Dynamical Loop Equation

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MSRI Connections and Introductory Workshop
Universality and Integrability
in Random Matrix Theory and Interacting Particle Systems

joint work with Vadim Gorin

Content

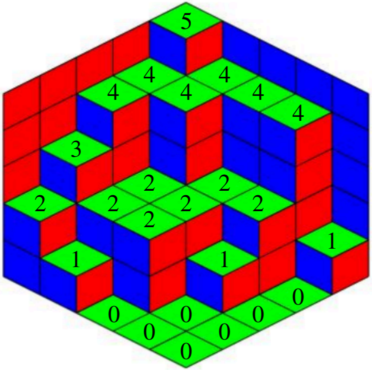
1. Motivating Examples
2. Loop Equations and Dynamical Loop Equations
3. Application of Dynamical Loop Equations

Part 1: Motivating Examples

Random Lozenge Tilling

Random Lozenge Tiling

uniform probability measure on the set of Lozenge tilings of a given domain.



three types of lozenges

Random Lozenge Tilling

Random Lozenge Tiling

uniform probability measure on the set of Lozenge tilings of a given domain.

One feature of the limit shapes of random lozenge tilings is the presence of frozen regions which contain only one type of lozenges and the liquid regions which contain all three types of lozenges. The boundary curves separate liquid region and frozen region are referred to as "arctic curves".

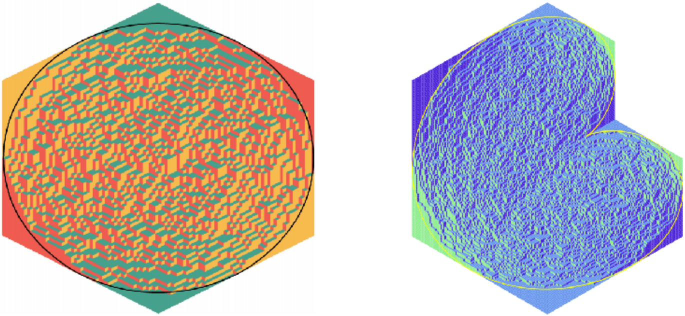


Photo credit: Gorin, Kenyon-Okounkov

Random Lozenge Tilling

Theorem (Cohn-Kenyon-Propp (2001))

Let $\mathfrak{R}$ *be a domain in the plane, and* R *is obtained from* $\mathfrak{R}$ *after rescaling by a factor* n*. The height function* H(x, t) *satisfies*

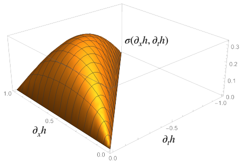
$$\#\left\{\frac{H(nx,nt)}{n}\approx h(x,t)\right\}=\exp\left(n^2\left(\int\int_{\mathfrak{R}}\sigma(\nabla h)dxdt+o(1)\right)\right).$$

Especially

$$\frac{H(nx,nt)}{n}\to h^*(x,t),\quad (x,t)\in\mathfrak{R},$$

where h^* *is the unique maximizer of the variational problem*

$$h^*=\text{argmax}_h\iint_{\mathfrak{P}}\sigma(\nabla h)dxdt.$$



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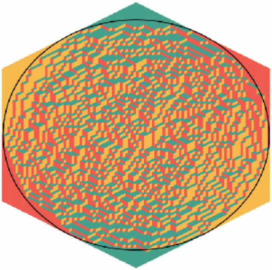
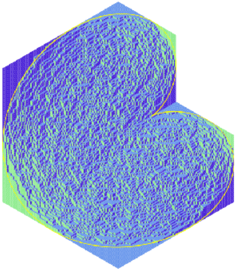
 

Photo credit: Gorin, Kenyon-Okounkov

Conjecture (Kenyon (2004), Kenyon-Okounkov (2005))

In the liquid region, the fluctuation of the height function converges to the Gaussian Free Field with complex structure given by the complex slope and zero boundary condition

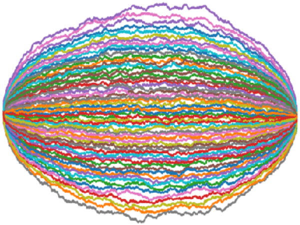
$$\sqrt{\pi}(H(nx, nt) - \mathbb{E}[H(nx, nt)]) \to GFF.$$

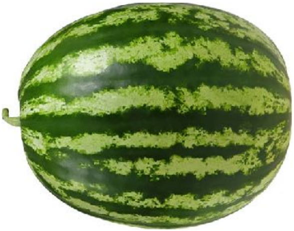
Gaussian Free Field fluctuation for domains without frozen regions (Kenyon, Berestycki, Laslier, Ray, Russkikh...); trapezoidal domains (Petrov); non-simply connected domains (Bufetov-Gorin).

Dyson's Nonintersecting Brownian Motion

Brownian Watermelon

n Brownian motions starting from the origin at time $t = 0$, conditioned to return to the origin at time $t = 1$ and stay nonintersecting.





Dyson's Brownian motion

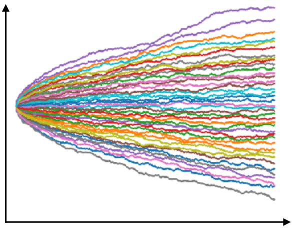
Matrix Valued Brownian motion:

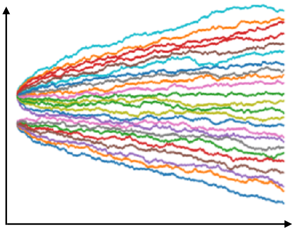
$$\mathrm{d}A_t = \frac{1}{\sqrt{n}}\mathrm{d}B_t, \quad A_0 = A$$

where B_t have complex Brownian motion entries ($\beta = 2$), real Brownian motion entries ($\beta = 1$). Integrating out the eigenvectors, the dynamic for the eigenvalues is called Dyson's Brownian motion

$$\mathrm{d}\lambda_i(t) = \sqrt{\frac{2}{\beta n}}\mathrm{d}W_i(t) + \frac{1}{n}\sum_{j:j\neq i}\frac{\mathrm{d}t}{\lambda_i(t)-\lambda_j(t)}, \quad 1 \leqslant i \leqslant n.$$

If $\beta = 2$, this is also the Brownian motions starting from A conditioned to be nonintersecting.

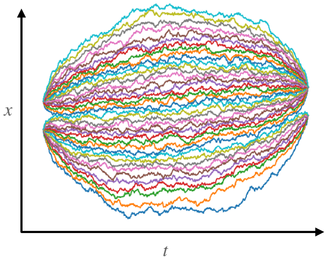


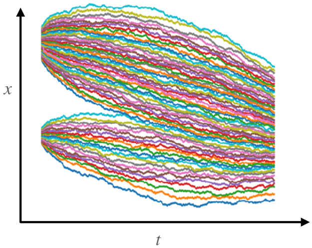


Dyson's Nonintersecting Brownian Motion

Nonintersecting Brownian Motion

We consider n particle non-intersecting Brownian motions $(x_1(t) \leqslant x_2(t) \leqslant \cdots x_n(t))$ with initial configuration $(a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n)$ at time $t = 0$ and terminal configuration $(b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n)$ at time $t = 1$.





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Theorem (Guionnet-Zeitouni (2002))

For n particle non-intersecting Brownian motion with boundary data $\mu_{A_n} = (1/n)\sum_{i=1}^n \delta_{a_i}$ *and* $\mu_{B_n} = (1/n)\sum_{i=1}^n \delta_{b_i}$ *converges weakly*

$$\lim_{n\to\infty} \mu_{A_n} = \mu_A, \quad \lim_{n\to\infty} \mu_{B_n} = \mu_B.$$

The empirical measure $\mu_t = (1/n)\sum_{i=1}^n \delta_{x_i(t)}$ *converging weakly to* $\rho_t^*(x)$*, which is the minimizer of the variational problem:*

$$\inf \int_0^1 \int u_t(x)^2 \rho_t(x) + \frac{\pi^2}{12}\rho_t(x)^3 \mathrm{d}x\mathrm{d}t,$$

where $\inf$ *is taken over all the pairs* (u_t, ρ_t) *with* $\partial_t \rho_t + \partial_x(\rho_t u_t) = 0$ *in the sense of distributions, and the initial and terminal data for* ρ_t *are given by*

$$\lim_{t\to 0} \rho_t(x)\mathrm{d}x = \mu_A, \quad \lim_{t\to 1} \rho_t(x)\mathrm{d}x = \mu_B.$$

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Gaussian Free Field fluctuation for Dyson's Brownian motion (Spohn 1998, Israelsson 2001, Bender 2008); for Brownian watermelon (Breuer-Duits 2013).

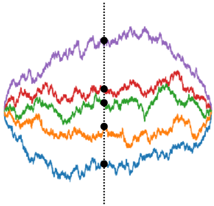
Part 2: Loop Equations and Dynamical Loop Equations

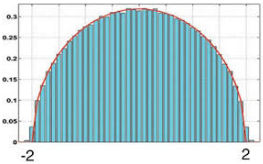
Loop Equation

The β-ensemble is an n-particle stochastic system

$$p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^\beta e^{-n \sum_{i=1}^n V(\lambda_i)}$$

For classical values of $\beta = 1, 2, 4$ and $V(x) = \beta x^2/4$, $\mathbb{P}_n$ corresponds to the joint law of the eigenvalues of the Gaussian Orthogonal (with real entries), Gaussian Unitary (with complex entries) or Gaussian Symplectic (with quaternion entries) Ensembles.





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We denote the empirical particle density and its Stieltjes transform as

$$d\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}, \quad m_n(z) = \int \frac{d\mu_n(\lambda)}{z - \lambda} = \frac{1}{n} \sum_{i=1}^n \frac{1}{z - \lambda_i}, \quad z \in \mathbb{C}_+.$$

The Stieltjes transform encodes all the information of the empirical particle density, we can recover the empirical particle density as $\mu_n = -\lim_{\eta \to 0+} \mathrm{Im}[m_n(x + \mathrm{i}\eta)]/\pi$.

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Under the assumption that $V(\lambda)$ grows sufficiently fast, the empirical measure μ_n converges almost surely to a deterministic measure μ_V, characterized by certain variational problem. The loop (Schwinger-Dyson) equation was use to study matrix models in physics literature (Migdal, Ambjorn-Makeenko), and used to study the fluctuations of $\mu_n - \mu_V$ by Johansson (1998).

Loop Equation

β-ensembles with analytic potential V

$$p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^\beta e^{-n \sum_{i=1}^n V(\lambda_i)}$$

Loop (Schwinger-Dyson) Equation

For any $p \geqslant 0$ and $z, z_1, z_2, \cdots, z_p \in \mathbb{C}$, the following holds

$$\mathbb{E}\left[\left(m_n(z)^2 - \frac{2}{n\beta}\sum_{i=1}^n \frac{V'(\lambda_i)}{z-\lambda_i} + \frac{\beta-2}{n\beta}\partial_z m_n(z)\right)\prod_{j=1}^p m_n(z_j)\right]$$

$$= -\frac{2}{n\beta}\mathbb{E}\left[\sum_{j=1}^p \partial_{z_j}\left(\frac{m_n(z)-m_n(z_j)}{z-z_j}\right)\prod_{k:k\neq j} m_n(z_k)\right].$$

Integration by part:

$$\sum_{i=1}^n \int \partial_{\lambda_i}\left(\frac{1}{z-\lambda_i}\right)\prod_{j=1}^p m_n(z_j) p_n(\lambda_1, \cdots, \lambda_n) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n$$

$$+\sum_{i=1}^n \int \frac{1}{z-\lambda_i}\partial_{\lambda_i}\left(\prod_{j=1}^p m_n(z_j) p_n(\lambda_1, \cdots, \lambda_n)\right)\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n = 0.$$

Loop Equation

By taking $p = 0$, the first order loop equation

$$\mathbb{E}_n\left[m_n^2 - \frac{2V'(z)}{\beta} m_n + \frac{2}{n\beta}\sum_{i=1}^{n}\frac{V'(z) - V'(\lambda_i)}{z - \lambda_i}\right]$$

$$= \mathbb{E}_n[m_n^2] - \frac{2V'(z)}{\beta}\mathbb{E}_n[m_n] + \text{analytic} = \frac{1}{n}(\cdots).$$

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$$= \mathbb{E}_n[m_n^2] - \frac{2V'(z)}{\beta}\mathbb{E}_n[m_n] + \text{analytic} = \frac{1}{n}(\cdots).$$

The limiting equation is

$$m_V^2 - \frac{2V'(z)}{\beta}m_V + Q = 0.$$

For the noncritical one-cut setting, it can be solved as

$$m_V - \frac{V'(z)}{\beta} = \sqrt{(V'(z)/\beta)^2 - Q} = S(z)\sqrt{(z-a)(z-b)},$$

where μ_V is supported on $[a, b]$ and $S(z)$ is analytic and nonzero in a neighborhood of $[a, b]$. By taking difference and divide by $S(z)$

$$(\mathbb{E}_n[m_n] - m_V)\left(m_V - \frac{V'(z)}{\beta}\right) + \text{analytic} = \frac{1}{n}(\cdots),$$
$$(\mathbb{E}_n[m_n] - m_V)\sqrt{(z-a)(z-b)} + \frac{\text{analytic}}{S(z)} = \frac{1}{n}\frac{(\cdots)}{S(z)},$$

This determines the mean shift $\mathbb{E}_n[m_n] - m_V$, via a contour integral.

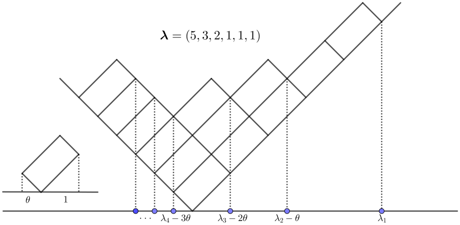
Loop Equation

- Macroscopic: Loop equations can be used to get the also the variance of $m_n(z) - m_V(z)$, and joint moments of $m_n(z_j) - m_V(z_j)$ for $1 \leqslant j \leqslant p$. For V sufficiently regular (not necessarily analytic), the β-ensemble can also be studied using the loop equation, taking an operator approach (Bekerman-Figalli-Guionnet 2013).
- Loop equation leads to Mesoscopic and Microscopic fluctuations.
- Loop equations are i) infinite hierarchy of equations, ii) relates $(k+1)$-point correlation functions with k-point correlation functions, iii) certain observable is analytic.

Discrete Loop Equation

Discrete β-ensembles as introduced by Borodin, Gorin and Guionnet, are probability measures on n-tuples of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, which we call Young diagrams (in the case $\lambda_n \geqslant 0$). They can be identified with a particle configuration as

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_n) = (\lambda_1, \lambda_2 - \theta, \lambda_3 - 2\theta, \cdots, \lambda_n - (n-1)\theta) \in W_\theta^n$$



Discrete β-ensemble with $\beta = 2\theta$ (Borodin-Gorin-Guionnet 2015)

$$\mathbb{P}_n(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \leqslant i < j \leqslant n} \frac{\Gamma(x_j - x_i + 1)\Gamma(x_j - x_i + \theta)}{\Gamma(x_j - x_i)\Gamma(x_j - x_i + 1 - \theta)} \prod_{i=1}^{n} w_n(x_i),$$

where $x_1 > x_2 > \cdots > x_n$ with $x_i \in \mathbb{Z} - (i-1)\theta$.

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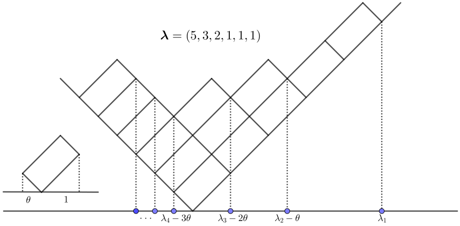

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where $x_1 > x_2 > \cdots > x_n$ with $x_i \in \mathbb{Z} - (i-1)\theta$.

$\beta = 2$ case gives the discrete orthogonal polynomial ensemble

$$\mathbb{P}_n(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \leqslant i < j \leqslant n} |x_i - x_j|^2 \prod_{i=1}^n w(x_i), \quad x_1, x_2, \cdots, x_n \in \mathbb{Z}.$$

Discrete Loop Equation

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where $x_1 > x_2 > \cdots > x_n$ with $x_i \in \mathbb{Z} - (i-1)\theta$.

Theorem (Borodin-Gorin-Guionnet (2015), Nekrasov (2012))

Consider the probability distribution (1)*, and assume that*

$$\frac{w(x)}{w(x-1)} = \frac{\phi_+(x)}{\phi_-(x)},$$

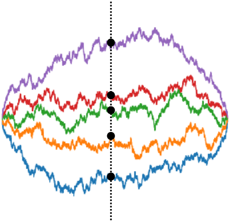
and $\phi_\pm(z)$ are analytic. Then

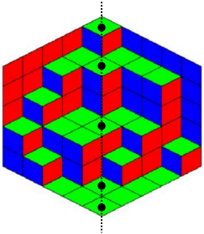
$$\phi_-(z)\mathbb{E}\left[\prod_{i=1}^{n}\left(1 - \frac{\theta}{z - x_i}\right)\right] + \phi_+(z)\mathbb{E}\left[\prod_{i=1}^{n}\left(1 + \frac{\theta}{z - x_i - 1}\right)\right]$$

is analytic.

Generalization to q-weighted measures and multi-level loop equations by Dimitrov-Knizel.

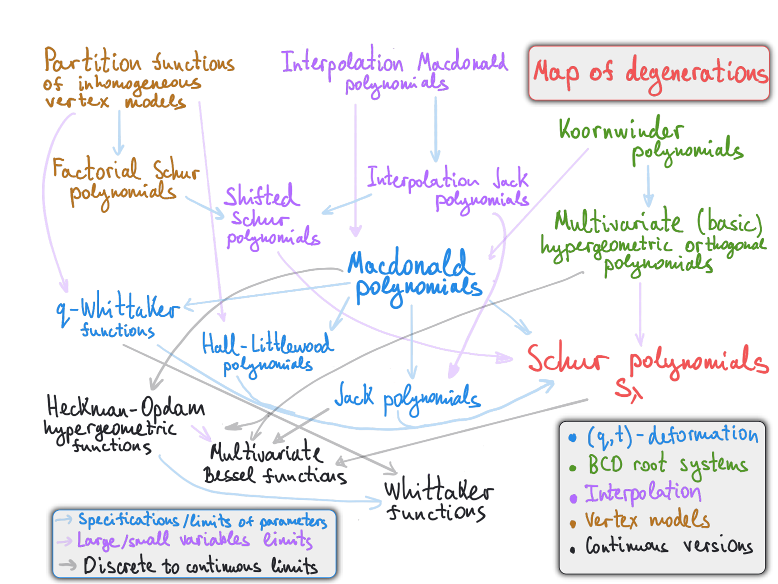
One slice of the Brownian watermelon and random lozenge tiling of hexagon is given by a (discrete) β-ensemble. They can be analyzed by loop equations, and have Gaussian fluctuation.





Need a loop equation for $1+1$ dimension stochastic systems!

Zoo of Symmetric Polynomial (by V. Gorin)



Part 3: Application of Dynamical Loop Equation

For the nonintersecting Bernoulli random walk:

$$\mathbb{P}(\boldsymbol{x}(t+1) = \boldsymbol{x}+\boldsymbol{e} | \boldsymbol{x}(t) = \boldsymbol{x}) = \frac{1}{2^n} \prod_{1 \leqslant i < j \leqslant n} \frac{x_i + e_i - x_j - e_j}{x_i - x_j} = \frac{1}{2^n} \frac{V(\boldsymbol{x}+\boldsymbol{e})}{V(\boldsymbol{x})},$$

where $\boldsymbol{e} = (e_1, e_2, \cdots, e_n) \in \{0,1\}^n$. Fix $\varepsilon = \mathrm{N}/n \ll 1$, and denote the normalized empirical particle density and its Stieltjes transform as

$$\mu_t = \frac{1}{n} \sum_{i=1}^{n} \delta_{\varepsilon x_i(t)}, \quad m_t(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{z - \varepsilon x_i(t)},$$

For the nonintersecting Bernoulli random walk:

$$\mathbb{P}(\boldsymbol{x}(t+1)=\boldsymbol{x}+\boldsymbol{e}|\boldsymbol{x}(t)=\boldsymbol{x})=\frac{1}{2^n}\prod_{1\leqslant i<j\leqslant n}\frac{x_i+e_i-x_j-e_j}{x_i-x_j}=\frac{1}{2^n}\frac{V(\boldsymbol{x}+\boldsymbol{e})}{V(\boldsymbol{x})},$$

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Dynamical loop equation gives

$$\frac{m_{t+1}(z)-m_t(z)}{\varepsilon}=\varepsilon^{1/2}\Delta\mathcal{M}_t(z)+\partial_z\left(e^{m_t(z)}+1\right)+\varepsilon\mathcal{E}_t(z)+\mathrm{O}(\varepsilon^2), \tag{2}$$

where $\Delta\mathcal{M}_t(z)$ is a martingale difference term.

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Dynamical loop equation gives

$$\frac{m_{t+1}(z) - m_t(z)}{\varepsilon} = \varepsilon^{1/2} \Delta \mathcal{M}_t(z) + \partial_z \left(e^{m_t(z)} + 1 \right) + \varepsilon \mathcal{E}_t(z) + \mathrm{O}(\varepsilon^2), \tag{2}$$

where $\Delta\mathcal{M}_t(z)$ is a martingale difference term.

The leading order term converges to the complex Burgers equation. By a tightness argument, we can show that the martingale

$$\mathcal{M}_t(z) = \sum_{s\leqslant t} \varepsilon^{1/2} \Delta \mathcal{M}_s(z),$$

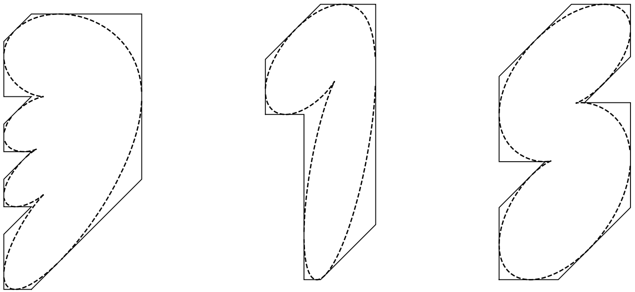
converges to a Gaussian process, and (2) converges to a stochastic differential equation. The hard part is to identify it with the Gaussian Free Field.

Application: Lozenge Tiling

Theorem (H. 2020)

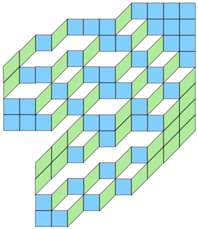
Fix a polygonal domain $\mathfrak{R}$, the arctic boundary does not have tacnode, and the cusp points to the right (tangent has slope in $(-\pi, \pi)$). Let $\mathrm{H}(\mathrm{x}, \mathrm{t})$ be the height function of random lozenge tilings of the domain R obtained from $\mathfrak{R}$ by rescaling a factor n, then as n goes to infinity, its fluctuation converges to a Gaussian Free Field on the liquid region with zero boundary condition,

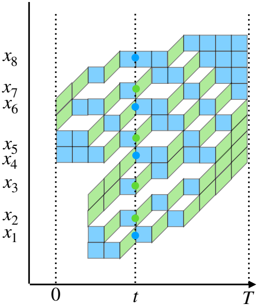
$$\sqrt{\pi}\,(\mathrm{H}(nx, nt) - \mathbb{E}[\mathrm{H}(nx, nt)]) \to GFF.$$



Application: Lozenge Tiling

We identify lozenge tilings with nonintersecting Bernoulli walks,





Nonintersecting means that at time t the particle configuration $x_1(t) < x_2(t) < \cdots < x_{n(t)}(t) \in \mathbb{Z}$, where $n(t)$ is the number of particles at time t and it is uniquely determined by the domain $\mathfrak{R}$.

Application: Lozenge Tiling

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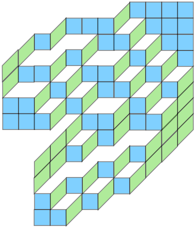

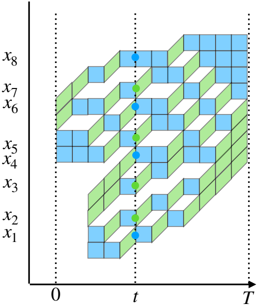

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We denote $N_t(x_1, x_2, \cdots, x_{n(t)})$ the number of nonintersecting Bernoulli walks staying in $\mathfrak{R}$ and starting from particle configuration $\boldsymbol{x} = (x_1, x_2, \cdots, x_{n(t)})$ at time t. We simply set $N_t(x_1, x_2, \cdots, x_{n(t)}) = 0$ for particle configurations not in $\mathfrak{R}$ or containing overlap particles. Then it satisfies the following discrete heat equation

$$N_t(\boldsymbol{x}) = \sum_{\boldsymbol{e}=(e_1, e_2, \cdots, e_{n(t)}) \in \{0,1\}^{n(t)}} N_{t+1}(\boldsymbol{x} + \boldsymbol{e}).$$

Application: Lozenge Tiling

We can solve the discrete heat equation

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by an ansatz

$$\begin{aligned} \frac{N_{t+1}(\boldsymbol{x}+\boldsymbol{e})}{N_t(\boldsymbol{x})} &= \frac{V(\boldsymbol{x}+\boldsymbol{e})}{V(\boldsymbol{x})} \frac{N_{t+1}(\boldsymbol{x}+\boldsymbol{e})/V(\boldsymbol{x}+\boldsymbol{e})}{N_t(\boldsymbol{x})/V(\boldsymbol{x})} \\ &= \frac{1}{Z_t(\boldsymbol{x})} \frac{V(\boldsymbol{x}+\boldsymbol{e})}{V(\boldsymbol{x})} \prod_{i=1}^{n(t)} (g(x_i;\boldsymbol{x},t))^{e_i} \frac{E_{t+1}(\boldsymbol{x}+\boldsymbol{e})}{E_t(\boldsymbol{x})}, \end{aligned} \tag{3}$$

where $g(z;\boldsymbol{x},t)$ is constructed using the complex Burgers equation.

Application: Lozenge Tiling

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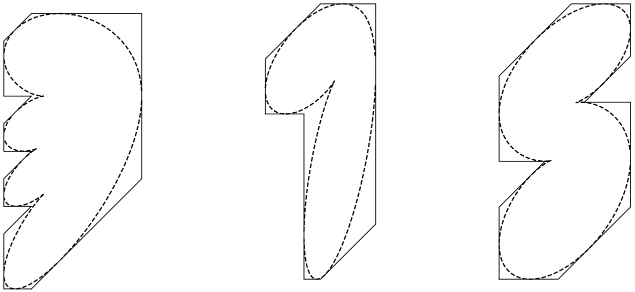
where $g(z;\boldsymbol{x},t)$ is constructed using the complex Burgers equation.

$g(z;\boldsymbol{x},t)$ is analytic except at cusps which point to the left, where it has a branch point. If the all the cusps point to the right, we can solve (3) and $E_{t+1/n}(\boldsymbol{x})/E_t(\boldsymbol{x}) \approx 1$ is negligible.



Application: Lozenge Tiling

We can use the partition function $N_t(\boldsymbol{x})$

$$N_t(\boldsymbol{x}) = \sum_{\boldsymbol{e}=(e_1,e_2,\cdots,e_{n(t)})\in\{0,1\}^{n(t)}} N_{t+1}(\boldsymbol{x}+\boldsymbol{e}).$$

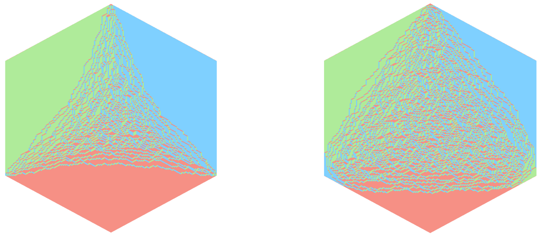
to define a nonintersecting Bernoulli random walk with transition probability given by

$$\begin{aligned}\mathbb{P}(\boldsymbol{x}(t+1) = \boldsymbol{x}+\boldsymbol{e} | \boldsymbol{x}(t) = \boldsymbol{x}) &= \frac{N_{t+1}(\boldsymbol{x}+\boldsymbol{e})}{N_t(\boldsymbol{x})} \\ &\approx \frac{1}{Z_t(\boldsymbol{x})} \frac{V(\boldsymbol{x}+\boldsymbol{e})}{V(\boldsymbol{x})} \prod_{i=1}^{n(t)} (g_t(x_i; \boldsymbol{x}, t))^{e_i},\end{aligned}$$

for any $t \in [0, T] \cap \mathbb{Z}$, and $\boldsymbol{e} = (e_1, e_2, \cdots, e_{n(t)}) \in \{0, 1\}^{n(t)}$.

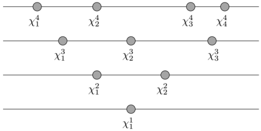
Application

Weighted Lozenge tiling



Corner Process

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$



Summary

- We study $1+1$ dimensional stochastic systems related to symmetric polynomials:

$$\mathbb{P}(\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{e}|\boldsymbol{x})=\frac{1}{Z(\boldsymbol{x})}\prod_{1\leqslant i<j\leqslant n}\frac{b(x_i+\theta e_i)-b(x_j+\theta e_j)}{b(x_i)-b(x_j)}\prod_{i=1}^{n}\phi^+(x_i)^{e_i}\phi^-(x_i)^{1-e_i},$$

- We introduced a dynamical version of loop equations: the first order loop equation is given by

$$\mathbb{E}\left[\phi^+(z)\prod_{j=1}^{n}\frac{b(z+\theta)-b(x_j+\theta e_j)}{b(z)-b(x_j)}+\phi^-(z)\prod_{j=1}^{n}\frac{b(z)-b(x_j+\theta e_j)}{b(z)-b(x_j)}\right].$$

  is analytic.

- The dynamical loop equations imply a decomposition of empirical particle measure as Gaussian fluctuation part and deterministic part. Stochastic differential equation can be identified with GFF.

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Thank you for listening!