Moments of Characteristic Polynomials and Integrability

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Two canonical examples of random matrix ensembles

Random Hermitian matrices: $N \times N$ Hermitian matrices whose off-diagonal entries A_{ij}, $i < j$, are i.i.d. complex random variables with real and imaginary parts that are independently Gaussian distributed, each with zero mean and variance $1/2$, and whose diagonal entries A_{ii} are i.i.d. real Gaussian random variables with zero mean and variance 1. Such matrices are said to form the *Gaussian Unitary Ensemble* (GUE).

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Random unitary matrices: $A \in U(N)$ with a probability measure given by Haar measure on the group. Such matrices are said to form the *Circular Unitary Ensemble* (CUE).

Characteristic Polynomials of Random Unitary Matrices

Let A be an $N \times N$ unitary matrix. Denote the eigenvalues of A by $e^{i\theta_n}$, $1 \leq n \leq N$, and the characteristic polynomial of A on the unit circle in the complex plane by

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}) = \prod_n (1 - e^{i\theta_n - i\theta}).$$

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Moments:

$$\begin{aligned} M_N(\beta) &= \mathbb{E}_{A \in U(N)} |P_N(A,\theta)|^{2\beta} \\ &= \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} |1 - e^{i(\theta_n - \theta)}|^{2\beta} \\ &\qquad \times \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \dots d\theta_N \end{aligned}$$

Analytical Approach – JPK & NC Snaith 2000

For $\mathrm{Re}\beta > -1/2$

$$M_N(\beta) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{\Gamma(j+\beta)^2} = \frac{G(1+\beta)^2 G(N+1) G(N+1+2\beta)}{G(1+2\beta) G(N+1+\beta)^2}$$

where $G(s)$ is the Barnes G-function, which satisfies $G(s+1) = \Gamma(s)G(s)$.

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and for $k \in \mathbb{N}$

$$M_N(k) \sim \left(\prod_{m=0}^{k-1} \frac{m!}{(m+k)!} \right) N^{k^2}$$

Representation-theoretic approach

A *partition* λ is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$. We call the maximum l such that $\lambda_l > 0$ the length of the partition $l(\lambda)$, $|\lambda| = \sum_{i=1}^{l} \lambda_i$ the weight, and denote by λ' the conjugate partition.

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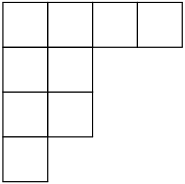
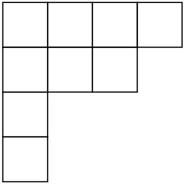


Young diagram of λ



Young diagram of λ'

In the above example $\lambda = (4, 2, 2, 1)$, $|\lambda| = 9$ and $l(\lambda) = 4$. We denote a sub-partition μ of λ by $\mu \subseteq \lambda$ if the Young diagram of μ is contained in the Young diagram of λ.

The *Schur polynomials* are symmetric polynomials indexed by partitions. Given a partition λ such that $l(\lambda) \leq N$,

$$S_\lambda(x_1, \dots, x_N) = \frac{1}{\Delta(\underline{x})} \begin{vmatrix} x_1^{\lambda_1+N-1} & x_2^{\lambda_1+N-1} & \dots & x_N^{\lambda_1+N-1} \\ x_1^{\lambda_2+N-2} & x_2^{\lambda_2+N-2} & \dots & x_N^{\lambda_2+N-2} \\ \vdots & \vdots & & \vdots \\ x_1^{\lambda_N} & x_2^{\lambda_N} & \dots & x_N^{\lambda_N} \end{vmatrix},$$

where $\Delta(\underline{x})$ is the Vandermonde determinant:

$$\Delta(\underline{x}) = \det \left[x_i^{N-j} \right]_{i,j=1}^{N} = \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

Cauchy Identity and Dual Cauchy identity

Let $t_1, t_2, \ldots$ and $x_1, x_2, \ldots$ be two finite or infinite sequences of independent variables. Then,

$$\prod_{i,j}(1 - t_i x_j)^{-1} = \sum_{\lambda} S_\lambda(\underline{t}) S_\lambda(\underline{x}).$$

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(1 + t_i x_j) = \sum_{\lambda} S_\lambda(t_1, \ldots, t_p) S_{\lambda'}(x_1, \ldots, x_q).$$

Since $S_\lambda = 0$ or $S_{\lambda'} = 0$ unless $l(\lambda) \leq p$ or $l(\lambda') \leq q$, λ runs over a finite number of partitions such that the Young diagram of λ fits inside a $p \times q$ rectangle.

Theorem (Bump & Gamburd 2006)

For $\beta \in \mathbb{N}$

$$M_N(\beta) = \mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} = S_{\langle N^\beta \rangle}(1^{2\beta})$$

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This also gives the interpretation that, for $\beta = k \in \mathbb{N}$, as $N \to \infty$

$$M_N(k) \sim \frac{g_k}{k^2!} N^{k^2}$$

where g_k is the number of ways of filling a $k \times k$ array with the integers $1, 2, \ldots, k^2$ in such a way that the numbers increase along each row and down each column (i.e. the number of $k \times k$ *Young tableaux*).

More generally

$$\mathbb{E}_{A\in U(N)} \prod_{l=1}^{L} P(A,\theta_l) \prod_{k=1}^{K} \overline{P(A,\theta_{L+k})} = \frac{S_{\langle N^L\rangle}(e^{i\theta_1},\ldots,e^{i\theta_{K+L}})}{\prod_{l=1}^{L} e^{iN\theta_l}}$$

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To what extent do these formulae extend to the GUE and to other related ensembles?

Multivariate orthogonal polynomials

Let $\varphi_i(x), i = 0, 1, \ldots$ be orthogonal polynomials with respect to a weight $w(x)$.

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Let $\varphi_i(x), i = 0, 1, \ldots$ be orthogonal polynomials with respect to a weight $w(x)$.
Multivariate orthogonal polynomials can be defined by the determinant formula

$$\Phi_\mu(\underline{x}) := \frac{1}{\Delta(\underline{x})} \begin{vmatrix} \varphi_{\mu_1+N-1}(x_1) & \varphi_{\mu_1+N-1}(x_2) & \cdots & \varphi_{\mu_1+N-1}(x_N) \\ \varphi_{\mu_2+N-2}(x_1) & \varphi_{\mu_2+N-2}(x_2) & \cdots & \varphi_{\mu_2+N-2}(x_N) \\ \vdots & \vdots & & \vdots \\ \varphi_{\mu_N}(x_1) & \varphi_{\mu_N}(x_2) & \cdots & \varphi_{\mu_N}(x_N) \end{vmatrix},$$

where $l(\mu) \leq N$.

$$\int \Phi_\mu(x_1,\ldots,x_N)\Phi_\nu(x_1,\ldots,x_N) \prod_{1\le i<j\le N}(x_i-x_j)^2 \prod_{j=1}^N w(x_j)\,dx_j = \delta_{\mu\nu} C_\mu$$

Here the lengths of the partitions μ and ν are less than or equal to the number of variables N, and C_μ is a constant which depends on N.

$$\int \Phi_\mu(x_1,\dots,x_N)\Phi_\nu(x_1,\dots,x_N) \prod_{1\le i<j\le N}(x_i-x_j)^2 \prod_{j=1}^{N} w(x_j)\, dx_j = \delta_{\mu\nu} C_\mu$$

Here the lengths of the partitions μ and ν are less than or equal to the number of variables N, and C_μ is a constant which depends on N. The following lemma generalises the dual Cauchy identity.

Lemma

Let Φ_μ be multivariate polynomials given as just defined. Let $p, q \in \mathbb{N}$ and for $\lambda \subseteq (q^p) \equiv (\underbrace{q,\dots,q}_{p})$ let $\tilde{\lambda} = (p-\lambda'_q,\dots,p-\lambda'_1)$. Then

$$\prod_{i=1}^{p}\prod_{j=1}^{q}(t_i - x_j) = \sum_{\lambda\subseteq(q^p)} (-1)^{|\tilde{\lambda}|}\Phi_\lambda(t_1,\dots,t_p)\Phi_{\tilde{\lambda}}(x_1,\dots,x_q).$$

Let us focus in particular on when $w(x)$ is a Gaussian, Laguerre and Jacobi weight:

$$w(x) = \begin{cases} e^{-\frac{x^2}{2}}, & x \in \mathbb{R}, & & \text{Gaussian}, \\ x^{\gamma} e^{-x}, & x \in \mathbb{R}_+, & \gamma > -1, & \text{Laguerre}, \\ x^{\gamma_1}(1-x)^{\gamma_2}, & x \in [0,1], & \gamma_1, \gamma_2 > -1, & \text{Jacobi}. \end{cases}$$

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Similarly, their multivariate generalizations are eigenfunctions of second-order partial differential operators, known as Calogero-Sutherland Hamiltonians, for example

$$H^{(H)} = \sum_{j=1}^{N} \left(\frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^{N} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}$$

$$H^{(L)} = \sum_{j=1}^{N} \left(x_j \frac{\partial^2}{\partial x_j^2} + (\gamma - x_j + 1) \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^{N} \frac{x_j}{x_j - x_k} \frac{\partial}{\partial x_j}$$

Theorem (Bhargavi Jonnadula, JPK, F. Mezzadri, 2020)

Let M be an $N \times N$ GUE, LUE or JUE matrix and $t_1, \ldots, t_p \in \mathbb{C}$. Then,

$$(a) \quad \mathbb{E}_N^{(H)}[\prod_{j=1}^{p} \det(t_j I - M)] = \mathcal{H}_{(N^p)}(t_1, \ldots, t_p)$$

$$(b) \quad \mathbb{E}_N^{(L)}[\prod_{j=1}^{p} \det(t_j I - M)] = \left(\prod_{j=N}^{p+N-1} (-1)^j j! \right) \mathcal{L}_{(N^p)}^{(\gamma)}(t_1, \ldots, t_p)$$

$$(c) \quad \mathbb{E}_N^{(J)}[\prod_{j=1}^{p} \det(t_j I - M)] = \left(\prod_{j=N}^{p+N-1} (-1)^j j! \frac{\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)} \right) \times \mathcal{J}_{(N^p)}^{(\gamma_1,\gamma_2)}(t_1, \ldots, t_p)$$

Here the subscripts $(H), (L), (J)$ indicate Hermite, Laguerre and Jacobi, respectively, and $\mathcal{H}_\lambda$, $\mathcal{L}_\lambda^\gamma$, $\mathcal{J}_\lambda^{(\gamma_1,\gamma_2)}$ are multivariate polynomials orthogonal with respect to the corresponding weights.

Focus now on the GUE. Let M be an $N \times N$ GUE matrix and let $M_R = M/\sqrt{N}$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, $\sum_j \lambda_j \leq N$, define

$$C_\lambda(N) = \prod_{j=1}^{N} \frac{(\lambda_j + N - j)!}{(N-j)!}.$$

Let $\lambda = (N^{2p})$. Then the moments of characteristic polynomial of a rescaled GUE matrix of size N are given by

Theorem (Bhargavi Jonnadula, JPK, F. Mezzadri, 2021)

$$\mathbb{E}_N^{(H)}\left[\det(tI - M_R)^{2p}\right] = C_\lambda(2p)\left(-\frac{1}{2N}\right)^{\frac{|\lambda|}{2}} \sum_{\nu\subseteq\lambda} \frac{(-2N)^{\frac{|\nu|}{2}}}{|\nu|!} \dim V_\nu D_{\lambda\nu}^{(H)} t^{|\nu|},$$

where

$$D_{\lambda\nu}^{(H)} = \det\left[\mathbb{I}_{\lambda_j-\nu_k-j+k=0(mod\ 2)}\left(\left(\frac{\lambda_j-\nu_k-j+k}{2}\right)!\right)^{-1}\right]_{j,k=1,\dots p}$$

and $\dim V_\nu$ *is the dimension of the irreducible representation labelled by* ν *of the symmetric group* $\mathcal{S}_{|\nu|}$.

Brezin and Hikami (2000): when $N \to \infty$

$$\mathbb{E}_N^{(H)} \left[\det(tI - M)^{2p}\right] \sim e^{-Np} e^{Np\frac{t^2}{2}} (2\pi N \rho(t))^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p+j)!},$$

where the asymptotic eigenvalue density is

$$\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}.$$

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$$\mathbb{E}_N^{(H)}\left[\det(M_R)^{2p}\right] \sim e^{-Np}(2N)^{p^2} \prod_{j=0}^{p-1} \frac{j!}{(p+j)!}$$

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What about $t \neq 0$?

Example: $p = 1$

Re-writing the Brezin-Hikami formula when for $p = 1$,

$$\lim_{N\to\infty} \frac{1}{2N} e^{N} \exp\left(-\frac{Nt^2}{2}\right) \mathbb{E}_N^{(H)} \left[\det(tI - M_R)^2\right] = \pi\rho(t).$$

Expanding the right-hand side in powers of t gives

$$1 - \frac{1}{8}t^2 - \frac{1}{128}t^4 + \frac{1}{1024}t^6 + O(t^8).$$

This is to be compared to the expression from the partition sum.

When N is even we find

$$\Big[1+\left(-\frac{5}{12}-\frac{1}{2}N\right)t^2+\left(-\frac{811}{77760}+\frac{17}{216}N+\frac{19}{72}N^2+\frac{1}{6}N^3\right)t^4 + \Big(-\frac{640879}{587865600}+\frac{799}{1749600}N-\frac{3667}{291600}N^2-\frac{323}{6480}N^3 - \frac{31}{540}N^4-\frac{1}{45}N^5\Big)t^6+O(t^8)\Big],$$

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and when N is odd

$$\Big[1+\left(\frac{1}{6}+\frac{1}{2}N\right)t^2+\left(-\frac{101}{19440}-\frac{17}{216}N-\frac{19}{72}N^2-\frac{1}{6}N^3\right)t^4 + \Big(-\frac{15853}{18370800}-\frac{799}{1749600}N+\frac{3667}{291600}N^2 + \frac{323}{6480}N^3+\frac{31}{540}N^4+\frac{1}{45}N^5\Big)t^6+O(t^8)\Big].$$

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- So neither case agrees with the Taylor expansion of the semicircle!
- But formally averaging the two expressions does give the Taylor expansion of the semicircle!!!

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The asymptotics of the moments of the characteristic polynomials of GUE matrices is therefore rather more subtle than might initially be assumed.

Joint moments of CUE characteristic polynomials

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Set

$$V_N(A,\theta) := \exp\left(\mathrm{i}N\frac{(\theta+\pi)}{2} - \mathrm{i}\sum_{n=1}^{N}\frac{\theta_n}{2}\right)P_N(A,\theta),$$

($V_N(A,\theta)$ is real-valued for $\theta \in [0, 2\pi)$).

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($V_N(A,\theta)$ is real-valued for $\theta \in [0,2\pi)$).
The joint moments of the function $V_U(\theta)$ and its derivative are

$$F_N(k,h) := \mathbb{E}_{A\in U(N)}|V_N(A,0)|^{2k-2h}|V_N'(A,0)|^{2h},$$

where it is assumed that

$$h > -\frac{1}{2} \quad \text{and} \quad k > h - \frac{1}{2}.$$

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These joint moments have been studied by many authors, including Hughes (2001), Conrey Rubinstein & Snaith (2006), Dehaye (2008, 2010), Winn (2012), Riedtmann (2018), Basor *et al.* (2018), Bailey *et al.* (2019).

Asymptotics

Conjecture (Hughes 2001)

When $N \to \infty$, for $k > -1/2$ and $0 \leq h < k + 1/2$

$$F_N(k, h) \sim F(k, h) N^{k^2+2h}$$

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i.e.

$$F(k,h) := \lim_{N\to\infty} \frac{F_N(k,h)}{N^{k^2+2h}}$$

exists and is non-zero for $k > -1/2$ and $0 \leq h < k + 1/2$

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For integer and half-integer values of h and k, $F_N(k, h)$ is equal to a sum over Young Tableaux, but with a complicated summand (Dehaye (2008, 2010), Winn (2012), and Riedtmann (2018)). The analysis of these formulae in general is a major challenge.

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It has so far not been possible to extend these approaches for a given $h \in \mathbb{N}$, to $k > h - 1/2$, or to non-integer values of h.

Aside – Painlevé Equations

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Homogeneous linear o.d.e.s, e.g.

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Which o.d.e.s have no movable critical points?

Fuchs (1884) showed that amongst the first-order equations of the form

$$\frac{dw}{dz} = F(w, z)$$

where $F(w, z)$ is rational in w and locally analytic in z, the only equations without movable critical points are Riccati equations, which have $F(w, z) = C_0(z) + C_1(z)w + C_2(z)w^2$ and which are linearizable to a second-order o.d.e.

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Around 1900, Painlevé and Gambier showed that only a finite number (circa 50) of second-order equations of the form

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$$\frac{d^2w}{dz^2} = 6w^2 + z$$

The general form of P_{V} is

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{1}{z}w' + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{z} + \delta\frac{w(w+1)}{z-1},$$

where α, β, γ and δ are complex constants.

The general form of $P_{\rm V}$ is

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where α, β, γ and δ are complex constants.

$P_{\rm V}$ has a Lax pair, namely it can be written as the compatibility condition of two linear systems of ODEs for the 2×2 matrix function $\Phi(z,x)$, $z, x, \in \mathbb{C}$, that satisfies the equations

$$\frac{d\Phi}{dz} = \left(\frac{x}{2}\sigma_3 + \frac{A_0}{z} + \frac{A_1}{z-1}\right)\Phi(z,x),$$
$$\frac{d\Phi}{dx} = \left(\frac{z}{2}\sigma_3 + \frac{B_0}{x}\right)\Phi(z,x),$$

where A_0, A_1 and B_0 are 2×2 matrices.

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Let $L_n^{(\alpha)}(t)$ be the generalized Laguerre polynomial

$$L_n^{(\alpha)}(t) := \frac{e^t}{t^\alpha n!}\frac{d^n}{dt^n}\left(t^{\alpha+n}e^{-t}\right) = \sum_{j=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(j+\alpha+1)(n-j)!}\frac{(-t)^j}{j!}$$

and define

$$K_n(\epsilon, y) := \frac{(-1)^n}{\pi}\frac{\partial^n}{\partial\epsilon^n}\left(\frac{\epsilon}{\epsilon^2+y^2}\right).$$

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Proposition (Winn 2012)

$$F_N(h,k) = \lim_{\epsilon\to 0}(-1)^{\frac{k(k-1)}{2}} 2^{-2h}\int_{-\infty}^{\infty} K_{2h}(\epsilon, y)e^{-N|y|} \times \det\left[L_{N+k-1-(i+j)}^{(2k-1)}(-2|y|)\right]_{i,j=0,\dots,k-1} dy,$$

with $N > k-1$.

Theorem – Basor, Bleher, Buckingham, Grava, Its, Its & Keating 2018

We have the representation

$$\det \left[L^{(2k-1)}_{N+k-1-(i+j)}(-2|y|) \right]_{i,j=0,\cdots,k-1} = \frac{e^{-2k|y|}}{(2\pi i)^k} H_k[w_0],$$

where $H_k[w_0] = H_n[w_0]|_{n=k}$, and $H_n[w_0]$ is the Hankel determinant

$$H_n = \det \left[\int_C w_0(t) t^{i+j} dt \right]_{i,j=0,\cdots,n-1}$$

with the weight

$$w_0(t) = \frac{e^{\frac{x}{1-t}}}{(1-t)^{2k} t^{N+k}}, \quad x = 2|y|.$$

Here C is a small (radius less than 1) positively oriented circle around zero.

Theorem (cont.)

Furthermore,

$$\frac{d}{dx}\ln H_k = \frac{\sigma(x) + kx + Nk}{x},$$

where $\sigma(x)$ is a solution of the σ-Painlevé V equation

$$\left(x\frac{d^2\sigma}{dx^2}\right)^2 = \left(\sigma - x\frac{d\sigma}{dx} + 2\left(\frac{d\sigma}{dx}\right)^2 - 2N\frac{d\sigma}{dx}\right)^2 - 4\frac{d\sigma}{dx}\left(-N + \frac{d\sigma}{dx}\right)\left(-k - N + \frac{d\sigma}{dx}\right)\left(k + \frac{d\sigma}{dx}\right).$$

with asymptotics

$$\sigma(x) = -Nk + \frac{N}{2}x + \mathcal{O}(x^2), \quad x \to 0.$$

Outline of Proof

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1. Formulate a Riemann-Hilbert problem for the generalised Laguerre polynomials and derive a system of related o.d.e.s;
2. a series of rational and gauge transformations reduces this system of o.d.e.s to the Lax pair of P_{V};
3. identify the Hankel determinant with a particular solution of the σ-form of P_{V}.

Large Matrix Asymptotics

The large-matrix asymptotics can be recovered by analysing the asymptotics of the solutions of the σ-Painlevé V equation

$$\left(x\frac{d^2\sigma}{dx^2}\right)^2 = \left(\sigma - x\frac{d\sigma}{dx} + 2\left(\frac{d\sigma}{dx}\right)^2 - 2N\frac{d\sigma}{dx}\right)^2 - 4\frac{d\sigma}{dx}\left(-N + \frac{d\sigma}{dx}\right)\left(-k - N + \frac{d\sigma}{dx}\right)\left(k + \frac{d\sigma}{dx}\right)$$

when $N \to \infty$.

Large-Matrix Limit

Theorem – Basor, Bleher, Buckingham, Grava, Its, Its & Keating 2018

For $h \in \mathbb{N}$, $k > h - 1/2$, in general

$$F(h,k) = (-1)^h \frac{G(k+1)^2}{G(2k+1)} \frac{d^{2h}}{dx^{2h}} \left[\exp \int_0^x \left(\frac{\xi(s)}{s} ds \right) \right] \Bigg|_{x=0},$$

where G is the Barnes function and $\xi(x)$ is a particular solution of the σ-Painlevé III equation

$$(x\xi'')^2 = -4x(\xi')^3 + \left(4k^2 + 4\xi\right)(\xi')^2 + x\xi' - \xi,$$

with the initial conditions

$$\xi(0) = 0\,, \quad \xi'(0) = 0.$$

c.f. Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein & Snaith (2019)

Non-integer joint moments and the Hua-Pickrell Measure

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Let $\mathbb{W}_N$ denote the Weyl chamber:

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For $N \geq 1$ and $s > -\frac{1}{2}$, the Hua-Pickrell probability measure $\mathrm{M}_N^{(s)}$ on $\mathbb{W}_N$ is

$$\mathrm{M}_N^{(s)}(d\mathbf{x}) = \frac{1}{\mathrm{c}_N^{(s)}} \prod_{j=1}^{N} \frac{1}{(1 + x_j^2)^{N+s}} \Delta_N(\mathbf{x})^2 dx_1 \cdots dx_N$$

where $\Delta_N(\mathbf{x}) = \prod_{1 \leq i < j \leq N}(x_j - x_i)$ and $\mathrm{c}_N^{(s)}$ is a normalisation constant.

Let $s > -\frac{1}{2}$. Then,

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{x}_i^{(N)} \xrightarrow{\mathrm{d}} \mathrm{X}(s), \quad \text{as } N \to \infty,$$

where $(\mathrm{x}_1^{(N)}, \dots, \mathrm{x}_N^{(N)})$ has law $\mathrm{M}_N^{(s)}$ and $\mathrm{X}(s)$ is a random variable that plays an important role in the work of Pickrell (1991), Vershik (1994), Olshanski & Vershik (1996), Borodin & Olshanski (2001), Qiu (2017), . . . , classifying the ergodic measures for the action of the infinite dimensional unitary group on the space of infinite Hermitian matrices.

Connection to joint moments [Assiotis, Keating & Warren (2020)]

Theorem Let $s > -\frac{1}{2}$ and $0 \leq h < s + \frac{1}{2}$. Then,

$$\lim_{N\to\infty} \frac{1}{N^{s^2+2h}} F_N(s,h) \stackrel{\text{def}}{=} F(s,h) = F(s,0) 2^{-2h} \mathbb{E}\left[|\mathsf{X}(s)|^{2h} \right]$$

with the limit $F(s,h)$ satisfying $0 < F(s,h) < \infty$. The function $F(s,0)$ is given by

$$F(s,0) = \frac{G(s+1)^2}{G(2s+1)},$$

where G is the Barnes G-function.