, and singular values of order $N^{-\gamma}\sqrt{M} = N^{1-2\gamma} \sim M/N$. *If* the 0 eigenvector of K is delocalized, with essentially uncorrelated entries, then the kernel of $\mathcal{P}^\delta$ is a combination (with uncorrelated weights) of the M bottom singular vectors of $T - z_N I$, which in the case $T_N = J$ are just the eigenfunctions of the Laplacian, ie sinusoids modulated by $(-1)^x$. Thus correlation window $\sim N/M$ (up to log terms).

Speculations on $\gamma < 1$

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- If the eigenfunction is $f(x) = M^{-1/2}\sum_{i=1}^{M} w_i e_i(x)$, the ansatz that $Ew_iw_j = \delta_{i=j}$ gives that

$$Ef(x)f(y) \sim \frac{(-1)^{x+y}}{2M}\sum_{i=1}^{M}\big(\sin((x-y)i/N) + \sin((x+y)i/N)\big)$$

which indeed decorrelates at scale N/M.

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In general, this requires QUE type results for matrices like K - a bit outside results of Benigni, Bourgade, Yau, ...

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• What about actual numerical algorithms/errors, as in case of random conjugation?

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