On the circle, $GMC^\gamma = \varprojlim C\beta E_n$ if $\gamma = \sqrt{\frac{2}{\beta}} \leq 1$

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Sommaire

A puzzling identity in law

Consider $\left(\mathcal{N}_1^{\mathbb{C}}, \mathcal{N}_2^{\mathbb{C}}, \dots\right)$ to be a sequence of i.i.d standard complex Gaussians i.e:

$$\mathbb{P}\left(\mathcal{N}_i^{\mathbb{C}} \in dxdy\right) = \frac{1}{\pi} e^{-x^2-y^2} dxdy \ ,$$

so that:

$$\mathbb{E}\mathcal{N}_k^{\mathbb{C}} = 0, \qquad \mathbb{E}|\mathcal{N}_k^{\mathbb{C}}|^2 = 1 \ .$$

Let $\left(\alpha_j\right)_{j \geq 0}$ be independent random variables with uniform phases and modulii as follows:

$$|\alpha_j|^2 \overset{\mathcal{L}}{=} Beta(1, \beta_j := \frac{\beta}{2}(j+1))$$

As a shadow of a more global correspondence between GMC and RMT:

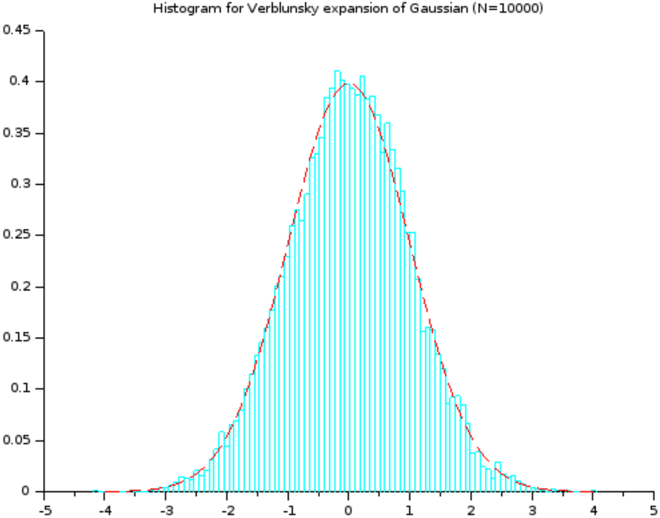
Proposition (Verblunsky expansion of Gaussians)

The following equality in law holds, while the RHS converges almost surely (!):

$$\sqrt{\frac{2}{\beta}} \mathcal{N}_1^{\mathbb{C}} \overset{\mathcal{L}}{=} \sum_{j=0}^{\infty} \alpha_j \overline{\alpha}_{j-1} \ .$$

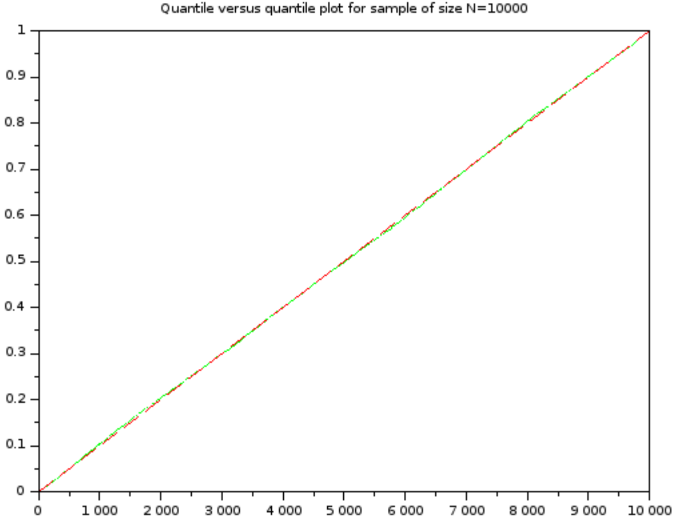
A puzzling identity in law (II)

"Numerical proof:" Histogram of $\Re\left(\sigma \sum_{j=0}^{\infty} \alpha_j \overline{\alpha}_{j-1}\right), |\sigma| = 1.$



A puzzling identity in law (III)

“Numerical proof:”



Introduction

The main player of this talk will be the random Gaussian distribution on S^1:

$$G(e^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \frac{\mathcal{N}_k^{\mathbb{C}}}{\sqrt{k}} e^{ik\theta} \ .$$

Remark

Given the decay of Fourier coefficients, this is a Schwartz distribution in negative Sobolev spaces $\cap_{\varepsilon>0} H^{-\varepsilon}(S^1)$ *where:*

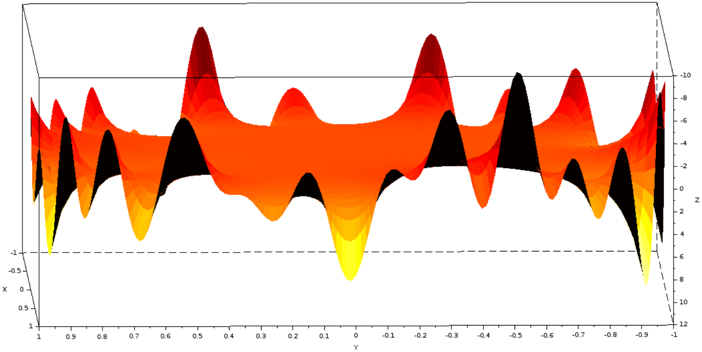
$$H^s(S^1) := \left\{ f \in \mathcal{D}'(S^1) \mid \sum_{k \in \mathbb{Z}^*} |k|^s |\widehat{f}(k)|^2 \right\} \ .$$

Harmonic extension of G

Consider the harmonic extension of G to the disc:

$$G(re^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \frac{\mathcal{N}_k^{\mathbb{C}}}{\sqrt{k}} r^k e^{ik\theta} = P_r * G_{|S^1}\left(e^{i\theta}\right) \ ,$$

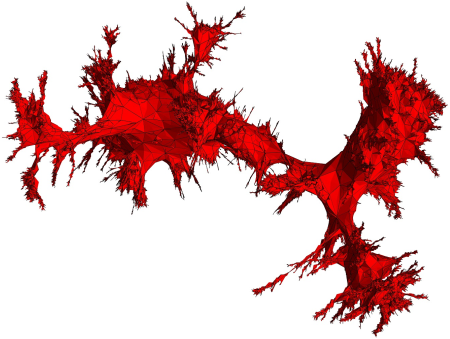
where P_r is the Poisson kernel.

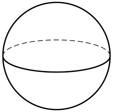


Sommaire

Modern motivations: “Liouville Conformal Field Theory” in 2D

Brownian Map (© Bettinelli)





$$\overset{Uniformization}{\rightsquigarrow}\left(\qquad , GMC^{\gamma}(dz)\right)$$

for $\gamma = \sqrt{\frac{8}{3}}$.

(Theorem by Miller-Sheffield)

Message

The GMC^{γ} is the natural Riemannian measure on random surfaces which model LCFT.

But please, ask someone else to tell you about this... E.g. Rhodes-Vargas, Miller-Sheffield and/or their students.

Our construction: On the circle, in 1d

A natural object (for Kahane and the LCFT crowd) is ($r < 1$):

$$GMC_r^\gamma(d\theta) := e^{\gamma G(re^{i\theta}) - \frac{1}{2}\gamma^2 \operatorname{Var}\left[G(re^{i\theta})\right]} \frac{d\theta}{2\pi} = e^{\gamma G(re^{i\theta})}(1-r^2)^{\gamma^2} \frac{d\theta}{2\pi} \ .$$

We have:

Theorem (Kahane, Rhodes-Vargas, Berestycki)

Define for every $f : S^1 = \partial\mathbb{D} \to \mathbb{R}_+$, and $\gamma < 1$:

$$GMC_r^\gamma(f) := \int_0^{2\pi} f(e^{i\theta}) GMC_r^\gamma(d\theta) \ .$$

Then the following convergence holds in $L^1(\Omega)$:

$$GMC_r^\gamma(f) \xrightarrow{r \to 1} GMC^\gamma(f) \ .$$

The limiting measure GMC^γ is called Kahane's Gaussian Multiplicative Chaos.

Sommaire

The model

- Consider the distribution of n points on the circle:

$$(C\beta E_n) \qquad \frac{1}{Z_{n,\beta}} \prod_{1\leq k<l\leq n} \left|e^{i\theta_k} - e^{i\theta_l}\right|^\beta d\theta = \frac{1}{Z_{n,\beta}} \left|\Delta(\theta)\right|^\beta d\theta$$

- For $\beta = 2$, one recognizes the Weyl integration formula for central functions on the compact group $U(n)$. Therefore, this nothing but the distribution of a Haar distributed matrix on the group $U(n)$. The study of this case is very rich in the representation theory of U_n (Bump-Gamburd, Borodin-Okounkov, ...)
- For general β, not as nice but still an integrable system: Jack polynomials in n variables are orthogonal for the $C\beta E_n$, Eigenvectors for the trigonometric Calogero-Sutherland system (n variables), "Higher" representation theory (Rational Cherednik algebras).
- The characteristic polynomial:

$$X_n(z) := \det\left(\mathrm{id} - zU_n^*\right) = \prod_{1\leq j\leq n} \left(1 - ze^{-i\theta_j}\right)$$

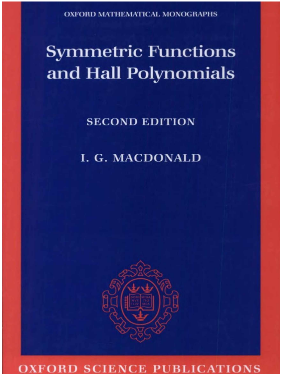
CBE as regularization of Gaussian Fock space

The $C\beta E_n$ is the regularization of a Gaussian space by n points at the level of symmetric functions. In fact:

$$tr\left(U_n^k\right) \overset{n\to\infty}{\to} \sqrt{\frac{2k}{\beta}}\mathcal{N}_k^{\mathbb{C}} \ ,$$

(Strong Szegö - $\beta = 2$, Diaconis-Shahshahani - $\beta = 2$, Matsumoto-Jiang)

Short proof: Open the bible of symmetric functions



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CBE as regularization of Gaussian Fock space: Proof

- Power sum polynomials: $p_k := p_k(U_n) = tr\left(U_n^k\right)$ and $p_\lambda := \prod_i p_{\lambda_i}$.

CBE as regularization of Gaussian Fock space: Proof

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- Scalar product for functions in n variables: $\langle f, g\rangle_n := \mathbb{E}_{C\beta E_n}\left(f(z_i)\overline{g(z_i)}\right)$.
- Fact 1: This scalar product approximates the Hall-Macdonald scalar product in infinitely many variables $\langle\cdot,\cdot\rangle_n \to \langle\cdot,\cdot\rangle$, where

$$\langle p_\lambda, p_\mu\rangle = z_\lambda\left(\frac{2}{\beta}\right)^{\ell(\lambda)}\delta_{\lambda,\mu} = \delta_{\lambda,\mu} Cste(\lambda) \ .$$

CBE as regularization of Gaussian Fock space: Proof

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- Fact 2: The Macdonald scalar product has a Gaussian space lurking behind as

$$\delta_{\lambda,\mu} Cste(\lambda) = \mathbb{E}\left(\prod_k \left(\sqrt{\frac{2k}{\beta}}\mathcal{N}_k^{\mathbb{C}}\right)^{m_k(\lambda)} \left(\sqrt{\frac{2k}{\beta}}\overline{\mathcal{N}_k^{\mathbb{C}}}\right)^{m_k(\mu)}\right) \ ,$$

  where $m_k(\lambda)$ multiplicity of k in partition λ.

⇝ the $C\beta E$ is the regularization of a Gaussian Fock space by restricting the symmetric functions to n variables.

Classical Gaussianity and log-correlation in RMT

Since:

$$\log X_n(z) = \sum_{k \geq 1} \frac{tr\left(U_n^k\right)}{k} z^k \ ,$$

it is conceivable that:

Proposition (O'C-H-K for $\beta = 2$, C-N for $\beta > 0$)

We have the convergence in law to the log-correlated field:

$$\left(\log |X_n(z)|\right)_{z \in \mathbb{D}} \overset{n \to \infty}{\longrightarrow} \left(\sqrt{\frac{2}{\beta}} G(z)\right)_{z \in \mathbb{D}}$$

- *uniformly in $z \in K \subset \mathbb{D}$, K compact.*
- *for $z \in \partial\mathbb{D}$, in the Sobolev space $H^{-\varepsilon}(\partial\mathbb{D})$.*

GMC from RMT: A convergence in law (I)

A step further, it is natural to construct a measure from the characteristic polynomial

$$\left(\log|X_n(z)|\right)_{z\in\mathbb{D}} \overset{n\to\infty}{\longrightarrow} \left(\sqrt{\frac{2}{\beta}}G(z)\right)_{z\in\mathbb{D}}$$

and compare it to the GMC.

Here is a result whose content is very different from ours but easily confused with it:

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Proposition (Nikula, Saksman and Webb (2018))

For $\beta = 2$ and for every $\alpha \in [0, 2)$, consider $X_n(z) = \det(I_n - zU_n^*)$ to be the characteristic polynomial of the $CUE = C2E$. Then, for all continuous $f : \partial\mathbb{D} \to \mathbb{R}$, we have the convergence in law as $n \to \infty$:

$$\int_{[0,2\pi]} \frac{d\theta}{2\pi}\ f\left(e^{i\theta}\right) \frac{\left|X_n(e^{i\theta})\right|^{\alpha}}{\mathbb{E}\left|X_n(e^{i\theta})\right|^{\alpha}} \stackrel{\mathcal{L}}{\to} GMC^{\alpha/2}(f)\ .$$

GMC from RMT: A convergence in law (II)

A few remarks are in order:

- In fact, for $\beta = 2$, there is an extremely fast convergence of traces of Haar matrices to Gaussians. For f polynomial on the circle, we have:

$$(Johansson) \quad d_{TV}\left(\mathrm{Tr}\ f(U_n), \sum_k c_k(f)\sqrt{k}\mathcal{N}_k^{\mathbb{C}}\right) \overset{n\to\infty}{\sim} C_f\, n^{-cn/\deg f} \ .$$

- Not true for general $\beta > 0 \rightsquigarrow \beta = 2$ critical in some sense.

Message (Take home message)

Our statement $GMC^\gamma = \varprojlim C\beta E_n$ *is non-asymptotic and an almost sure equality for all* $\beta > 0$ *and* $n \in \mathbb{N}$, *via a non-trivial coupling. We are saying for* $\gamma < 1$:

"GMC^γ *is the object whose finite n approximations are given by* $C\beta E_n$*'s."*

Sommaire

OPUC and Szegö recurrence

- OPUC : “Orthogonal Polynomials on the Unit Circle”
- Consider a probability measure μ on the circle and apply the Gram-Schmidt procedure:
$$\left\{1, z, z^2, \dots\right\} \rightsquigarrow \left\{\Phi_0(z), \Phi_1(z), \Phi_2(z), \dots\right\}$$
- Szegö recurrence:
$$\begin{cases} \Phi_{j+1}(z) & = & z\Phi_j(z) - \overline{\alpha_j}\Phi_j^*(z) \\ \Phi_{j+1}^*(z) & = & -\alpha_j z\Phi_j(z) + \Phi_j^*(z) \ . \end{cases}$$

  Here:
$$\Phi_j^*(z) := z^j\overline{\Phi_j(1/\bar{z})}$$
  is the polynomial with reversed and conjugated coefficients. The α_j's are inside the unit disc, known as Verblunsky coefficients.

The work of Killip, Nenciu

Killip and Nenciu have discovered an explicit distribution for Verblunsky coefficients so that X_n, the characteristic polynomial of $C\beta E_n$, is a Φ_n^*!

Theorem (Killip, Nenciu)

- *Let* $(\alpha_j)_{j\geq 0}$*, as before and* η *uniform on the circle.*
- *Let* $(\Phi_j, \Phi_j^*)_{j\geq 0}$ *be a sequence of OPUC obtained from the coefficients* $(\alpha_j)_{j\geq 0}$ *and the Szegö recurrence.*

Then we have the equality in law between random polynomials:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

Proof.

Essentially computation of a Jacobian - with two important subtleties! ☐

IMPORTANT: Projective family. Notice the consistency. A priori, a realization of CBE_n has no reason to share the first Verblunsky coefficients with CBE_{n+1}.

A puzzling question

If a measure defines Verblunsky coefficients, the converse is also true:

Theorem (Verblunsky 1930)

Let $\mathcal{M}_1(\partial\mathbb{D})$ be the simplex of probability measures on the circle, endowed with the weak topology. The set $\mathbb{D}^{\mathbb{N}}$ is endowed with the topology of point-wise convergence. The Verblunsky map

$$\begin{array}{rccc} \mathbb{V}: & \mathcal{M}_1(\partial\mathbb{D}) & \to & \mathbb{D}^{\mathbb{N}} \sqcup \left(\sqcup_{n\in\mathbb{N}}\mathbb{D}^n \times \partial\mathbb{D}\right) \\ & \mu & \mapsto & (\alpha_j(\mu); j \in \mathbb{N}) \end{array}$$

is an homeomorphism. Atomic measures with n atoms have n Verblunsky coefficients, the last one being of modulus one.

This begs the question:

Question

The Verblunsky coefficients are consistent. Since the obvious coupling respects the Verblunksy map, we define a limiting measure $\varprojlim CBE_n$, whose n-point approximation/projection is the CBE_n. What is this measure?

Sommaire

Statement

Theorem (C-Najnudel, arXiv:1904.00578)

For $\gamma = \sqrt{\frac{2}{\beta}} \leq 1$, *we have equality between*

- *the measure* μ^β *whose Verblunsky coefficients are the* $(\alpha_n; n \in \mathbb{N})$ *from* $C\beta E$.
- *Kahane's* GMC^γ, *renormalized into a probability measure.*

$$\mu^\beta(d\theta) = \frac{1}{GMC^\gamma(\partial\mathbb{D})} GMC^\gamma(d\theta) \ .$$

⇝ One can theoretically sample the GMC^γ. Then upon considering the best approximating measure on n points, the quadrature points are nothing but the RMT ensembles CBE_n.

⇝ One could write a projective limit:

$$\boxed{GMC^\gamma := \varprojlim_n C\beta E_n \ .}$$

Ideas of proof

Finitely many Verblunsky coeff

- RMT regularization of Gaussians

Gaussian fields

$$\begin{tikzcd}
\mu^{\beta}_{n,r}(d\theta) \propto \frac{1}{|\Phi_n^*(re^{i\theta})|^2} d\theta \arrow[r, "n \to \infty"] \arrow[d, "r \to 1"'] & \mu^{\beta}_r(d\theta) = \frac{e^{\omega_r(\theta)}}{C_0} GMC_r^{\gamma}(d\theta) \arrow[d, "r \to 1"] \\
\mu^{\beta}_n(d\theta) = \frac{\prod_{j=0}^{n-1}(1-|\alpha_j|^2)}{|\Phi_n^*(e^{i\theta})|^2} d\theta \arrow[r, "n \to \infty"'] & \mu^{\beta} \Big\backslash \frac{K_\beta}{C_0} GMC^{\gamma=\sqrt{\frac{2}{\beta}}}(d\theta)
\end{tikzcd}$$

Poisson kernel regularization

On the circle

Bernstein-Szegö approx.

Difficult points:

- Filtrations by Gaussians and Verblunsky coefficients ($\mathbb{F}$) have bad overlap. Top $n \to \infty$ limit is built to be a martingale limit, with parameter r.
- Doob decomposition w.r.t $\mathbb{F}$: $\omega_r = \sum_{k=0}^{\infty}(1-r^2)\frac{Y_k^r}{k+1}$. $\left(Y^r_{t/(1-r^2)}\ ;\ t \geq 0\right)$ has has a non-trivial limiting SDE as $r \to 1$. SDE is ill-behaved at time 0.
- SDE = Crossing mechanism, which quickly forgets initial Verblunsky coefficients, thanks to non-trivial entrance law. Crucial for $r \to 1$ limit.

Consequences

- $(C\beta E_n \ ; \ \beta \geq 2, n \in \mathbb{N}^*)$ can all be coupled from $(GMC^\gamma \ ; \ 0 \leq \gamma < 1)$.
- Conjecture of B. Virag (ICM 2014, Seoul): GMC^γ and $\varprojlim C\beta E_n$ should have the same multifractal spectrum.
- (Fyodoroff-Bouchaud Conjecture) Another proof of G. Rémy's identity:

$$GMC^\gamma(\partial\mathbb{D}) = K_\beta \prod_{j=0}^{\infty} \left(1 - |\alpha_j|^2\right)^{-1} e^{-\frac{2}{\beta(j+1)}} \stackrel{\mathcal{L}}{=} K'_\beta \ \mathbf{e}^{-\frac{2}{\beta}} \ .$$

- (Beyond Fyodoroff-Bouchaud) One can also describe all moments

$$c_k = \frac{1}{GMC^\gamma(\partial\mathbb{D})} \int_0^{2\pi} e^{ik\theta} GMC^\gamma(d\theta) \ .$$

via universal expressions in terms of the Verblunsky coefficients. For example:

$$\left\{ \begin{array}{lcc} c_1 & = & \alpha_0 \ , \\ c_2 & = & \alpha_0^2 + \alpha_1(1 - |\alpha_0|^2) \ , \\ c_3 & = & (\alpha_0 - \alpha_1\overline{\alpha_0})[\alpha_0^2 + \alpha_1(1 - |\alpha_0|^2)] \\ & & +\alpha_1\alpha_0 + \alpha_2(1 - |\alpha_0|^2)(1 - |\alpha_1|^2). \end{array} \right.$$

Open questions

Our result brings forth other questions:

- What happens in the supercritical phase $\beta < 2 \Leftrightarrow \gamma > 1$? Our intuition suggests no freezing. Conjectural answer: the KPZ dual measure.
- At critical $\beta = 2$, relate back our result to the Fyodorov-Hiary-Keating conjecture on the maximum of the characteristic polynomial X_n.
- $C\beta E$ has an intimate relationship to algebraic structures: Jack polynomials, the integrable Calogero-Sutherland system ($\sim$ Wick-rotated circular Dyson dynamics), Vertex algebras... Bridge between the Liouville CFT/GMC and the algebra?
- Linking dynamics in RMT and dynamics in conformal growth: papers of Cardy on multiple SLEs and Calogero, hints in work of Norris-Turner-Silvestri with Loewner-(Kufarev) Evolutions...
- Question to physicists: Role of $\beta_{critical} = \beta = 2$? This is where the geometry and rep. theory of unitary groups lies.

Acknowledgements

Thank you for your attention!