Transfer matrix approach to random band matrices

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Local statistics, localization and delocalization

One of the key physical parameter of models is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length ℓ is comparable with the matrix size, and it is called localized otherwise.

- Localized eigenvectors: lack of transport (insulators), and Poisson local spectral statistics (typically strong disorder)
- Delocalization: diffusion (electric conductors), and GUE/GOE local statistics (typically weak disorder).

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory.

From the RMT point of view, the main objects of the local regime are k-point correlation functions R_k $(k = 1, 2, \ldots)$, which can be defined by the equalities:

$$\mathbb{E}\left\{\sum_{j_1\neq\ldots\neq j_k}\varphi_k(\lambda_{j_1}^{(N)},\ldots,\lambda_{j_k}^{(N)})\right\}$$
$$=\int_{\mathbb{R}^k}\varphi_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})R_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})d\lambda_1^{(N)}\ldots d\lambda_k^{(N)},$$

where $\varphi_k : \mathbb{R}^k \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments.

Universality conjecture in the bulk of the spectrum (hermitian case, deloc.eg.s.) (Wigner – Dyson):

$$(N\rho(E))^{-k}R_k\big(\{E + \xi_j/N\rho(E)\}\big) \quad \xrightarrow{N\to\infty} \quad \det\left\{\frac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^{k}.$$

- Wigner matrices, β-ensembles with $\beta = 1, 2$, sample covariance matrices, etc.: delocalization, GUE/GOE local spectral statistics
- Anderson model (Random Schrödinger operators):

$$H_{RS} = -\triangle + \lambda V,$$

where $\triangle$ is the discrete Laplacian in lattice box $\Lambda = [1, n]^d \cap \mathbb{Z}^d$, V is a random potential (i.e. a diagonal matrix with i.i.d. entries). In $d = 1$: narrow band matrix with i.i.d. diagonal

$$H_{RS} = \begin{pmatrix} \lambda V_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \lambda V_2 & 1 & 0 & \dots & 0 \\ 0 & 1 & \lambda V_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \lambda V_{n-1} & 1 \\ 0 & \dots & 0 & 0 & 1 & \lambda V_n \end{pmatrix}.$$

Localization, Poisson local spectral statistics

Random band matrices

Can be defined in any dimension, but we will speak about $\mathrm{d} = 1$.

Entries are independent (up to the symmetry) but not identically distributed.

$$\mathrm{H} = \{\mathrm{H}_{\mathrm{jk}}\}_{\mathrm{j,k}=1}^{\mathrm{N}}, \quad \mathrm{H} = \mathrm{H}^*, \quad \mathbb{E}\{\mathrm{H}_{\mathrm{jk}}\} = 0.$$

Variance is given by some function J (even, compact support or rapid decay)

$$\mathbb{E}\{|\mathrm{H}_{\mathrm{jk}}|^2\} = \mathrm{W}^{-1}\,\mathrm{J}\big(|\mathrm{j}-\mathrm{k}|/\mathrm{W}\big), \quad \sum_{\mathrm{k}=1}^{\mathrm{N}} \mathrm{J}_{\mathrm{jk}} = 1.$$

Main parameter: band width $\mathrm{W} \in [1; \mathrm{N}]$.

If both W, N goes to infinity, then the density of states is the Wigner semi-circle law (Bogachev, Molchanov, Pastur '91; Molchanov, Pastur, Khorunzhii '92)

1d case

$$H = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

$W = O(1)$ [$\sim$ random Schrödinger] $\longleftrightarrow$ $W = N$ [Wigner matrices]

Anderson transition in random band matrices

Varying W, we can see the transition between localization and delocalization

Conjecture (in the bulk of the spectrum):

$d = 1:$	$\ell \sim W^2$	$W \gg \sqrt{N}$	Delocalization, GUE statistics
		$W \ll \sqrt{N}$	Localization, Poisson statistics
$d = 2:$	$\ell \sim e^{W^2}$	$W \gg \sqrt{\log N}$	Delocalization, GUE statistics
		$W \ll \sqrt{\log N}$	Localization, Poisson statistics
$d \geq 3:$	$\ell \sim N$	$W \geq W_0$	Delocalization, GUE statistics

Anderson model: expected to have the same qualitative properties when $\lambda \sim W^{-1}$: $\ell \sim \lambda^{-2}$ for 1d (Fröhlich, Spencer; Aizenman, Molchanov), $\ell \sim \exp(\lambda^{-2})$ (conjectured)

Partial results for general RBM ($d = 1$):

- Schenker (2009): $\ell \leq W^8$ localization techniques; improved to W^7;
- Erdős, Yau, Yin (2011): $\ell \geq W$ – RM methods;
- Erdős, Knowles (2011): $\ell \gg W^{7/6}$ (in a weak sense);
- Erdős, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$ (in a weak sense, not uniform in N);
- Bourgade, Erdős, Yau, Yin (2016): gap universality for $W \sim N$;
- Bourgade, Yang, Yau, Yin (2018): $W \gg N^{3/4}$ (quantum unique ergodicity);

Another method, which allows to work with random operators with non-trivial spatial structures, is supersymmetry techniques (SUSY), which based on the representation of the determinant as an integral over the Grassmann (anticommuting) variables.

The method allows to obtain an integral representation for the main spectral characteristic (such as density of states, second correlation functions, or the average of an elements of the resolvent) as the averages of certain observables in some SUSY statistical mechanics models (so-called dual representation in terms of SUSY). This is basically an algebraic step, and usually can be done by the standard algebraic manipulations. The real mathematical challenge is a rigour analysis of the obtained integral representation.

In the context of RBM: Efetov; Fyodorov, Mirlin (early 90th).

"Generalised" correlation functions

$$\mathcal{R}_1(z_1, z_1') := \mathbb{E}\Big\{\frac{\det(H - z_1')}{\det(H - z_1)}\Big\};$$

$$\mathcal{R}_2(z_1, z_1'; z_2, z_2') := \mathbb{E}\Big\{\frac{\det(H - z_1')\det(H - z_2'))}{\det(H - z_1)\det(H - z_2))}\Big\}$$

We study these functions for $z_{1,2} = E + \xi_{1,2}/\rho(E)N$, $z'_{1,2} = E + \xi'_{1,2}/\rho(E)N$, $E \in (-2, 2)$.

Link with the spectral correlation functions:

$$\frac{d}{dz_1}\mathcal{R}_1\Big|_{z_1'=z_1} = \mathbb{E}\Big\{\mathrm{Tr}\,(H - z_1)^{-1}\Big\} = \sum \frac{1}{\lambda_j - z} = N\int \frac{\mathcal{N}_N(d\lambda)}{\lambda - z}$$

Correlation function of the characteristic polynomials:

$$\mathcal{R}_0(\lambda_1, \lambda_2) = \mathbb{E}\Big\{\det(H - \lambda_1)\det(H - \lambda_2)\Big\}, \quad \lambda_{1,2} = E \pm \xi/\rho(E)N$$

Integral representation for characteristic polynomials

- GUE case

$$\mathcal{R}_0(\lambda_1,\lambda_2)=C_N\int\limits_{\mathcal{H}_2}\exp\Big\{-\frac{N}{2}\mathrm{Tr}\,X^2\Big\}\det{}^N\big(X-i\Lambda/2\big)d\overline{X},$$

- RBM case

$$\mathcal{R}_0(\lambda_1,\lambda_2)=C_N\int\limits_{\mathcal{H}_2^N}\exp\Big\{-\frac{1}{2}\sum_{j,k}J_{jk}^{-1}\mathrm{Tr}\,X_jX_k\Big\}\prod_j\det\big(X_j-i\Lambda/2\big)d\overline{X},$$

where X, $\{X_j\}$ are hermitian 2×2 matrices, $\Lambda=\mathrm{diag}\{\lambda_1,\lambda_2\}$.

For the density of states or the second correlation function X_j will be super-matrices

$$X_{1,j}=\begin{pmatrix} a_j & \rho_j\\ \tau_j & b_j\end{pmatrix},\quad X_{2,j}=\begin{pmatrix} A_j & \bar\rho_j\\ \bar\tau_j & B_j\end{pmatrix}$$

with real variables a_j, b_j and Grassmann variables ρ_j, τ_j, or hermitian A_j, hyperbolic B_j and Grassmann 2×2 matrices $\bar\rho_j$, $\bar\tau_j$.

We consider the following two models:

- Random band matrices: specific covariance

$$J_{ij} = \left(-W^2\Delta + 1\right)^{-1}_{ij} \approx C_1 W^{-1} \exp\{-C_2|i-j|/W\}$$

- Block band matrices

Only 3 block diagonals are non zero.

$$H = \begin{pmatrix} A_1 & B_1 & 0 & 0 & 0 & \dots & 0 \\ B_1^* & A_2 & B_2 & 0 & 0 & \dots & 0 \\ 0 & B_2^* & A_3 & B_3 & 0 & \dots & 0 \\ . & . & B_3^* & . & . & . & . \\ . & . & . & . & . & A_{n-1} & B_{n-1} \\ 0 & . & . & . & 0 & B_{n-1}^* & A_n \end{pmatrix}$$

A_j – GUE-matrices with variance $(1-2\alpha)/W$, $\quad \alpha < \frac{1}{4}$; B_j - Ginibre matrices with variance α/W

transition here is expected at $W \sim n$ (n is a number of blocks)

Why we need a specific J?

SUSY formulas can be obtained in any dimension and for any J, although the specific $\mathrm{J} = \left(-\mathrm{W}^2\Delta + 1\right)^{-1}$ gives a nearest neighbor model. In particular, it becomes accessible for transfer matrix approach.

For the specific covariance $(-\mathrm{W}^2\triangle + 1)^{-1}$:

$$\mathcal{R}_0(\lambda_1, \lambda_2) = \mathrm{C_N} \int\limits_{\mathcal{H}_2^{\mathrm{N}}} \exp\Big\{ -\frac{\mathrm{W}^2}{2} \sum_{\mathrm{j}=2}^{\mathrm{N}} \mathrm{Tr}\,(\mathrm{X_j} - \mathrm{X_{j-1}})^2 \Big\} \times$$

$$\exp\Big\{ -\frac{1}{2} \sum_{\mathrm{j}=1}^{\mathrm{N}} \mathrm{Tr}\,\Big(\mathrm{X_j} + \frac{\mathrm{iE}\cdot\mathrm{I}}{2} + \frac{\mathrm{i}\hat{\xi}}{2\mathrm{N}\rho(\lambda_0)}\Big)^2 \Big\} \prod_{\mathrm{j}=1}^{\mathrm{N}} \det\,(\mathrm{X_j} - \mathrm{iE}\cdot\mathrm{I}/2)\mathrm{d}\overline{\mathrm{X}},$$

with $\hat{\xi} = \mathrm{diag}\{\xi, -\xi\}$

Rigorous SUSY results for Gaussian RBM of a certain types (specific band profile or block band structure):

- characteristic polynomials, all three regimes (delocalized/threshold/localized)
  - Shcherbina T., Shcherbina M. (2013, 2016, 2019): hermitian case;
  - Shcherbina T. (2015, 2020): real symmetric case;
- Shcherbina T. (2014): universality for $W \sim N$;
- Bao, Erdős (2015): delocaization for $W \gg N^{6/7}$ (specific form, but general distribution (sub-Gaussian));
- Shcherbina T., Shcherbina M. (2018): universality for sigma-model approximation for $W \gg N^{1/2}$;
- Shcherbina T., Shcherbina M. (2019): universality of the correlation functions (full model) $W \gg N^{1/2}$;

Universality for the block band matrices

Theorem [M. Shcherbina, TS, 2019]

In the dimension $d = 1$ the behavior of the second order correlation function of the Gaussian block band matrices, as $W \gg n$, in the bulk of the spectrum coincides with those for the GUE. More precisely, if $\Lambda = [1, n] \cap \mathbb{Z}$ and H_N, $N = Wn$ are block RBM with $J = 1/W + \alpha\Delta/W$, $\alpha < 1/4$, then for any $E \in (-2, 2)$

$$(N\rho(E))^{-2}\mathcal{R}_2\left(E + \frac{\xi_1}{\rho(E)\,N}, E + \frac{\xi_2}{\rho(E)\,N}\right) \longrightarrow 1 - \frac{\sin^2(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2},$$

in the limit $W, n \to \infty$, with $W \geq n \log^5 n$.

The techniques can be applied for localization case as well (in progress).

Characteristic polynomials for GUE/GOE:

Hermitian case:

$$F_{2k}\left(\Lambda_0+\hat{\xi}/N\rho(E)\right)=C_N\frac{\det\left\{\dfrac{\sin(\pi(\xi_i-\xi_{j+k}))}{\pi(\xi_i-\xi_{j+k})}\right\}_{i,j=1}^{2k}}{\triangle(\xi_1,\dots,\xi_k)\triangle(\xi_{k+1},\dots,\xi_{2k})}(1+o(1)),$$

Real symmetric case:

$$F_{2k}\left(\Lambda_0+\hat{\xi}/N\rho(E)\right)=C_N\frac{\mathrm{Pf}\left\{DS(\pi(\xi_i-\xi_j))\right\}_{i,j=1}^{2k}}{\triangle(\xi_1,\dots,\xi_{2k})}(1+o(1)),$$

where

$$DS(x)=-\frac{3}{x}\frac{d}{dx}\frac{\sin x}{x}=3\Big(\frac{\sin x}{x^3}-\frac{\cos x}{x^2}\Big),$$

$\triangle(\xi_1,\dots,\xi_k)$ is the Vandermonde determinant of $\xi_1,\dots,\xi_k$, and $\hat{\xi}=\mathrm{diag}\left\{\xi_1,\dots,\xi_{2k}\right\}$, $\quad\Lambda_0=E\cdot I.$

Brézin-Hikami'01 ($k=1$), Borodin-Strahov'06

Characteristic polynomials of RBM, Hermitian case:

Let $D_2 = F_2(E, E)$, $\bar{F}_2(E, \xi) = D_2^{-1} \cdot F_2\Big(E + \hat{\xi}/2N\rho(E)\Big)$.

$$\lim_{n\to\infty} \bar{F}_2(E, \xi) = \begin{cases} \dfrac{\sin \pi\xi}{\pi\xi}, & W \geq N^{1/2+\theta}; \\ (e^{-C_* t_* \Delta_U - i\pi\xi\nu} \cdot 1, 1), & N = C_* W^2 \\ 1, & 1 \ll W \leq \sqrt{\dfrac{N}{C \log N}}, \end{cases}$$

where $t_* = (2\pi\rho(E))^2$,

$$\Delta_U = -\frac{d}{dx} x(1-x) \frac{d}{dx}, \quad \nu(U) = 1 - 2x, \quad x = |U_{12}|^2.$$

- Delocalization part: TS, 2013 – saddle-point analysis;
- Localization part: M. Shcherbina, TS, 2016 – transfer matrix approach.
- Near the crossover: TS, 2019
- Real symmetric case: TS, 2020

Some heuristic

Recall the formula for the second correlation function of characteristic polynomials of RBM with $J = (-W^2\triangle + 1)^{-1}$

$$\mathcal{R}_0(\lambda_1, \lambda_2) = C_N \int\limits_{\mathcal{H}_2^N} \exp\Big\{ -\frac{W^2}{2}\sum_{j=2}^{N} \mathrm{Tr}\,(X_j - X_{j-1})^2 \Big\} \times$$

$$\exp\Big\{ -\frac{1}{2}\sum_{j=1}^{N} \mathrm{Tr}\,\Big(X_j + \frac{iE \cdot I}{2} + \frac{i\hat{\xi}}{2N\rho(\lambda_0)}\Big)^2 \Big\} \prod_{j=1}^{N} \det\big(X_j - iE \cdot I/2\big) d\overline{X},$$

with $\hat{\xi} = \mathrm{diag}\{\xi, -\xi\}$

Now do the change of variables $X_j = U_j^* A_j U_j$, where U_j is a 2×2 unitary matrix and $A_j = \mathrm{diag}\{a_j, b_j\}$, and integrate out a_j, b_j (i.e. put them to be equal to their saddle-point values $a_\pm = \pm\pi\rho(E)$, so write the sigma-model approximation). Then if we use a standard parametrization of $U_j \in U(2)$ as a vector on the two-dimensional sphere, we obtain a classical Heisenberg model:

$$\int \exp\left\{\pi^2\rho(\lambda_0)^2 W^2 \sum_{j=2}^{N}(S_j S_{j-1} - 1) + \frac{i\pi\xi}{2N}\sum_{j=1}^{N} S_j\sigma_3\right\}\prod_{j=1}^{N} dS_j$$

$$\longrightarrow \int e^{i\pi\xi S_0\sigma_3/2} dS_0 = \frac{\sin(\pi\xi)}{\pi\xi}, \quad W^2 \gg N,$$

where $S_j \in \mathbb{S}^2$ corresponds to $U_j^* L U_j$, and $\sigma_3 = (0, 0, 1)$.

Transfer matrix approach

The idea of the transfer operator approach is very simple and natural. Let $\mathcal{K}(X, Y)$ be the compact integral operator in some L^2 space. Then

$$\int g(X_1)\mathcal{K}(X_1, X_2)\ldots\mathcal{K}(X_{N-1}, X_n)f(X_N)\prod d\mu(X_i) = (\mathcal{K}^{N-1}f, \bar{g})$$

$$= \sum_{j=0}^{\infty} \lambda_j^{N-1}(\mathcal{K})c_j, \quad \text{with} \quad c_j = (f, \psi_j)(g, \tilde{\psi}_j).$$

Here $|\lambda_0| \geq |\lambda_1| \geq \ldots$ are the eigenvalues of $\mathcal{K}$, ψ_j are corresponding eigenvectors, and $\tilde{\psi}_j$ are the eigenvectors of $\mathcal{K}^*$. In our cases $\lambda_0 = 1$. If we can show $|\lambda_1| = 1 - C/W^2$, then in the regime $N \gg W^2$ we have $\lambda_1^N \to 0$, so only λ_0 gives the contribution (which corresponds to localization).

Transfer matrix approach for characteristic polynomials:

$$\mathcal{R}_0\Big(\mathrm{E}\cdot\mathrm{I}+\frac{\widehat{\xi}}{\mathrm{N}\rho(\mathrm{E})}\Big)=-\mathrm{W}^{-4\mathrm{N}}\mathrm{det}^{-2}\mathrm{J}\cdot(\mathrm{K}_\xi^{\mathrm{N}-1}\mathcal{F},\overline{\mathcal{F}}),$$

$$\mathcal{K}_\xi(\mathrm{X},\mathrm{Y})=\frac{\mathrm{W}^4}{2\pi^2}\,\mathcal{F}_\xi(\mathrm{X})\,\exp\Big\{-\frac{\mathrm{W}^2}{2}\mathrm{Tr}\,(\mathrm{X}-\mathrm{Y})^2\Big\}\,\mathcal{F}_\xi(\mathrm{Y}),$$

where $\mathcal{F}_\xi(\mathrm{X})$ is the operator of multiplication by

$$\mathcal{F}_\xi(\mathrm{X})=\mathcal{F}(\mathrm{X})\cdot\exp\Big\{-\frac{\mathrm{i}}{2\mathrm{n}\rho(\mathrm{E})}\,\mathrm{Tr}\,\mathrm{X}\hat{\xi}\Big\}$$

with

$$\mathcal{F}(\mathrm{X})=\exp\Big\{-\frac{1}{4}\,\mathrm{Tr}\,\Big(\mathrm{X}+\frac{\mathrm{i}\Lambda_0}{2}\Big)^2+\frac{1}{2}\,\mathrm{Tr}\,\log\big(\mathrm{X}-\mathrm{i}\Lambda_0/2\big)-\mathrm{C}_+\Big\}$$

and some specific C_+

Saddle-points: $\mathrm{X_j}=\pi\rho(\mathrm{E})\cdot\mathrm{U_j^*LU_j}$, $\mathrm{X_j}=\pm\pi\rho(\mathrm{E})\cdot\mathrm{I}_2$

The main difficulties:

1. the transfer operator is not self-adjoint, and thus the perturbation theory is not easily applied in a rigorous way;
2. the transfer operator has a complicated structure including a part that acts on unitary and hyperbolic groups, hence we need to work with corresponding special functions;
3. the kernel of the transfer operator for the density of states and for the second correlation function contains not only only the complex, but also some Grassmann variables. Therefore, for the density of states $\mathcal{K}_1$ is a 2×2 matrix kernel, containing the Jordan cell, and for the second correlation function $\mathcal{K}_2$ is a $2^8 \times 2^8$ matrix kernel, containing 4×4 Jordan cell in the main block.
   Using the symmetry of the problem, $\mathcal{K}_2$ could be replaced by 70×70 matrix kernel, but it is still very complicated.

Step by step project

- characteristic polynomials (continuous symmetry, but no Grassmann variables): we can prove the transition at $W \sim \sqrt{n}$, and can study the behavior near the threshold
- density of states (2 Grassmann variables, but no continuous symmetry): we can show the local semicircle for the average density of states
- σ-model approximation for second correlation function (4 Grassmann variables & continuous symmetry): we have done the delocalization side
- second correlation function (8 Grassmann variables & continuous symmetry): we have done the delocalization side

Sketch of the proof

It appears that it is simpler to work with the resolvent

$$(K_\xi^{n-1}f, g) = -\frac{1}{2\pi i}\oint_{\mathcal{L}} z^{n-1}(G_\xi(z)f, g)dz, \quad G_\xi(z) = (K_\xi - z)^{-1},$$

where $\mathcal{L}$ is any closed contour which enclosed all eigenvalues of K_ξ.
We say that the operator $\mathcal{A}_{n,W}$ is equivalent to $\mathcal{B}_{n,W}$ ($\mathcal{A}_{n,W} \sim \mathcal{B}_{n,W}$) on some contour $\mathcal{L}$ if

$$\int_{\mathcal{L}} z^{n-1}((\mathcal{A}_{n,W} - z)^{-1}f, \bar{g})dz = \int_{\mathcal{L}} z^{n-1}((\mathcal{B}_{n,W} - z)^{-1}f, \bar{g})dz\,(1 + o(1)),$$

$n, W \to \infty$, with some particular functions f, g depending of the problem.

The aim is to find some operator equivalent to K_ξ whose spectral analysis is more accessible.

Set

$$\lambda_* = \lambda_0(\mathcal{K}), \quad (\lambda_* \approx 1),$$

then it suffices to choose L as $L_0 = \{z : |z| = |\lambda_*|(1 + O(n^{-1}))\}$.
We choose $L = L_1 \cup L_2$ where $L_2 = \{z : |z| = |\lambda_*|(1 - \log^2 n/n)\}$, and L_1 is some special contour, containing all eigenvalues between L_0 and L_2. Then

$$(\mathcal{K}^{n-1}f, \bar{g}) = -\frac{1}{2\pi i}\oint_{L_1} z^{n-1}(\mathcal{G}(z)f, \bar{g})dz \\ - \frac{1}{2\pi i}\oint_{|z|=|\lambda_*|(1-\log^2 n/n)} z^{n-1}(\mathcal{G}(z)f, \bar{g})dz$$

The second integral is small comparing with $|\lambda_*|^{n-1}$, since

$$|z|^{n-1} \le |\lambda_*|^{n-1} \cdot e^{-\log^2 n}$$

Mechanism of the crossover for $\mathcal{R}_0$

Key technical steps

$\mathcal{K}_\xi \sim \mathcal{K}_{\xi,\pm}$ (projection to the neighborhoods of saddle-points)

$\mathcal{K}_{\xi,\pm} \sim \mathcal{K}_{*\xi} \otimes \mathcal{A}$,

$\mathcal{K}_{*\xi}(U_1, U_2) = e^{-i\xi\nu(U_1)/N} K_{*0}(U_1 U_2^*) e^{-i\xi\nu(U_2)/N}$, $\mathcal{K}_{*0} : L_2(\mathring{U}(2)) \to L_2(\mathring{U}(2))$,

$\mathcal{A}(x_1, x_2, y_1, y_2) = A_1(x_1, x_2) A_2(y_1, y_2)$, $L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$.

Here $\xi_1 = -\xi_2 = \xi$, and $\nu(U) = \pi(1 - 2|U_{12}|^2)$

Then

$$\mathcal{R}_0 = (\mathcal{K}_{*\xi}^N \otimes \mathcal{A}^N f, \bar{g})(1 + o(1)) = (\mathcal{K}_{*\xi}^N \cdot 1, 1)(\mathcal{A}^N f_1, \bar{g}_1)(1 + o(1)).$$

Here we used that both f, g asymptotically can be replaced by $1 \otimes f_1(x, y)$.
After normalization we get:

$$D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(\mathcal{K}_{*\xi}^N \cdot 1, 1)}{(\mathcal{K}_{*0}^N \cdot 1, 1)}(1 + o(1))$$

Spectral analysis of $\mathcal{K}_{*\xi}$

A good news is that $\mathcal{K}_{*0}$ with a kernel

$$\mathcal{K}_{*0} = t_* W^2 e^{-t_* W^2 |(U_1 U_2^*)_{12}|^2}$$

is a self-adjoint "difference" operator. It is known that his eigenfunctions are Legendre polynomials P_j. Moreover, it is easy to check that corresponding eigenvalues have the form:

$$\lambda_j = 1 - t_* j(j+1)/W^2 + O((j(j+1)/W^2)^2), \quad j = 0, 1 \ldots .$$

Besides,

$$\mathcal{K}_{*\xi} = \mathcal{K}_{*0} - 2i\xi\hat{\nu}/N + O(N^{-2})$$

where $\hat{\nu}$ is the operator of multiplication by ν. Thus the eigenvalues of $\mathcal{K}_{*\xi}$ are in the N^{-1}-neighbourhood of λ_j.

Mechanism of the Poisson behavior for $W^2 \ll N$

For $W^{-2} \gg N^{-1}$ (the spectral gap is much bigger than the perturbation norm)

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 - 2N^{-1} i\xi(\nu \cdot 1, 1) + o(N^{-1}),$$
$$|\lambda_1(\mathcal{K}_{*\xi})| \le 1 - O(W^{-2}) \quad \Rightarrow \quad |\lambda_j(\mathcal{K}_{*\xi})|^N \to 0, \ (j = 1, 2, \dots).$$

Since

$$(\nu \cdot 1, 1) = 0,$$

we obtain that

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 + o(N^{-1}),$$

and

$$D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{\lambda_0^N(\mathcal{K}_{*\xi})}{\lambda_0^N(\mathcal{K}_{*0})}(1 + o(1)) \to 1$$

The relation corresponds to the Poisson local statistics.

Mechanism of the GUE behavior for $\mathrm{W}^2 \gg \mathrm{N}$

In the regime $\mathrm{W}^{-2} \ll \mathrm{N}^{-1}$ we have $\mathcal{K}_{*0}^{\mathrm{N}} \to \mathrm{I}$ in the strong vector topology, hence one can prove that

$$\mathcal{K}_{*\xi} \sim 1 + \mathrm{O}(\mathrm{W}^{-2}) - \mathrm{N}^{-1} 2\mathrm{i}\xi\nu \Rightarrow (\mathcal{K}_{*\xi}^{\mathrm{N}} \cdot 1, 1) \to (\mathrm{e}^{-2\mathrm{i}\xi\hat{\nu}} \cdot 1, 1)$$

and

$$\mathrm{D}_2^{-1}\mathcal{R}_0\Big(\mathrm{E} + \frac{\xi}{\mathrm{N}\rho(\mathrm{E})}, \mathrm{E} - \frac{\xi}{\mathrm{N}\rho(\mathrm{E})}\Big) = \frac{(\mathrm{e}^{-2\mathrm{i}\xi\mathrm{t}^*\hat{\nu}} \cdot 1, 1)}{(1,1)}(1 + \mathrm{o}(1)) \to \frac{\sin(2\pi\xi)}{2\pi\xi}.$$

The expression for $\mathrm{D}_2^{-1}\mathcal{R}_0$ coincides with that for GUE.

In the regime $W^{-2} = C_* N^{-1}$ observe that $\mathcal{K}_{*\xi}$ is reduced by the subspace $\mathcal{E}_0$ of the functions depending only on $|U_{12}|^2$.
Recall also that the Laplace operator on $\mathring{U}(2)$ is reduced by $\mathcal{E}_0$ and have the form

$$\Delta_U = -\frac{d}{dx} x(1-x) \frac{d}{dx}, \quad x = |U_{12}|^2.$$

Besides, the eigenvectors of Δ_U and $\mathcal{K}_{*0}$ coincide (they are Legendre's polynomials P_j) and corresponding eigenvalues of Δ_U are

$$\lambda_j^* = j(j+1).$$

Hence we can write $\mathcal{K}_{*\xi}$ as

$$\mathcal{K}_{*\xi} \sim 1 - N^{-1}(C_* t_* \Delta_U + 2i\xi\nu) + o(N^{-1}) \Rightarrow (\mathcal{K}_{*\xi}^N \cdot 1, 1) \to (e^{-C\Delta_U - 2i\xi\hat{\nu}} \cdot 1, 1)$$