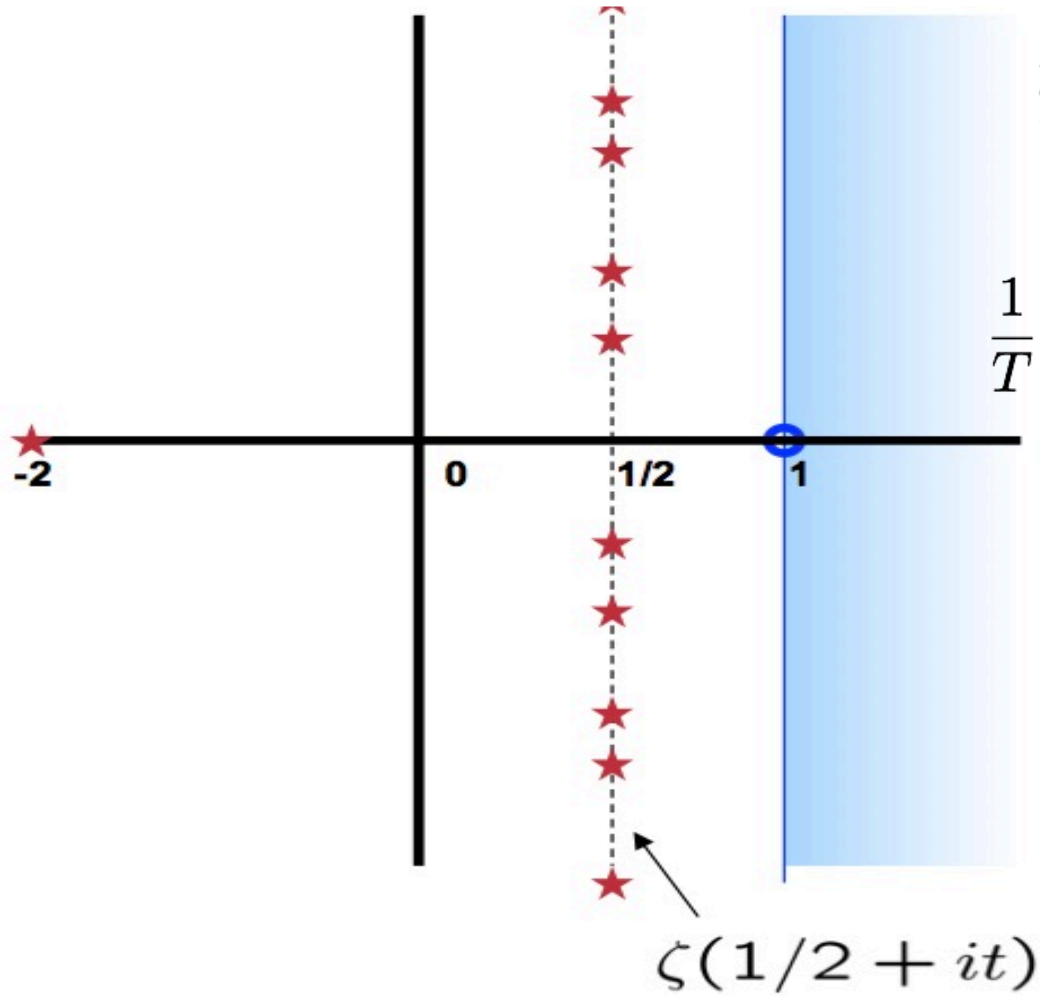
University of BRISTOL

Unearthing random matrix theory in the statistics of L-functions: the story of Beauty and the Beast

MSRI, August 24th, 2021

Nina Snaith (with Brian Conrey and Amy Mason)

Moments of the Riemann Zeta Function



$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 dt \sim \log T$$

(Hardy and Littlewood, 1918)

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 dt \sim \frac{1}{2\pi^2}\log^4 T$$

(Ingham, 1926)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}s > 1$$
$$= \prod_p (1-1/p^s)^{-1}$$

Moments of the Riemann Zeta Function: Conjecture

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt \sim a_\lambda f_\lambda \log^{\lambda^2} T$$

Moments of characteristic polynomials: Theorem

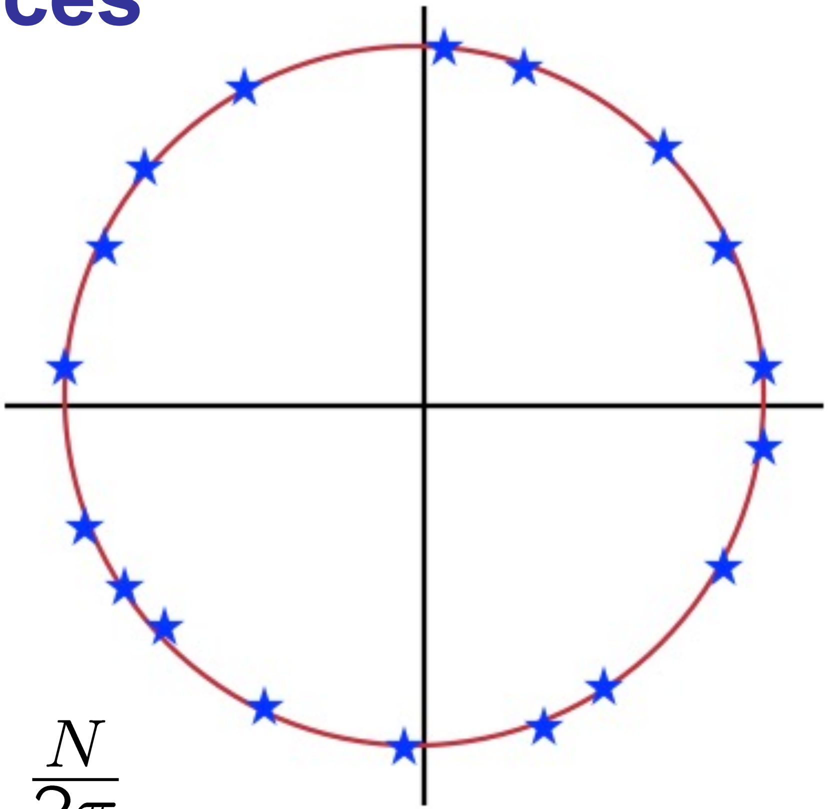
$$\int_{U(N)} |\Lambda_A(1)|^{2\lambda} dA_{Haar} \sim f_{\lambda,U(N)} N^{\lambda^2}$$

$$N \sim \log T$$

Conjecture (Keating and Snaith, 2000): $f_\lambda = f_{\lambda,U(N)}$

Random Unitary Matrices

$N \times N$ unitary matrix



$e^{i\theta_n}$ - eigenvalues

chosen randomly with respect to Haar measure on U(N)

density of eigenphases: $\frac{N}{2\pi}$

Characteristic polynomial: $$\Lambda_X(e^z) = \prod_{j=1}^{N} (1 - e^z e^{-i\theta_j})$$

Moments of the Riemann Zeta Function: Conjecture

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda}dt \sim a_\lambda f_\lambda \log^{\lambda^2} T$$

Random matrix theory conjectures:

$$f_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!},$$

for $\lambda = k$, an integer (Keating and Snaith, 2000)

Moments of the Riemann Zeta Function: Conjecture

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt \sim a_\lambda f_\lambda \log^{\lambda^2} T$$

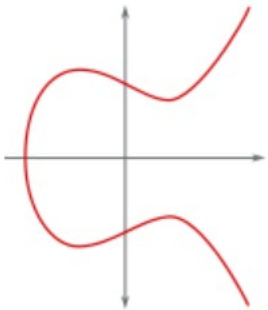
Random matrix theory

Random matrix theory conjectures:

$$f_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!},$$

for $\lambda = k$, an integer (Keating and Snaith, 2000)

Elliptic curve L-functions:

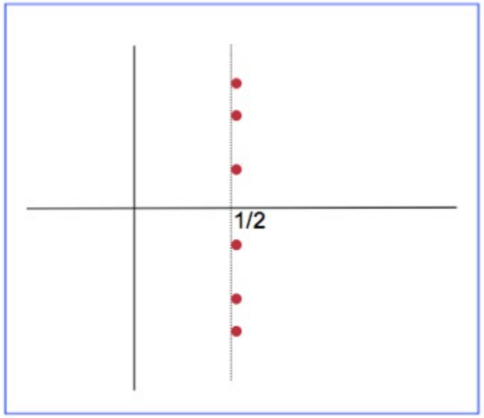


eg.

$$E_{11}: \; y^2 = 4x^3 - 4x^2 - 40x - 79$$

L-function:

$$L_{E_{11}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$



a_n determined by E_{11}

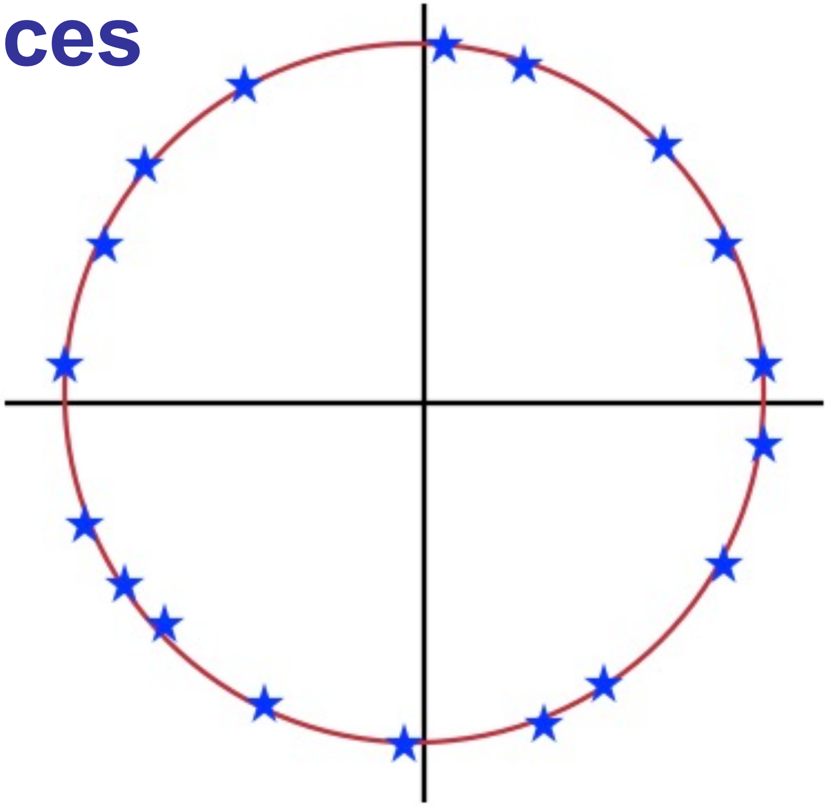
Conjecture: (Conrey, Keating, Rubinstein, Snaith)

Fraction of elliptic curves in the family that have rank 2 or higher

$$\sim c_E T^{-1/4} (\log T)^{3/8}$$

Random matrix theory conjectures this exponent

Random Unitary Matrices



$N \times N$ unitary matrix

$e^{i\theta_n}$ - eigenvalues

chosen randomly with respect to Haar measure on U(N)

density of eigenphases: $\frac{N}{2\pi}$

Averages over random unitary matrices

$$\int_{U(N)} f(\theta_1, \theta_2, \ldots, \theta_N) d\mu_{\mathrm{Haar}}$$

$$= \frac{1}{(2\pi)^N \; N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_N$$

$$= \frac{1}{N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) \det_{N \times N} [S_N(\theta_k - \theta_j)] d\theta_1 \cdots d\theta_N$$

with

$$S_N(\theta) = \frac{1}{2\pi} \frac{\sin \frac{N\theta}{2}}{\sin \frac{\theta}{2}}$$

Statistics of eigenvalues

$$\lim_{N\to\infty}\int_{U(N)}\sum_{i_1,\ldots,i_n}^{*} f\left(\tfrac{N}{2\pi}\theta_{i_1},\ldots,\tfrac{N}{2\pi}\theta_{i_n}\right)d\mu_{\text{Haar}}$$

$$= \frac{1}{n!}\int_0^\infty\cdots\int_0^\infty f(\theta_1,\ldots,\theta_n)\det_{n\times n}[S(\theta_k-\theta_j)]d\theta_1\cdots d\theta_n$$

with

$$S(\theta)=\frac{\sin\pi\theta}{\pi\theta}$$

eg. mean density $R_1 = 1$

eg. 2-point correlation:

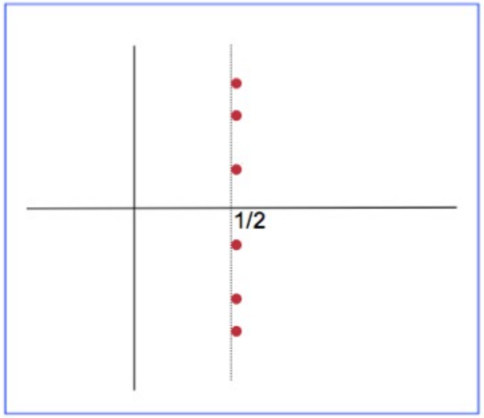
$$R_2(\theta_1,\theta_2) = 1-\frac{\sin^2(\pi(\theta_2-\theta_1))}{\pi^2(\theta_2-\theta_1)^2}$$

Emma Watson in Beauty and the Beast

Statistics of Riemann zeros

Density of zeros:

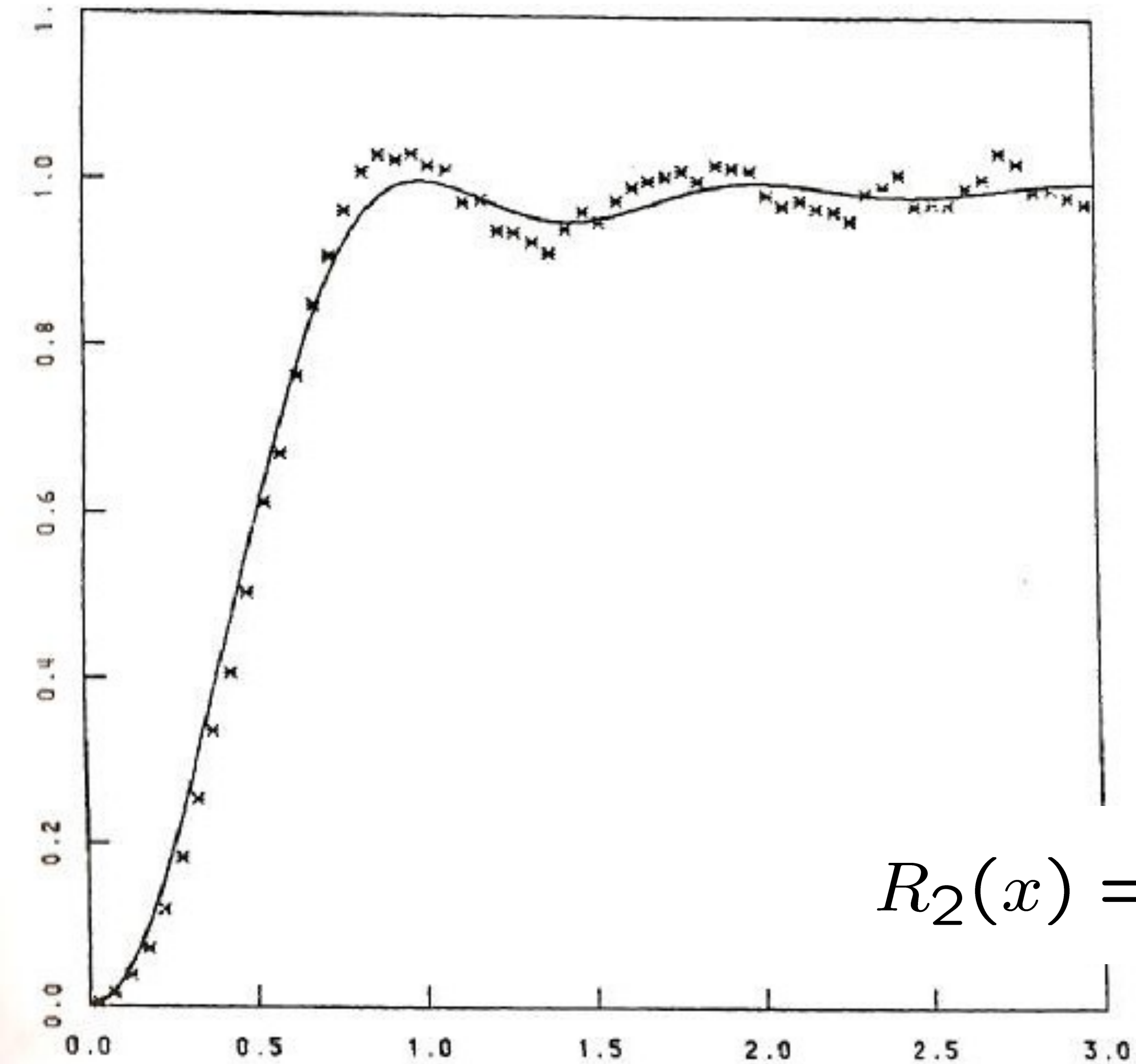
$$d(t) \sim \frac{1}{2\pi} \log \frac{t}{2\pi}$$



$$w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi}, \qquad t_n = \text{n}^{\text{th}} \text{ Riemann zero}$$

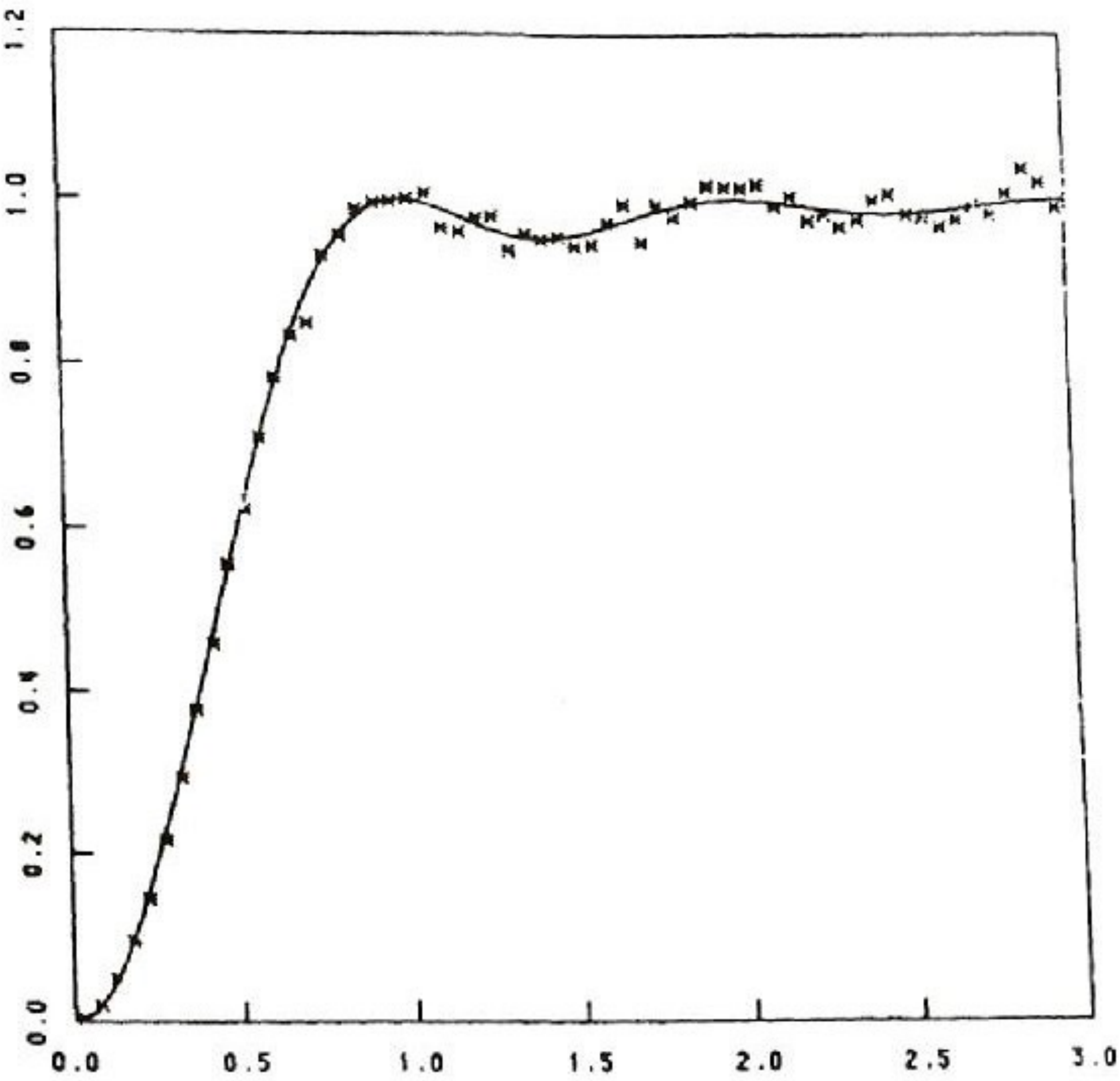
scale the Riemann zeros so that their average spacing is 1

Two point correlation function of the Riemann zeros (Odlyzko)

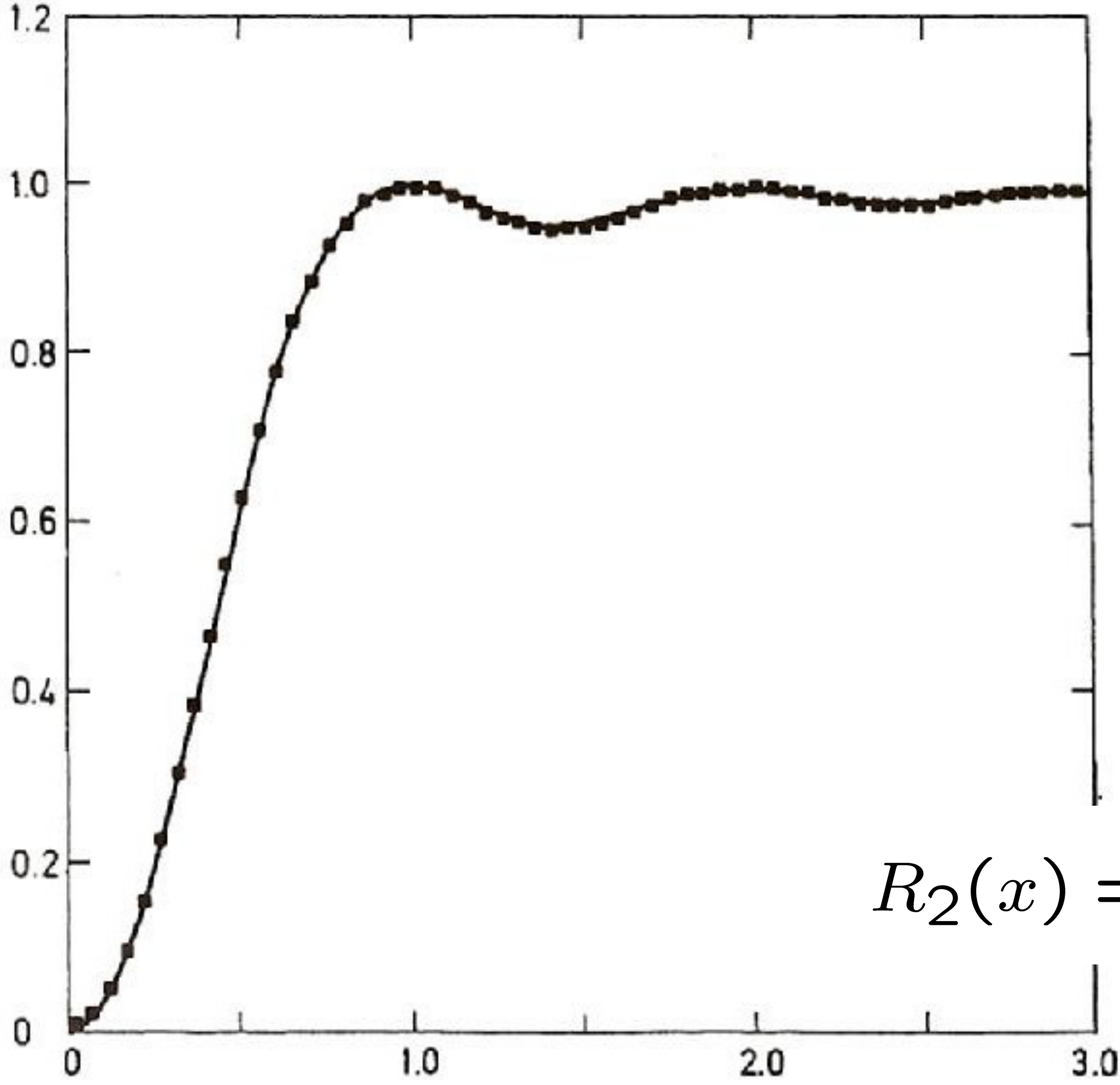


First 100000 zeros

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$



10^5 zeros around the 10^{12}th zero



Picture by
A. Odlyzko

79 million zeros around the 10^{20}th zero

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

Proving the connection??

Rudnick and Sarnak (1996) showed for the Riemann zeta function (and other individual L-functions) that for test functions with restricted support, the n-point correlation functions of zeros high on the critical line agree with those of the eigenvalues of large, random unitary matrices.

So, for example, with the *t*'s the heights of the Riemann zeros and f a suitable test function:

$$\sum_{0<t_1,\ldots,t_n\leq T} f\left(\frac{\log T}{2\pi}t_1,\ldots,\frac{\log T}{2\pi}t_n\right)$$

Statistics of eigenvalues

$$\lim_{N\to\infty}\int_{U(N)}\sum_{i_1,\ldots,i_n}^{*} f\left(\tfrac{N}{2\pi}\theta_{i_1},\ldots,\tfrac{N}{2\pi}\theta_{i_n}\right)d\mu_{\mathrm{Haar}}$$

$$=\frac{1}{n!}\int_0^\infty\cdots\int_0^\infty f(\theta_1,\ldots,\theta_n)\det_{n\times n}[S(\theta_k-\theta_j)]d\theta_1\cdots d\theta_n$$

with $$S(\theta)=\frac{\sin\pi\theta}{\pi\theta}$$

eg. mean density $R_1 = 1$

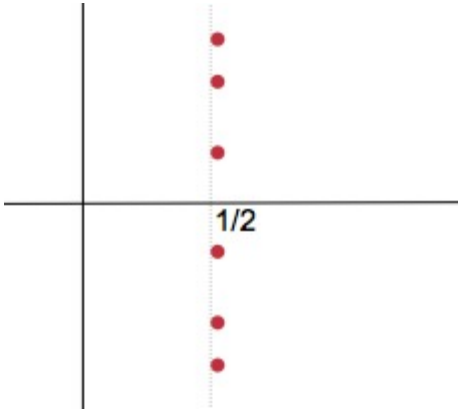
eg. 2-point correlation:

$$R_2(\theta_1,\theta_2)=1-\frac{\sin^2(\pi(\theta_2-\theta_1))}{\pi^2(\theta_2-\theta_1)^2}$$

Emma Watson in Beauty and the Beast

Families of L-functions:

Each L-function like Riemann zeta:
- Dirichlet series and Euler product
- functional equation
- Riemann hypothesis

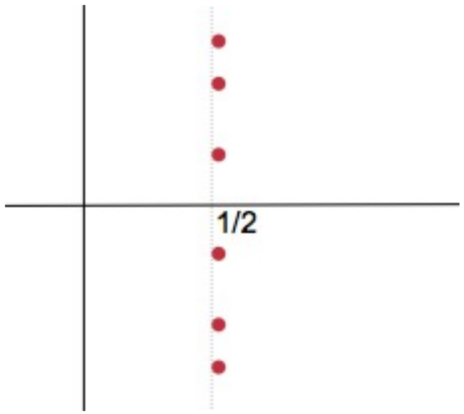


Natural families of L-functions:
- vary parameter(s) to obtain different L-functions
- family ordered by the parameter
- look at statistics of zeros averaged over the family

Statistics of zeros in families:

Katz and Sarnak (1999):

Averaged over a family of L-functions, zeros close to $s = 1/2$ show statistics like ONE of



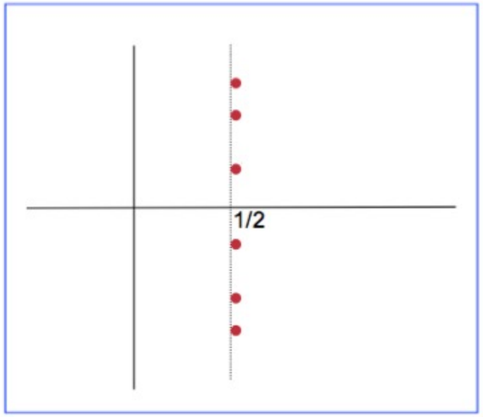
$U(N), O(N), USp(2N),$

depending on the family, when the ordering parameter becomes large.

Families of L-functions

Eg. Dirichlet L-functions:

$$L(s,\chi_d)=\sum_{n=1}^{\infty}\frac{\chi_d(n)}{n^s}$$



Kronecker symbol:

$$\chi_d(n)=\left(\frac{d}{n}\right)$$

extension of Legendre symbol:

$$\left(\frac{d}{p}\right)=\begin{cases}0 & \text{if } p|d\\ 1 & \text{if } d\equiv x^2 \mod p\\ -1 & \text{else}\end{cases}$$

Unitary symplectic matrices

For $X \in USp(2N)$

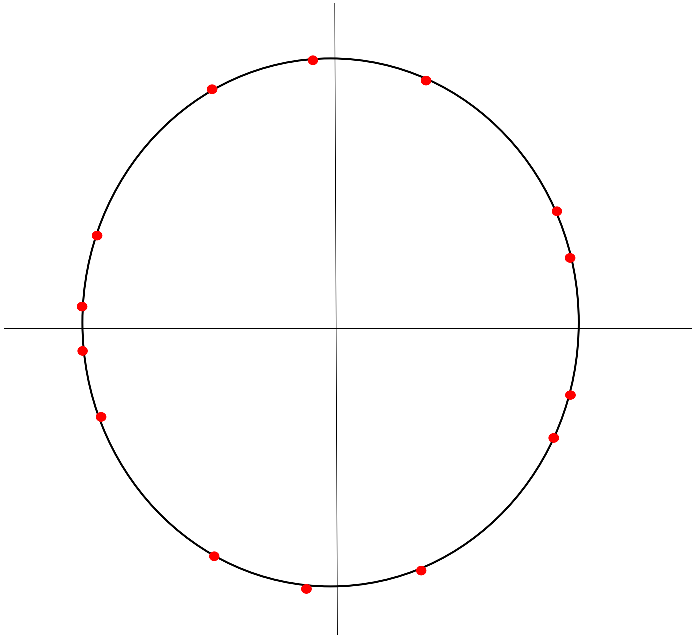
$$XX^\dagger = 1$$

and

$$X^\dagger J X = J$$

with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$



Eigenvalues:

$$e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_N}$$

Statistics of eigenvalues

$$\lim_{N\to\infty}\int_{USp(2N)}\sum_{i_1,\ldots,i_n}^{*} f(\tfrac{N}{\pi}\theta_{i_1},\ldots,\tfrac{N}{\pi}\theta_{i_n})d\mu_{\text{Haar}}$$

$$=\frac{1}{n!}\int_0^\infty\cdots\int_0^\infty f(\theta_1,\ldots,\theta_n)\det_{n\times n}[S(\theta_k-\theta_j)-S(\theta_k+\theta_j)]d\theta_1\cdots d\theta_n$$

with $$S(\theta)=\frac{\sin\pi\theta}{\pi\theta}$$

eg. 1-level density:

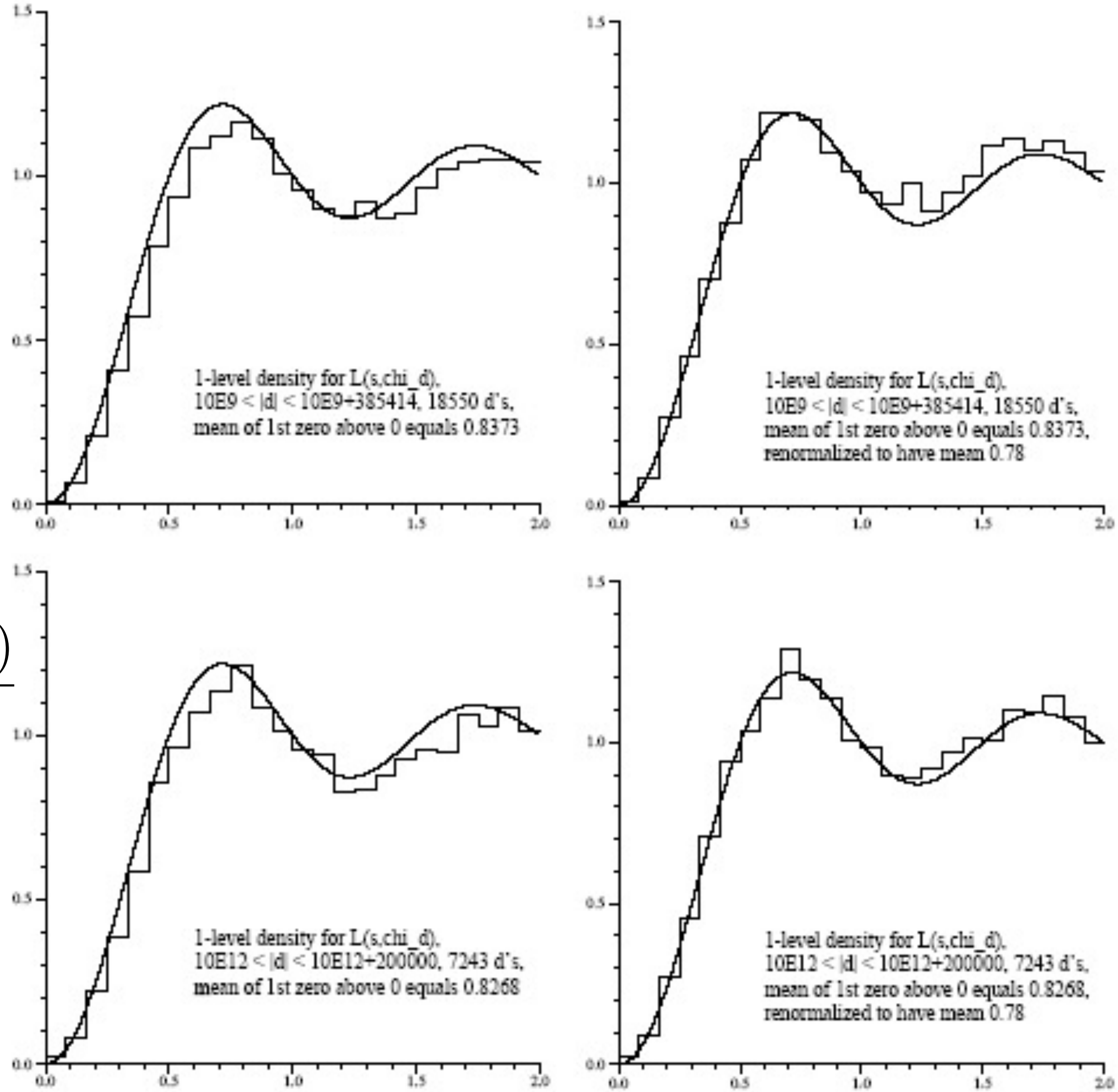
$$R_1(\theta)=1-\frac{\sin(2\pi\theta)}{2\pi\theta}$$

Emma Watson in Beauty and the Beast

University of BRISTOL

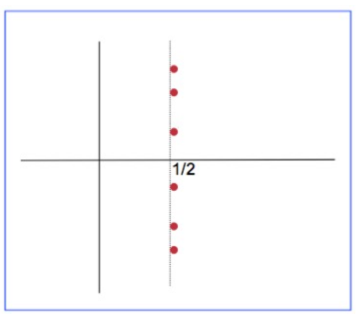
One level density of Dirichlet L-functions:

Figures by Michael Rubinstein



Solid curve:

$$R_1(\theta) = 1 - \frac{\sin(2\pi\theta)}{2\pi\theta}$$



Proving the connection??

Theorem (Ozluk and Snyder,1999): 1-level density where the support of the Fourier transform of the test function f lies in [-2,2]:

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{d\leq X}\sum_{\gamma_d} f\left(\gamma_d\frac{\log\frac{d}{\pi}}{2\pi}\right)$$

$$= \int_{-\infty}^{\infty} f(x)\left(1-\frac{\sin(2\pi\theta)}{2\pi\theta}\right)dx$$

γ_d: heights of zeros of $L(s,\chi_d)$

X^*: number of terms in the sum over d

Proving the connection??

Theorem (Ozluk and Snyder,1999): 1-level density where the support of the Fourier transform of the test function f lies in [-2,2]:

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{d\leq X}\sum_{\gamma_d} f\left(\gamma_d \frac{\log\frac{d}{\pi}}{2\pi}\right)$$

$$= \int_{-\infty}^{\infty} f(x)\left(1-\underbrace{\frac{\sin(2\pi\theta)}{2\pi\theta}}_{\text{RMT one-level density}}\right)dx$$

γ_d: heights of zeros of $L(s,\chi_d)$

X^*: number of terms in the sum over d

Theorem (Rubinstein, 2001):

With the product of the Fourier transforms of the test functions, $\prod_{i=1}^n \hat{f}_i(u_i)$, having support in $\sum_{i=1}^n |u_i| < 1$,

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \sum_{j_1,\dots,j_n=-\infty}^{\infty} f_1\left(\frac{\log X}{2\pi}\gamma_d^{(j_1)}\right)\cdots f_n\left(\frac{\log X}{2\pi}\gamma_d^{(j_n)}\right)$$

$$= \sum_{Q\cup M=\{1,\dots,n\}} \left(\prod_{m\in M}\int_{-\infty}^{\infty} f_m(x)dx\right)$$

$$\times\left(\sum_{\substack{S_2\subseteq Q\\ |S_2| \text{ even}}} \left(\left(-\frac{1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{-\infty}^{\infty}\hat{f}_\ell(u)du\right)\right.$$

$$\times\left.\left(\sum_{(A;B)} 2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{-\infty}^{\infty}|u|\hat{f_{a_j}}(u)\hat{f_{b_j}}(u)du\right)\right).$$

University of BRISTOL

n-level density of unitary symplectic matrices

$$\lim_{N\to\infty}\int_{USp(2N)}\sum_{i_1,\ldots,i_n}^{*} f\left(\tfrac{N}{\pi}\theta_{i_1},\ldots,\tfrac{N}{\pi}\theta_{i_n}\right)d\mu_{\mathrm{Haar}}$$

$$=\frac{1}{n!}\int_0^\infty\cdots\int_0^\infty f(\theta_1,\ldots,\theta_n)\det_{n\times n}[S(\theta_k-\theta_j)-S(\theta_k+\theta_j)]d\theta_1\cdots d\theta_n$$

Emma Watson in
Beauty and the Beast

Eigenvalue statistics from ratios

$$\int_{USp(2N)} \frac{\prod_{\alpha\in A} \Lambda_X(e^{-\alpha})}{\prod_{\beta\in B} \Lambda_X(e^{-\beta})} dX$$

where

$$\Lambda_X(e^z) = \prod_{j=1}^{N} (1 - e^z e^{i\theta_j})(1 - e^z e^{-i\theta_j})$$

Theorem (Mason and Snaith, 2015):

"The Beast"

$$\int_{USp(2N)} \sum_{\substack{j_1,\cdots,j_n=-\infty \\ j_1,\ldots,j_n\neq 0}}^{\infty} F\left(\theta_{j_1},\cdots,\theta_{j_n}\right) d\mu_{\mathrm{Haar}}$$

$$= \frac{1}{(2\pi i)^n} \sum_{Q\cup M=\{1,\cdots,n\}} (2N)^{|M|}$$

$$\times \int_{(\delta)^{|Q|}} \int_{(0)^{|M|}} 2^{|Q|} J^*_{USp(2N)}\left(z_Q\right) F\left(iz_1,\cdots,iz_n\right) dz_1\cdots dz_n.$$

where

$$J^*_{USp(2N)}(A) = \sum_{D\subseteq A} e^{-2N\sum_{d\in D} d} (-1)^{|D|} \sqrt{\frac{Z(D,D)Z(D^-,D^-)Y(D^-)}{Y(D)Z^\dagger(D^-,D)^2}}$$

$$\times \sum_{\substack{A/D=W_1\cup\cdots\cup W_R \\ |W_r|\leq 2}} \prod_{r=1}^{R} H_D(W_r)$$

(and previously with Brian Conrey for U(N))

Theorem (Rubinstein, 2001):

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \sum_{j_1,\ldots,j_n=-\infty}^{\infty} f_1\left(\frac{\log X}{2\pi}\gamma_d^{(j_1)}\right)\cdots f_n\left(\frac{\log X}{2\pi}\gamma_d^{(j_n)}\right)$$

$$= \sum_{Q\cup M=\{1,\ldots,n\}} \left(\prod_{m\in M}\int_{-\infty}^{\infty} f_m(x)dx\right)$$

$$\times\left(\sum_{\substack{S_2\subseteq Q\\ |S_2|\text{ even}}} \left(\left(-\frac{1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{-\infty}^{\infty}\hat{f}_\ell(u)du\right)\right.$$

$$\left.\times\left(\sum_{(A;B)} 2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{-\infty}^{\infty}|u|\hat{f}_{a_j}(u)\hat{f}_{b_j}(u)du\right)\right).$$

$\prod_{i=1}^n \hat{f}_i(u_i)$, having support in $\sum_{i=1}^n |u_i| < 1$

Theorem (Mason and Snaith, 2016):

$$\lim_{N\to\infty}\int_{USp(2N)}\sum_{\substack{j_1,\ldots,j_n=-\infty\\ j_1,\ldots,j_n\neq 0}}^{\infty} f_1\Big(\frac{N}{\pi}\theta_{j_1}\Big)\cdots f_n\Big(\frac{N}{\pi}\theta_{j_n}\Big)d\mu_{\mathrm{Haar}}$$

$$=\sum_{Q\cup M=\{1,\ldots,n\}}\Bigg(\prod_{m\in M}\int_{-\infty}^{\infty} f_m(x)dx\Bigg)$$

$$\times\Bigg(\sum_{\substack{S_2\subseteq Q\\ |S_2|\ \mathrm{even}}}\Bigg(\Big(-\frac{1}{2}\Big)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{-\infty}^{\infty}\hat{f}_\ell(u)du\Bigg)$$

$$\times\Bigg(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{-\infty}^{\infty}|u|\hat{f}_{a_j}(u)\hat{f}_{b_j}(u)du\Bigg)\Bigg).$$

$\prod_{i=1}^n \hat{f}_i(u_i)$, having support in $\sum_{i=1}^n |u_i|<1$

Theorem (Gao, 2014): Assuming GRH

Also Entin, Roditty-Gershon and Rudnick

$$\lim_{X\to\infty} \frac{\pi^2}{4X} \sum_{d\in D(X)} \sum_{j_1,\ldots,j_n} f_1(L\gamma_{8d}^{(j_1)}) \cdots f_n(L\gamma_{8d}^{(j_n)})$$

$$= \sum_{Q\cup M=\{1,\ldots,n\}} \left(\prod_{m\in M} \int_{-\infty}^{\infty} f_m(x)dx\right) \Bigg[\sum_{S_2\subseteq Q} \left(\left(\frac{-1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{-\infty}^{\infty} \hat{f}_\ell(u)du \right)$$

$$\times \left(\left(\frac{1+(-1)^{|S_2|}}{2}\right) 2^{|S_2|/2} \sum_{S_2=(A:B)} \prod_{i=1}^{|S_2|/2} \int_{-\infty}^{\infty} |u_i| \hat{f_{a_i}}(u_i) \hat{f_{b_i}}(u_i) du_i \right.$$

$$- \frac{1}{2} \sum_{\substack{S_3 \subsetneq S_2 \\ |S_3| \text{ even}}} 2^{|S_3|/2} \left(\sum_{S_3=(C:D)} \prod_{i=1}^{|S_3|/2} \int_{-\infty}^{\infty} |u_i| \hat{f_{c_i}}(u_i) \hat{f_{d_i}}(u_i) du_i \right)$$

$$\left. \times \sum_{I\subsetneq S_3^c} (-1)^{|I|} (-2)^{|S_3^c|} \int_{\substack{(\mathbb{R}\geq 0)^{S_3^c} \\ \sum_{i\in I} u_i \leq (\sum_{i\in I^c} u_i) - 1}} \prod_{i\in S_3^c} \hat{f}_i(u_i) \prod_{i\in S_3^c} du_i \right) \Bigg].$$

$\prod_{i=1}^n \hat{f}_i(u_i)$, having support in $\sum_{i=1}^n |u_i| < 2$

Theorem (Mason and Snaith, 2016):

$$\lim_{N\to\infty}\int_{USp(2N)}\sum_{\substack{j_1,\ldots,j_n=-\infty\\ j_1,\ldots,j_n\neq 0}}^{\infty} f_1\left(\frac{N}{\pi}\theta_{j_1}\right)\cdots f_n\left(\frac{N}{\pi}\theta_{j_n}\right)d\mu_{\text{Haar}}$$

$$=\sum_{Q\cup M=\{1,\ldots,n\}}\left(\prod_{m\in M}\int_{-\infty}^{\infty}f_m(x)dx\right)\Bigg[\sum_{S_2\subseteq Q}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{-\infty}^{\infty}\hat{f}_\ell(u)du\right)$$

$$\times\Bigg(\left(\frac{1+(-1)^{|S_2|}}{2}\right)2^{|S_2|/2}\sum_{S_2=(A:B)}\prod_{i=1}^{|S_2|/2}\int_{-\infty}^{\infty}|u_i|\hat{f_{a_i}}(u_i)\hat{f_{b_i}}(u_i)du_i$$

$$-\frac{1}{2}\sum_{\substack{S_3\subsetneq S_2\\ |S_3|\text{ even}}}2^{|S_3|/2}\left(\sum_{S_3=(C:D)}\prod_{i=1}^{|S_3|/2}\int_{-\infty}^{\infty}|u_i|\hat{f_{c_i}}(u_i)\hat{f_{d_i}}(u_i)du_i\right)$$

$$\times\sum_{I\subsetneq S_3^c}(-1)^{|I|}(-2)^{|S_3^c|}\int_{\substack{(\mathbb{R}\geq 0)^{S_3^c}\\ \sum_{i\in I}u_i\leq(\sum_{i\in I^c}u_i)-1}}\prod_{i\in S_3^c}\hat{f}_i(u_i)\prod_{i\in S_3^c}du_i\Bigg)\Bigg].$$

$\prod_{i=1}^{n}\hat{f}_i(u_i)$, having support in $\sum_{i=1}^{n}|u_i|<2$

Theorem (Mason and Snaith, 2015):

"The Beast"

$$\int_{USp(2N)} \sum_{\substack{j_1,\cdots,j_n=-\infty \\ j_1,\ldots,j_n\neq 0}}^{\infty} F\left(\theta_{j_1},\cdots,\theta_{j_n}\right) d\mu_{\text{Haar}}$$

$$= \frac{1}{(2\pi i)^n} \sum_{Q\cup M=\{1,\cdots,n\}} (2N)^{|M|}$$

$$\times \int_{(\delta)^{|Q|}} \int_{(0)^{|M|}} 2^{|Q|} J^*_{USp(2N)}(z_Q)\, F\left(iz_1,\cdots,iz_n\right) dz_1\cdots dz_n.$$

where

$$J^*_{USp(2N)}(A) = \sum_{D\subseteq A} e^{-2N\sum_{d\in D} d} (-1)^{|D|} \sqrt{\frac{Z(D,D)Z(D^-,D^-)Y(D^-)}{Y(D)Z^\dagger(D^-,D)^2}}$$

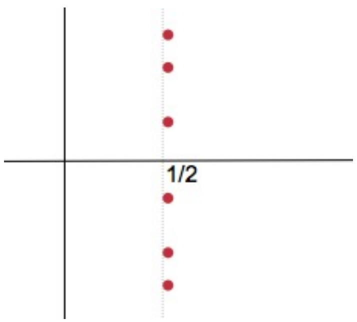
$$\times \sum_{\substack{A/D=W_1\cup\cdots\cup W_R \\ |W_r|\leq 2}} \prod_{r=1}^{R} H_D(W_r)$$

$|D| < q$ when $\sum_{j=1}^n |u_j| < q.$

(and previously with Brian Conrey for U(N))

Conjecture (Conrey, Keating, Rubinstein, Snaith 2002):

Let E be an elliptic curve defined over $\mathbb{Q}$. Then there is a constant $c_E \geq 0$ such that



$$\frac{\sum_{\substack{p\leq T,\text{ prime}\\ L_E(1/2,\chi_p)=0\\ L_E(s,\chi_p)\in\mathcal{F}_{E+}}}}{\sum_{\substack{p\leq T,\text{ prime}\\ L_E(s,\chi_p)\in\mathcal{F}_{E+}}}} 1 \sim c_E T^{-1/4}(\log T)^{3/8}$$

Random matrix theory conjectures this exponent

Conjecture (Birch and Swinnerton-Dyer):

$L_E(1/2,\chi_d) = 0$ if and only if E_d has infinitely many rational points (ie. rank greater than zero)