Fluctuations, around its mean, of the Stieltjes transform of the empirical spectral distribution of selfadjoint polynomials in Wigner and deterministic diagonal matrices

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joint work with S. Belinschi, S. Dallaporta and M. Février
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Notation

$B = B^* \in \mathcal{M}_N(\mathbb{C})$

Eigenvalues of B: $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_N(B)$,

The empirical distribution of these eigenvalues:

$$\mu_B := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(B)}$$

μ probability measure on $\mathbb{R}$, $z \in \mathbb{C} \setminus \mathrm{supp}(\mu)$,

$$\text{Stieltjes transform of } \mu\text{: } g_\mu(z) = \int \frac{d\mu(x)}{z - x}$$

Tr_N: non normalized trace,
$\mathrm{tr}_N = \frac{1}{N}\mathrm{Tr}$: normalized trace.

Linear eigenvalue statistics $\mathcal{N}_N(\varphi) = \sum_{i=1}^N \varphi(\lambda_i(M_N))$

In recent years, several authors have derived CLT for centred linear eigenvalue statistics $\mathcal{N}_N(\varphi) - \mathbb{E}\left(\mathcal{N}_N(\varphi)\right)$ with various hypotheses on the entries of the random matrix M_N and the test function φ.

- **Wigner matrices/ band matrices**: Costin-Lebowitz(95), Khorunzhy-Khorunzhenko-Pastur (96), Johansson (98), Sinai-Soshnikov (98), Guionnet (02), Bai-Yao (05), Anderson-Zeitouni (06), Chatterjee (07), Bai-Wang-Zhou (09), Lytova-Pastur (09), Cabanal-Duvillard(11), Dallaporta-Vu (11), Shcherbina (11), Li-Soshnikov (13), Sosoe-Wong (13), Shcherbina (11), Benaych-Georges-Guionnet-Male (14), Kopel (15), Shcherbina (15), Bao-Xie (16), Benaych-Georges-Maltsev (16), Jana-Saha-Soshnikov (16), Landon-Sosoe (18), Adhikari-Jana-Saha (19), Bao-He 2021...
- **Deformed Wigner matrices**: Khorunzhy (94), Guionnet (02), Mingo-Speicher (06), Dallaporta-Février (19), Ji-Lee (19)...

A line of attack to study linear spectral statistics: based on Stieltjes transforms, developped by Bai-Yao (05), Bai-Silverstein (04), Girko (90), Khorunzhy-Khoruzhenko-Pastur (96).
$\Longrightarrow$ CLT for test functions: $\varphi_z, z \in \mathbb{C} \setminus \mathbb{R}$, $\varphi_z : x \mapsto \frac{1}{z-x}$.

CLT may be then extended to a wider class of test functions by Shcherbina's extension density argument, Cauchy's formula, Helffer-Söjstrand formula...

Aim of this talk

For any selfadjoint noncommutative polynomial P involving both a Wigner matrix W_N and a real deterministic diagonal matrix D_N, study of the convergence in distribution of the process on $\mathbb{C} \setminus \mathbb{R}$,

$$\xi_N(z) := N \left\{ g_{\mu_{P(W_N, D_N)}}(z) - \mathbb{E}\left(g_{\mu_{P(W_N, D_N)}}(z) \right) \right\}$$

$$= \mathrm{Tr}_N \left((z - P(W_N, D_N))^{-1} \right) - \mathbb{E}\left(\mathrm{Tr}_N \left((z - P(W_N, D_N))^{-1} \right) \right).$$

Wigner matrix W_N

$W_N = (W_{ij})_{1\leq i,j\leq N}$ a $N \times N$ Hermitian matrix.

1. The entries $\{W_{ij}\}_{1\leq i\leq j\leq N}$ are independent random variables;
2. **off-diagonal entries** $\{W_{ij}\}_{1\leq i<j\leq N}$ are i.i.d. complex r. v., $\mathbb{E}[W_{ij}] = 0$, for some $\varepsilon > 0$, $\mathbb{E}[|\sqrt{N}W_{ij}|^{6(1+\varepsilon)}] \leq C_6$,

$$\sigma_N^2 := \mathbb{E}[|W_{ij}|^2], \lim_{N\to+\infty} N\sigma_N^2 = \sigma^2 > 0;$$

$$\theta_N := \mathbb{E}[W_{ij}^2] \in \mathbb{C}, \lim_{N\to+\infty} N\theta_N = \theta \in \mathbb{R}$$

$$\kappa_N := \mathbb{E}[|W_{ij}|^4] - 2\sigma_N^4 - |\theta_N|^2 \in \mathbb{R}, \lim_{N\to+\infty} N^2\kappa_N = \kappa \in \mathbb{R}.$$

3. **diagonal entries** $\{W_{ii}\}_{1\leq i\leq N}$ are i.i.d. real random variables, $\mathbb{E}[W_{ii}] = 0$, for some $\varepsilon > 0$, $\mathbb{E}[|\sqrt{N}W_{ii}|^{4(1+\varepsilon)}] \leq C_4$,

$$\tilde{\sigma}_N^2 := \mathbb{E}[W_{ii}^2], \quad \lim_{N\to+\infty} N\tilde{\sigma}_N^2 = \tilde{\sigma}^2 > 0.$$

We will also assume that all entries of W_N are almost surely bounded by δ_N where δ_N is a sequence of positive numbers slowly converging to 0 (at rate less than $N^{-\eta}$ for any $\eta > 0$); this may be assumed without loss of generality for our purposes, by a truncation and centering argument.

Theorem (Girko 88, Khorunzhy-Khoruzhenko-Pastur 1996, Bai-Yao 2005, Bao-Xie 2016)

For any $z \in \mathbb{C} \setminus \mathbb{R}$, $N \left(g_{\mu_{W_N}}(z) - \mathbb{E}(g_{\mu_{W_N}}(z)) \right) \xrightarrow{\mathcal{D}} \mathcal{N}$,

$\mathcal{N}$: *centered Gaussian variable*, $\mathbb{E}\left(|\mathcal{N}|^2\right) = C(z, \bar{z})$, $\mathbb{E}\left(\mathcal{N}^2\right) = C(z, z)$

$$\text{with } C(z_1, z_2) := \frac{\partial^2}{\partial z_1 \partial z_2} \gamma(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \mathbb{R},$$

$$\begin{aligned} \gamma(z_1, z_2) = & -\log\left[1 - \sigma^2 T_{\{z_1,z_2\}}\right] \quad - \log\left[1 - \theta T_{\{z_1,z_2\}}\right] \\ & + (\tilde{\sigma}^2 - \sigma^2 - \theta) T_{\{z_1,z_2\}} \quad + \kappa/2\, T^2_{\{z_1,z_2\}}, \end{aligned}$$

$$T_{\{z_1,z_2\}} = g_{\mu_\sigma}(z_1) g_{\mu_\sigma}(z_2), \quad \frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma, 2\sigma]}(x).$$

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$$\sigma_N^2 := \mathbb{E}[|W_{ij}|^2], \theta_N := \mathbb{E}[W_{ij}^2] \in \mathbb{C}, \kappa_N := \mathbb{E}[|W_{ij}|^4] - 2\sigma_N^4 - |\theta_N|^2,$$

$$\lim_{N\to+\infty} N\sigma_N^2 = \sigma^2 > 0, \lim_{N\to+\infty} N\theta_N = \theta \in \mathbb{R}, \lim_{N\to+\infty} N^2\kappa_N = \kappa \in \mathbb{R}.$$

$\tilde{\sigma}_N^2 := \mathbb{E}[W_{ii}^2] \geq 0$, $\lim_{N\to+\infty} N\tilde{\sigma}_N^2 = \tilde{\sigma}^2 > 0$.

Strategy of Bai-Yao, Bao-Xie

$$N\left(g_{\mu_{W_N}}(z)-\mathbb{E}(g_{\mu_{W_N}}(z))\right)=\sum_{k=1}^{N}(\mathbb{E}_{\le k}-\mathbb{E}_{\le k-1})\left[\mathrm{Tr}_N(zI_N-W_N)^{-1}\right].$$

For each N, $(\mathbb{E}_{\le k}-\mathbb{E}_{\le k-1})\left[\mathrm{Tr}_N(zI_N-W_N)^{-1}\right]$ is a martingale difference: apply the CLT for sums of martingale differences.
$\mathbb{E}_{\le k}$: conditional expectation on $\mathcal{F}_k := \sigma(W_{ij}, 1\le i\le j\le k)$.

Suppose that, for all $N\ge 1$, $(M_k^{(N)})_{k\in\mathbb{N}}$ is a square integrable complex martingale and define, for $k\ge 1$, $\Delta_k^{(N)} := M_k^{(N)} - M_{k-1}^{(N)}$. If

$$\forall\varepsilon>0,\ \sum_{k=1}^{N}\mathbb{E}[|\Delta_k^{(N)}|^2\mathbf{1}_{|\Delta_k^{(N)}|\ge\varepsilon}]\underset{N\to+\infty}{\longrightarrow}0,$$

$$\sum_{k=1}^{N}\mathbb{E}_{\le k-1}[|\Delta_k^{(N)}|^2]\xrightarrow[N\to+\infty]{\mathbb{P}} v\ge 0,\ \sum_{k=1}^{N}\mathbb{E}_{\le k-1}[(\Delta_k^{(N)})^2]\xrightarrow[N\to+\infty]{\mathbb{P}} w\in\mathbb{C},$$

then $\sum_{k=1}^{N}\Delta_k^{(N)}\xrightarrow[N\to+\infty]{\mathcal{D}}\mathcal{N}$,
$\mathcal{N}$: complex centered Gaussian variable, $\mathbb{E}\left(|\mathcal{N}|^2\right)=v$, $\mathbb{E}\left(\mathcal{N}^2\right)=w$.

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$$T_{\{z_1,z_2\}} = g_{\mu_\sigma}(z_1) g_{\mu_\sigma}(z_2), \; \frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma, 2\sigma]}(x).$$

Some backgrounds on Free Probability theory

Free probability = non commutative probability + freeness.

Voiculescu in the 80's

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If $\mathcal{A}$ is a $\mathcal{C}^*$- algebra endowed with a state ϕ, then for any selfadjoint element a in $\mathcal{A}$, there exists a probability measure μ_a on $\mathbb{R}$ such that, for every polynomial P, we have $\phi(P(a)) = \int P(t) d\mu_a(t)$.

Example: $\mathcal{A}_N$ algebra of $N \times N$ complex matrices.

$$\phi_N : \begin{cases} \mathcal{A}_N \to \mathbb{C} \\ A \to \mathrm{tr}_N\, A \end{cases}$$

For each $N \geq 1$, $(\mathcal{A}_N, \phi_N)$ is a non commutative probability space.
$A = A^* \in \mathcal{A}_N$

$$\begin{aligned} \phi_N(A^k) &= \mathrm{tr}_N(A^k) \\ &= \frac{1}{N}\sum_{i=1}^{N}[\lambda_i(A)]^k \\ &= \int_{\mathbb{R}} t^k d\mu_A(t) \end{aligned}$$

$\Longrightarrow$ The distribution of A in $(\mathcal{A}_N, \phi_N)$ is $\mu_A = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(A)}$

Notion of freeness

$(\mathcal{A}, \phi)$: a non commutative probability space

Definition

(D. Voiculescu 80') $a, b \in (\mathcal{A}, \phi)$. $\mathcal{A}_1 = alg(1_{\mathcal{A}}, a)$, $\mathcal{A}_2 = alg(1_{\mathcal{A}}, b)$. a and b are free if $\forall x_1, \ldots, x_n$ such that $\forall k$, $x_k \in \mathcal{A}_{i_k}$ with $i_k \neq i_{k+1}$ and $\phi(x_k) = 0$, then $\phi(x_1 \cdots x_n) = 0$.

If a and b are free in $(\mathcal{A}, \phi)$:

- **classical non correlation of a and b** : the mixed moments $\phi(a^{n_1} b^{k_1} \cdots a^{n_p} b^{k_p})$ are completely determined by the $\phi(a^n)$ and $\phi(b^k)$
- **algebraic independence**: a and b do not commute (in non degenerate cases).

Free additive convolution $\boxplus$

Definition

If a and b are free in $(\mathcal{A}, \phi)$, $\mu_a \boxplus \mu_b := \mu_{a+b}$.

$$\Downarrow$$

Binary operation on compactly supported probability measures on $\mathbb{R}$ (actually even on probability measures on $\mathbb{R}$).

There exist analytic tools (R-transform) to compute $\mu \boxplus \nu$, given probability measures μ and ν.

For a probability measure τ on $\mathbb{R}$, $z \in \mathbb{C}\backslash\mathbb{R}$, $g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$.

Theorem (D.Voiculescu, P. Biane)

Let μ and ν be two probability measures on $\mathbb{R}$, there exists a unique analytic map $\omega(= \omega_{\mu,\nu}) : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$\forall z \in \mathbb{C}^+, g_{\mu \boxplus \nu}(z) = g_\nu(\omega(z)),$$

$$\forall z \in \mathbb{C}^+, \Im\omega(z) \geq \Im z$$

and

$$\lim_{y \uparrow +\infty} \frac{\omega(iy)}{iy} = 1.$$

ω is called the subordination map of $\mu \boxplus \nu$ with respect to ν.

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When $\mu = \mu_\sigma$ (semi-circular distribution), $\omega(z) = z - \sigma^2 g_{\mu_\sigma\boxplus\nu}(z)$.

Asymptotic freeness

Several models of independent $N \times N$ Hermitian random matrices A_N and B_N provide asymptotically free random variables when N goes to infinity:

(almost surely or in expectation or in probability...)

$$\mathrm{tr}_N(A_N^{n_1} B_N^{k_1} \cdots A_N^{n_p} B_N^{k_p}) \longrightarrow_{N \to +\infty} \phi(a^{n_1} b^{k_1} \cdots a^{n_p} b^{k_p})$$

a and b are free in $(\mathcal{A}, \phi)$.

Pionnering work 90' of D. Voiculescu with independent GUE matrices. Extended by several authors to other classical models as independent unitarily invariant Hermitian matrices, general Wigner matrices and deterministic matrices...

Asymptotic freeness

D_N deterministic real diagonal $N \times N$ matrix s.t.

$$\sup_{N\in\mathbb{N}} \|D_N\| < \infty \ \nu_N := \frac{1}{N} \sum_{\lambda\in \mathrm{sp}(D_N)} \delta_\lambda \Rightarrow \nu, \nu \text{ probability measure on } \mathbb{R}$$

W_N: Wigner matrix.

Dykema 93, Ryan 98:

$$\mathbb{E}\left(\frac{1}{N}\mathrm{Tr}_N({W_N}^{n_1}{D_N}^{k_1}\cdots {W_N}^{n_p}{D_N}^{k_p})\right) \longrightarrow_{N\to+\infty} \phi(x^{n_1}d^{k_1}\cdots x^{n_p}d^{k_p})$$

x, d free in $(\mathcal{A}, \phi)$, $x = x^*, d = d^*$,

$x \sim \mu_\sigma$, $\frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,1_{[-2\sigma,2\sigma]}(x)$, $d \sim \nu$.

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$P = P^*$: a polynomial in two n.c. variables.
$\tau_P :=$ distribution of $P(x, d)$ in $(\mathcal{A}, \phi)$.

Deformed Wigner matrices

D_N deterministic real diagonal $N \times N$ matrix s.t. $\sup_{N \in \mathbb{N}} \|D_N\| < \infty$ and

$$\nu_N := \frac{1}{N} \sum_{\lambda \in \mathrm{sp}(D_N)} \delta_\lambda \Rightarrow \nu,$$

where ν is a probability measure on $\mathbb{R}$.

W_N: Wigner matrix.

By asymptotic freeness (Dykema, Ryan):

$$\mathbb{E}\left(\mu_{W_N + D_N}\right) \to \mu_\sigma \boxplus \nu \quad \text{weakly.}$$

μ_σ: the centered semi-circular distribution with variance σ^2;

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_\sigma \boxplus \nu}(z) = g_\nu(\omega(z)); \quad \omega(z) = z - \sigma^2 g_{\mu_\sigma \boxplus \nu}(z)$$

Theorem (Dallaporta-Février 2019, Ji-Lee 2019)

For any $z \in \mathbb{C} \setminus \mathbb{R}$, $N\left(g_{\mu_{W_N+D_N}}(z) - \mathbb{E}(g_{\mu_{W_N+D_N}}(z))\right) \xrightarrow{\mathcal{D}} \mathcal{N}$,

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$$\textit{with } C(z_1,z_2) := \frac{\partial^2}{\partial z_1 \partial z_2}\gamma(z_1,z_2), \quad z_1, z_2 \in \mathbb{C}\setminus\mathbb{R},$$

$$\gamma(z_1,z_2) = -\log\left[1 - \sigma^2 T_{\{z_1,z_2\}}\right] \quad -\log\left[1 - \theta T_{\{z_1,z_2\}}\right]$$
$$+ (\tilde{\sigma}^2 - \sigma^2 - \theta) T_{\{z_1,z_2\}} \quad + \kappa/2\, T^2_{\{z_1,z_2\}},$$

$T_{\{z_1,z_2\}} = \int \frac{\nu(dx)}{(\omega(z_1)-x)(\omega(z_2)-x)}$, $\omega(z) = z - \sigma^2 g_{\mu_\sigma \boxplus \nu}(z)$ *(subordination function)*,

Theorem (Dallaporta-Février 2019, Ji-Lee 2019)

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$\sigma_N^2 := \mathbb{E}[|W_{ij}|^2], \theta_N := \mathbb{E}[W_{ij}^2] \in \mathbb{C}, \kappa_N := \mathbb{E}[|W_{ij}|^4] - 2\sigma_N^4 - |\theta_N|^2$,

$$\lim_{N\to+\infty} N\sigma_N^2 = \sigma^2 > 0, \lim_{N\to+\infty} N\theta_N = \theta \in \mathbb{R}, \lim_{N\to+\infty} N^2\kappa_N = \kappa \in \mathbb{R}.$$

$\tilde{\sigma}_N^2 := \mathbb{E}[W_{ii}^2] \geq 0$, $\lim_{N\to+\infty} N\tilde{\sigma}_N^2 = \tilde{\sigma}^2 > 0$.

When $D_N \equiv 0$, $\nu \equiv \delta_0$, $\omega(z) = z - \sigma^2 g_{\mu_\sigma}(z) = \frac{1}{g_{\mu_\sigma}(z)}$.

$$T_{\{z_1,z_2\}} = \int \frac{\nu(dx)}{(\omega(z_1) - x)(\omega(z_2) - x)} = g_{\mu_\sigma}(z_1) g_{\mu_\sigma}(z_2).$$

$\Longrightarrow$ We recover the covariance for a non-deformed Wigner matrix.

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$\Longrightarrow$ We recover the covariance for a non-deformed Wigner matrix.

Same strategy:

$$N\left(g_{\mu_{W_N+D_N}}(z) - \mathbb{E}(g_{\mu_{W_N+D_N}}(z))\right) = \sum_{k=1}^{N} (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})\left[\operatorname{Tr}_N(zI_N - W_N - D_N)^{-1}\right]$$

$\mathbb{E}_{\leq k}$: conditional expectation on $\mathcal{F}_k := \sigma(W_{ij}, 1 \leq i \leq j \leq k)$.

For each N, $(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})\left[\operatorname{Tr}_N(zI_N - W_N - D_N)^{-1}\right]$ is a martingale difference: apply the CLT for sums of martingale differences (making use of linear algebra tools, including various consequences of Schur inversion formula).

Remark on second order free probability theory

An analog of the free probability framework for dealing with fluctuations of the process of traces of noncommutative polynomials in Gaussian Wigner and deterministic matrices was designed in a series of papers Mingo Speicher (06), Mingo-Sniady-Speicher (07), Collins-Mingo-Sniady-Speicher (07): the so-called **second order free probability theory**.

When $\sigma^2 = \tilde{\sigma}^2$, $\theta = \kappa = 0$, the covariance $C(z_1, z_2)$ coincides with the so-called second order Cauchy transform that may be computed from the R-transform machinery.

Nevertheless, this second order free probability theory does not seem to be relevant framework to describe fluctuations dealing with more general Wigner matrices (Male-Mingo-Péché-Speicher (20)).

($\mathbb{C}\langle X_1, X_2\rangle$ becomes a *-algebra by anti-linear extension of $(X_{i_1}X_{i_2}\ldots X_{i_m})^* = X_{i_m}\ldots X_{i_2}X_{i_1}$.)

Aim of this talk

For any selfadjoint polynomial P involving both two noncommutative selfadjoint variables, study of the convergence in distribution of the process

$$\xi_N \in \mathcal{H}(\mathbb{C}\setminus\mathbb{R}),$$

$$\xi_N(z) := N\left\{g_{\mu_{P(W_N,D_N)}}(z) - \mathbb{E}\left(g_{\mu_{P(W_N,D_N)}}(z)\right)\right\}$$

$$= \mathrm{Tr}_N\left((z-P(W_N,D_N))^{-1}\right) - \mathbb{E}\left(\mathrm{Tr}_N\left((z-P(W_N,D_N))^{-1}\right)\right).$$

$\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$: the space of complex analytic functions on $\mathbb{C}\setminus\mathbb{R}$, endowed with the topology of uniform convergence on compact sets.

Key tools

Linearization

$$+$$

Operator-valued subordination functions in free probability theory

Anderson's linearization (2013)

Definition

$P \in \mathbb{C}\langle X_1, \ldots, X_k \rangle$.

$$L_P := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k \rangle,$$

- $m \in \mathbb{N}$, $Q \in M_{m-1}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k \rangle$ is invertible,
- u is a row vector and v is a column vector, both of size $m-1$

is called a **linearization of P**, if

(i) there are matrices $\gamma_0, \gamma_1, \ldots, \gamma_k \in M_m(\mathbb{C})$, such that

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \cdots + \gamma_k \otimes X_k$$

(ii) $P = -uQ^{-1}v$.

Anderson's linearization (2013)

Proposition

Any polynomial $P \in \mathbb{C}\langle X_1, \ldots, X_k \rangle$ *admits a linearization* L_P *(with an explicit algorithm for finding one that we will call canonical).*

Proposition

Let $P \in \mathbb{C}\langle X_1, \ldots, X_k \rangle$ *be a selfadjoint polynomial. Then* P *admits a selfadjoint linearization* L_P*.*

Example

- $P(X_1, X_2) = X_1X_2 + X_2X_1 + X_1^2$.

$$L_P = \begin{pmatrix} 0 & X_1 & X_2 + \frac{1}{2}X_1 \\ X_1 & 0 & -1 \\ X_2 + \frac{1}{2}X_1 & -1 & 0 \end{pmatrix}$$ is a selfadjoint linearization of P.

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2,$$

where

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2, \quad \gamma_0, \gamma_1, \gamma_2 \in M_m^{sa}(\mathbb{C}), m \in \mathbb{N}.$$

$$(ze_{11} \otimes I_N - \gamma_0 \otimes I_N - \gamma_1 \otimes W_N - \gamma_2 \otimes D_N)^{-1} = \begin{pmatrix} (z - P(W_N, D_N))^{-1} & \star \\ \star & \star \end{pmatrix}.$$

$$e_{11} \in M_m(\mathbb{C}), (e_{11})_{ij} = \delta_{i1}\delta_{j1}.$$

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2, \quad \gamma_0, \gamma_1, \gamma_2 \in M_m^{sa}(\mathbb{C}), m \in \mathbb{N}.$$

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$$e_{11} \in M_m(\mathbb{C}), (e_{11})_{ij} = \delta_{i1}\delta_{j1}.$$

$$\begin{aligned} &\mathrm{Tr}_N \left((z - P(W_N, D_N))^{-1}\right) \\ &= \mathrm{Tr}_m \otimes \mathrm{Tr}_N \left[((ze_{11} - \gamma_0) \otimes I_N - \gamma_1 \otimes W_N - \gamma_2 \otimes D_N)^{-1} (e_{11} \otimes I_N)\right], \end{aligned}$$

$\Longrightarrow$ Study of the fluctuations of

$$\frac{1}{N}\mathrm{Tr}_m \otimes \mathrm{Tr}_N \left[((ze_{11} - \gamma_0) \otimes I_N - \gamma_1 \otimes W_N - \gamma_2 \otimes D_N)^{-1} (e_{11} \otimes I_N)\right]$$

around its mean.

General non commutative polynomial matrix model

$\downarrow$ (Linearisation)

a LINEAR polynomial with MATRIX coefficients

$\downarrow$

We can adapt the strategy based on CLT for martingales differences and Schur inversion formula

Intuition from the deformed additive models

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2, \quad \gamma_0, \gamma_1, \gamma_2 \in M_m^{sa}(\mathbb{C}), m \in \mathbb{N}.$$

x, d free in $(\mathcal{A}, \phi)$, $x \sim \mu_\sigma$, $d \sim \nu$.

$$(ze_{11} \otimes 1_{\mathcal{A}} - \gamma_0 \otimes 1 - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1} = \begin{pmatrix} (z1_{\mathcal{A}} - P(x,d))^{-1} & \star \\ \star & \star \end{pmatrix}.$$

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$$g_{\tau_P}(z) = \phi\left((z1_{\mathcal{A}} - P(x,d))^{-1}\right) = \left[G\left(ze_{11} - \gamma_0\right)\right]_{11},$$

$$G(\beta) = id_m \otimes \phi\left((\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1}\right) \in M_m(\mathbb{C}).$$

Intuition from the deformed additive models

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2, \quad \gamma_0, \gamma_1, \gamma_2 \in M_m^{sa}(\mathbb{C}), m \in \mathbb{N}.$$

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$$G(\beta) = id_m \otimes \phi\left((\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1}\right) \in M_m(\mathbb{C}).$$

Example m=2:

$$M_2(\mathcal{A}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathcal{A} \right\}, \quad E := id_2 \otimes \phi : M_2(\mathcal{A}) \to M_2(\mathbb{C}),$$

$$E\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix}.$$

$(M_2(\mathcal{A}), E)$ is an $M_2(\mathbb{C})$-valued probability space. $(\mathbb{C} \approx \mathbb{C}1_{\mathcal{A}})$

Intuition from the deformed additive models

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2, \quad \gamma_0, \gamma_1, \gamma_2 \in M_m^{sa}(\mathbb{C}), m \in \mathbb{N}.$$

x, d free in $(\mathcal{A}, \phi)$, $x \sim \mu_\sigma$, $d \sim \nu$.

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$$G(\beta) = id_m \otimes \phi\left((\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1}\right) \in M_m(\mathbb{C}).$$

x, d free in $(\mathcal{A}, \phi) \Longrightarrow \begin{cases} \gamma_1 \otimes x \text{ and } \gamma_2 \otimes d \text{ free with amalgamation over } M_m(\mathbb{C}) \\ \text{with respect to } E = (\mathrm{id_m} \otimes \phi) : M_m(\mathcal{A}) \to M_m(\mathbb{C}) \subset M_m(\mathcal{A}) \end{cases}$

Needed: **a matrix-valued subordination property**, with respect to the matrix-valued conditional expectation $E = id_m \otimes \phi$.

$\mathcal{B}$-valued probability space

Definition

$\mathcal{M}$: a unital algebra, $\mathcal{B} \subset \mathcal{M}$: a unital subalgebra.
A linear map $E : \mathcal{M} \to \mathcal{B}$ is a conditional expectation if
$\forall b \in \mathcal{B},\ E(b) = b, \quad \forall a \in \mathcal{M}, \forall b_1, b_2 \in \mathcal{B},\ E(b_1 a b_2) = b_1 E(a) b_2.$
Then $(\mathcal{M}, E)$ is called a $\mathcal{B}$-valued probability space.
If in addition $\mathcal{M}$ is a C^*-algebra, $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{M}$ and E is completely positive, then we have a $\mathcal{B}$-valued C^*-probability space.

The $\mathcal{B}$-valued distribution of $a \in \mathcal{M}$: all $\mathcal{B}$-valued moments
$E(ab_1ab_2 \cdots b_{n-1}a) \in \mathcal{B},\ n \in \mathbb{N}, b_0, \ldots, b_{n-1} \in \mathcal{B}.$

Freeness with amalgamation over $\mathcal{B}$ (Voiculescu 95)

Definition

$(\mathcal{M}, E : \mathcal{M} \to \mathcal{B})$: an operator valued probability space.
$(A_i)_{i \in I}$: a family of subalgebras with $\mathcal{B} \subset A_i$ for all $i \in I$.
$(A_i)_{i \in I}$ are free with amalgamation over $\mathcal{B}$ if $E(a_1 \cdots a_n) = 0$ whenever $a_j \in A_{i_j}$, $i_j \in I$, $E(a_j) = 0$, for all j and $i_1 \neq i_2 \neq \cdots \neq i_n$.

Random variables in $\mathcal{M}$ are free with amalgamation over $\mathcal{B}$ if the algebras generated by $\mathcal{B}$ and the variables are so.

Operator-valued free subordination property

Theorem (Voiculescu (2000); Belinschi-Mai-Speicher (2013))

$(\mathcal{M}, E : \mathcal{M} \to \mathcal{B})$: *an operator-valued* C^* *-probability space*
($\mathcal{B} \subset \mathcal{M}$ C^* *algebras,* $E : \mathcal{M} \to \mathcal{B}$ *c.p. conditional expectation).*
$x_1, x_2 \in \mathcal{M}$: *selfadjoint variables,* **free with amalgamation over** $\mathcal{B}$*.*
$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \colon \Im b > 0\}$.

- *There exists an analytic map* $\omega \colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ *such that*

$$E\left[(\omega(b) - x_1)^{-1}\right] = E\left[(b - (x_1 + x_2))^{-1}\right] \quad \textit{for all} \;\; b \in \mathbb{H}^+(\mathcal{B}).$$

Operator-valued free subordination property

Theorem (Voiculescu (2000); Belinschi-Mai-Speicher (2013))

$(\mathcal{M}, E : \mathcal{M} \to \mathcal{B})$: *an operator-valued C^* -probability space* *($\mathcal{B} \subset \mathcal{M}$ C^* algebras, $E : \mathcal{M} \to \mathcal{B}$ c.p. conditional expectation).*
$x_1, x_2 \in \mathcal{M}$: *selfadjoint variables,* **free with amalgamation over** $\mathcal{B}$.
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$$E\left[(\omega(b) - x_1)^{-1}\right] = E\left[(b - (x_1 + x_2))^{-1}\right] \quad \text{for all } \; b \in \mathbb{H}^+(\mathcal{B}).$$

- *For* $b \in \mathbb{H}^+(\mathcal{B})$, $\omega(b)$ *is the unique fixed point of the map*

$$f_b : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \quad f_b(\kappa) = h_{x_2}(h_{x_1}(\kappa) + b) + b$$

$$\text{where } \; h_{x_i}(\kappa) = E\left[(\kappa - x_i)^{-1}\right]^{-1} - \kappa$$

$$\text{and } \; \omega(b) = \lim_{k \to +\infty} f_b^{\circ k}(\kappa), \quad \text{for any } \; \kappa \in \mathbb{H}^+(\mathcal{B}).$$

Operator-valued free subordination property

In the particular case where x, d free in $(\mathcal{A}, \phi)$, $x = x^*, d = d^*$, $x \sim \mu_\sigma$, $d \sim \nu$,

$$E = id_m \otimes \phi : M_m(\mathcal{A}) \to M_m(\mathbb{C}),$$

for $\beta \in M_m(\mathbb{C}), \Im\beta > 0$,

$$id_m \otimes \phi \left((\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1} \right) = id_m \otimes \phi \left((\omega_m(\beta) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} \right)$$

the $M_m(\mathbb{C})$-valued subordination function is explicitely given by

$$\omega_m(\beta) = \beta - \gamma_1 id_m \otimes \phi \left[(\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1} \right] \gamma_1 \in M_m(\mathbb{C}).$$

Results

Theorem (Belinschi, C., Dallaporta, Février 2021)

$$\xi_N(z) = \mathrm{Tr}_N\left((z - P(W_N, D_N))^{-1}\right) - \mathbb{E}\left(\mathrm{Tr}_N\left((z - P(W_N, D_N))^{-1}\right)\right).$$

The sequence $(\xi_N)_{N\in\mathbb{N}}$ of $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$-valued random variables converges in distribution towards a complex centred Gaussian process $\{\mathcal{G}(z), z \in \mathbb{C}\setminus\mathbb{R}\}$ defined by $\overline{\mathcal{G}(z)} = \mathcal{G}(\bar{z})$ and

$$\mathbb{E}\left(\mathcal{G}(z_1)\mathcal{G}(z_2)\right) := \frac{\partial^2}{\partial z_1 \partial z_2}\gamma(z_1, z_2), \quad z_1, z_2 \in \mathbb{C}\setminus\mathbb{R},$$

$\gamma(z_1, z_2)$

$$P = P^* \in \mathbb{C} < X_1, X_2 >$$

$\gamma(z_1, z_2)$

$$P = P^* \in \mathbb{C} < X_1, X_2 >$$

$$\downarrow \text{(Linearisation)}$$

Compute the canonical selfadjoint linearization by Anderson's algorithm

$$L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes X_1 + \gamma_2 \otimes X_2 \in M_m(\mathbb{C} < X_1, X_2 >).$$

$\gamma(z_1, z_2)$

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$\downarrow$ (subordination function)

Compute $z \mapsto \omega_m(ze_{11} - \gamma_0) \in M_m(\mathbb{C})$ defined on $\mathbb{C} \setminus \mathrm{supp}(\tau_P)$ by

$$\omega_m(ze_{11}-\gamma_0)=ze_{11}-\gamma_0-\gamma_1 id_m \otimes \phi \left[((ze_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1}\right] \gamma_1,$$

$x = x^* \sim \mu_\sigma$, $d = d^* \sim \nu$ free in $(\mathcal{A}, \phi)$,

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$x = x^* \sim \mu_\sigma$, $d = d^* \sim \nu$ free in $(\mathcal{A}, \phi)$,

$$\downarrow$$

Define $T_{\{z_1,z_2\}} : M_m(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ by

$$T_{\{z_1,z_2\}}(u)$$

$$= \int_{\mathbb{R}} ((\omega_m(z_1 e_{11} - \gamma_0) - t\gamma_2)^{-1} \gamma_1 \otimes I_m) u (I_m \otimes \gamma_1 (\omega_m(z_2 e_{11} - \gamma_0) - t\gamma_2)^{-1}) d\nu(t).$$

$\gamma(z_1, z_2)$

$$P = P^* \in \mathbb{C} < X_1, X_2 >$$

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Compute the canonical selfadjoint linearization by Anderson's algorithm

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$$\omega_m(ze_{11}-\gamma_0)=ze_{11}-\gamma_0-\gamma_1 id_m\otimes\phi\left[((ze_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_1 \otimes x - \gamma_2 \otimes d)^{-1}\right]\gamma_1,$$

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$$T_{\{z_1,z_2\}}(u)$$

$$= \int_{\mathbb{R}} ((\omega_m(z_1 e_{11}-\gamma_0)-t\gamma_2)^{-1}\gamma_1\otimes I_m)u(I_m\otimes\gamma_1(\omega_m(z_2 e_{11}-\gamma_0)-t\gamma_2)^{-1})d\nu(t).$$

Proposition (Belinschi, C., Dallaporta, Février 2021)

For any $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$, the spectrum of the operator

$$T_{\{z_1,z_2\}} : M_m(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$$

is included in the open disk of radius σ^{-2}.

$\Longrightarrow$ The operators $\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \theta\, T_{\{z_1,z_2\}}\right]$ and $\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \sigma^2\, T_{\{z_1,z_2\}}\right]$ are well defined.

Finally

$$\begin{aligned}\gamma(z_1, z_2) = &- \mathsf{Tr}_m \otimes \mathsf{Tr}_m \left\{\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \sigma^2 T_{\{z_1,z_2\}}\right](I_m \otimes I_m)\right\} \\ &- \mathsf{Tr}_m \otimes \mathsf{Tr}_m \left\{\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \theta T_{\{z_1,z_2\}}\right](I_m \otimes I_m)\right\} \\ &+ (\tilde{\sigma}^2 - \sigma^2 - \theta)\, \mathsf{Tr}_m \otimes \mathsf{Tr}_m\{T_{\{z_1,z_2\}}(I_m \otimes I_m)\} \\ &+ \kappa/2\, \mathsf{Tr}_m \otimes \mathsf{Tr}_m\{T^2_{\{z_1,z_2\}}(I_m \otimes I_m)\},\end{aligned}$$

Finally

$$\begin{aligned}\gamma(z_1,z_2) = &- \mathrm{Tr}_m \otimes \mathrm{Tr}_m \left\{\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \sigma^2 T_{\{z_1,z_2\}}\right](I_m \otimes I_m)\right\} \\ &- \mathrm{Tr}_m \otimes \mathrm{Tr}_m \left\{\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \theta T_{\{z_1,z_2\}}\right](I_m \otimes I_m)\right\} \\ &+ (\tilde{\sigma}^2 - \sigma^2 - \theta)\,\mathrm{Tr}_m \otimes \mathrm{Tr}_m\{T_{\{z_1,z_2\}}(I_m \otimes I_m)\} \\ &+ \kappa/2\,\mathrm{Tr}_m \otimes \mathrm{Tr}_m\{T^2_{\{z_1,z_2\}}(I_m \otimes I_m)\},\end{aligned}$$

$\sigma_N^2 := \mathbb{E}[|W_{ij}|^2], \theta_N := \mathbb{E}[W_{ij}^2] \in \mathbb{C}, \kappa_N := \mathbb{E}[|W_{ij}|^4] - 2\sigma_N^4 - |\theta_N|^2 \in \mathbb{R},$

$\lim_{N\to+\infty} N\sigma_N^2 = \sigma^2 > 0, \lim_{N\to+\infty} N\theta_N = \theta \in \mathbb{R}, \lim_{N\to+\infty} N^2\kappa_N = \kappa \in \mathbb{R}.$

$\tilde{\sigma}_N^2 := \mathbb{E}[W_{ii}^2] \geq 0, \lim_{N\to+\infty} N\tilde{\sigma}_N^2 = \tilde{\sigma}^2 > 0.$

$T_{\{z_1,z_2\}} : M_m(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ defined by

$T_{\{z_1,z_2\}}(u)$

$$= \int_{\mathbb{R}} ((\omega_m(z_1 e_{11} - \gamma_0) - t\gamma_2)^{-1} \gamma_1 \otimes I_m) u (I_m \otimes \gamma_1 (\omega_m(z_2 e_{11} - \gamma_0) - t\gamma_2)^{-1}) d\nu(t)$$

If $m = 1$, $\gamma_1 = \gamma_2 = 1$, $\gamma_0 = 0$, then

$$T_{\{z_1,z_2\}} = \int \frac{\nu(dx)}{(\omega(z_1) - x)(\omega(z_2) - x)}$$

$\Longrightarrow$ We recover the covariance for a deformed Wigner matrix.

Tasks in the near future

- Study of the mean function and extension of the CLT of linear statistics to a wider class of test functions than $x \mapsto \frac{1}{z-x}$ by Shcherbina's argument, Cauchy's formula, Helffer-Söjstrand's formula....
- Connect our result with Male-Mingo-Péché-Speicher's result (20) on CLT for

$$\operatorname{Tr} Q(\mathbf{X}, \mathbf{A}) - \mathbb{E}\left(\operatorname{Tr} Q(\mathbf{X}, \mathbf{A})\right), \; Q \text{ polynomials},$$

  for a collection $\mathbf{X}$ of independent Wigner matrices, and a collection $\mathbf{A}$ of deterministic matrices.

For general deterministic matrices, the fluctuations may not depend only on the limiting distribution...
(Male-Mingo-Péché-Speicher (20))

Algorithmic computation of ω_m.

If $\kappa \in \mathbb{H}^+(M_m(\mathbb{C}))$, then $\omega_m(\kappa)$ is the unique fixed point of the map

$$f_\kappa : \mathbb{H}^+(M_m(\mathbb{C})) \to \mathbb{H}^+(M_m(\mathbb{C})), \quad f_\kappa(\zeta) = h_{\gamma_1 \otimes x}(h_{\gamma_2 \otimes d}(\zeta) + \kappa) + \kappa$$

$$\text{where} \quad h_{\gamma \otimes y}(\zeta) = G_{\gamma \otimes y}(\zeta)^{-1} - \zeta,$$

$$G_{\gamma \otimes y}(\zeta) := \mathrm{id_m} \otimes \phi \left[(\zeta \otimes 1_{\mathcal{A}} - \gamma \otimes y)^{-1} \right],$$

$$G_{\gamma \otimes y}(\zeta) = \lim_{\epsilon \downarrow 0} -\frac{1}{\pi} \int_{\mathbb{R}} (\zeta - t\gamma)^{-1} \Im m(g_{\mu_y}(t + i\epsilon)) dt.$$

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$$G_{\gamma\otimes y}(\zeta) := \mathrm{id_m} \otimes \phi\left[(\zeta \otimes 1_{\mathcal{A}} - \gamma \otimes y)^{-1}\right],$$

$$G_{\gamma\otimes y}(\zeta) = \lim_{\epsilon \downarrow 0} -\frac{1}{\pi} \int_{\mathbb{R}} (\zeta - t\gamma)^{-1} \Im m(g_{\mu_y}(t + i\epsilon))dt.$$

$$\omega_m(\kappa) = \lim_{k \to +\infty} f_\kappa^{\circ k}(\theta), \quad \text{for any} \quad \theta \in \mathbb{H}^+(\mathcal{B}).$$

A concrete algorithm for linearisation (Anderson, Mai)

$$L_{X_j} = \begin{pmatrix} 0 & X_j \\ 1 & -1 \end{pmatrix}.$$

$$i_1, \ldots, i_m \in \{1, \ldots, k\}, \quad L_{X_{i_1} X_{i_2} \ldots X_{i_m}} = \begin{pmatrix} 0 & \ldots & 0 & X_{i_1} \\ 0 & \ldots 0 & X_{i_2} & -1 \\ (0) & \iddots & \iddots & (0) \\ X_{i_m} & -1 & (0) & \end{pmatrix}.$$

$$Q = \begin{pmatrix} 0 & \ldots 0 & X_{i_2} & -1 \\ (0) & \iddots & \iddots & (0) \\ X_{i_{m-1}} & -1 & & (0) \\ -1 & 0 & (0) & \end{pmatrix}.$$

$$Q^{-1} =$$

$$\begin{pmatrix} 0 & \dots 0 & 0 & 0 & -1 \\ 0 & \dots 0 & 0 & -1 & -X_{i_{m-1}} \\ 0 & \dots 0 & -1 & -X_{i_{m-2}} & -X_{i_{m-2}}X_{i_{m-1}} \\ 0 & -1 & -X_{i_{m-3}} & -X_{i_{m-3}}X_{i_{m-2}} & -X_{i_{m-3}}X_{i_{m-2}}X_{i_{m-1}} \\ \iddots & \iddots & \iddots & \iddots & \\ -1 & -X_{i_2} & -X_{i_2}X_{i_3} & -X_{i_2}X_{i_3}X_{i_4} & \dots - X_{i_2}X_{i_3}X_{i_4}\dots X_{i_{m-1}} \end{pmatrix}$$

$$\alpha \in \mathbb{C}, \quad L_{\alpha x_{i_1} x_{i_2} \dots x_{i_m}} = \begin{pmatrix} 0 & \dots & 0 & X_{i_1} \\ 0 & \dots 0 & X_{i_2} & -1 \\ (0) & \iddots & \iddots & (0) \\ \alpha X_{i_m} & -1 & (0) & \end{pmatrix}.$$

If the polynomials $p_1, \ldots, p_l$ in $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ have linearizations for $j = 1, \ldots, l$,

$$L_{p_j} := \begin{pmatrix} 0 & u_j \\ v_j & Q_j \end{pmatrix} \in M_{n_j}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k\rangle,$$

then

$$L_P = \begin{pmatrix} 0 & u_1 & \cdots & u_l \\ v_1 & Q_1 & 0\cdots & 0 \\ \vdots & (0) & \ddots & (0) \\ v_l & 0 & \cdots 0 & Q_l \end{pmatrix} \in M_n(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k\rangle$$

with $n = \sum_{j=1}^{l} n_j - l + 1$, is a linearization of $p_1 + \ldots + p_l$. Note that the entries of $Q^{-1} = \begin{pmatrix} Q_1^{-1} & \cdots & 0 \\ (0) & \ddots & (0) \\ 0 & \cdots 0 & Q_l^{-1} \end{pmatrix}$ are still polynomials.

$\mathbb{C}\langle X_1, \ldots, X_k \rangle$ becomes a *-algebra by anti-linear extension of $(X_{i_1} X_{i_2} \ldots X_{i_m})^* = X_{i_m} \ldots X_{i_2}\ X_{i_1}$.

Let q be in $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ having linearization $L_q := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix}$.

Then $\begin{pmatrix} 0 & u & v^* \\ u^* & (0) & Q^* \\ v & Q & (0) \end{pmatrix}$ is a linearization of $q + q^*$.

In particular, if $p = p^*$ and if $L_{p/2} := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix}$ is a linearisation of $p/2$, then $\begin{pmatrix} 0 & u & v^* \\ u^* & (0) & Q^* \\ v & Q & (0) \end{pmatrix}$ is a **selfadjoint** linearization of p.

Moreover, since $\begin{bmatrix} (0) & Q^* \\ Q & (0) \end{bmatrix}^{-1} = \begin{bmatrix} (0) & Q^{-1} \\ (Q^*)^{-1} & (0) \end{bmatrix}$, the matrix $\begin{bmatrix} (0) & Q^* \\ Q & (0) \end{bmatrix}^{-1}$ has entries which are polynomials.

Thank you for your attention!