Eigenstate Thermalisation Hypothesis and Gaussian Fluctuations for Wigner matrices

joint with Giorgio Cipolloni (IST Austria) *and* Dominik Schröder (ITS-ETH)

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Universality and Integrability in Random Matrix Theory and Interacting Particle Systems (MSRI)

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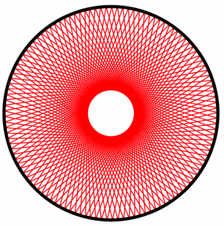
Dominik Schröder

Eigenstate Thermalisation Hypothesis
$=$ Quantum Unique Ergodicity

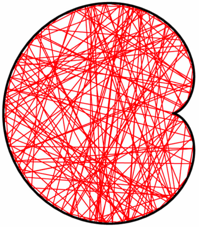
Quantization of classical systems: $p \to -i\hbar\nabla_x$

Motto:

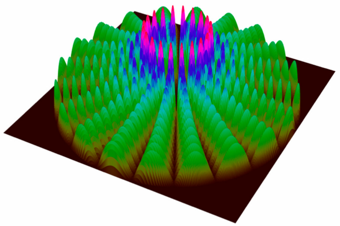
Eigenfunctions of the quantization of a chaotic classical dynamics are uniformly distributed.



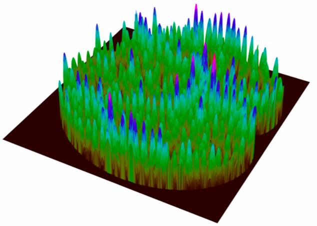
Regular (integrable) billiard



Chaotic billiard



Wavefunctions with symmetries



Chaotic wavefunctions

Most prominent example:

ψ_i: efn's of Laplace-Beltrami operator on a surface with ergodic geodesic flow, then

$$\langle \psi_i, A\psi_j \rangle \to \delta_{ij} \int_{S^*} \sigma(A), \qquad i,j \to \infty$$

holds for any appropriate pseudo-differential operator A with symbol $\sigma(A)$ (defined on the unit tangent bundle).

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Quantum Unique Ergodicity (QUE) conjecture (Rudnick, Sarnak 1994): it holds for all pairs.

Only special cases are proven on arithmetic surfaces (Lindenstrauss 2006), (Soundararajan 2010)

Convergence rate in Quantum Ergodicity

$$\langle \psi_i, A\psi_j \rangle \to \delta_{ij} \int_{S^*} \sigma(A), \qquad i,j \to \infty$$

Physics prediction for generic systems (Feingold, Peres 1986), (Eckhardt et al. 1995)

$$\mathrm{Var}\left[\langle \psi_i, A\psi_i \rangle\right] \sim \text{(local ev. spacing)}$$

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Much slower (log) decay is proven in averaged sense (Zelditch 1994), (Schubert 2006): optimal for highly degenerate spectrum.

Polynomial decay for special arithmetic surfaces (Luo, Sarnak 1995), linear maps on the torus (Marklof, Rudnick 2000).

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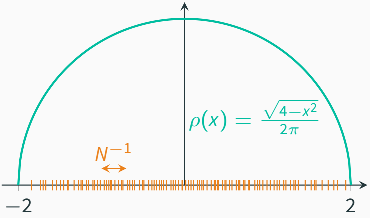

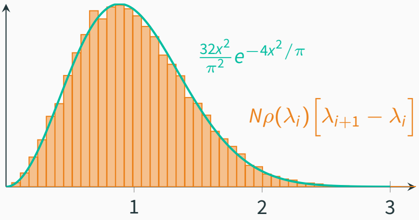

semicircular density of states ρ; Bulk level spacing $\sim N^{-1}$

Histogram of rescaled gaps and Wigner surmise

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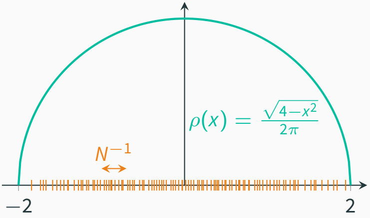

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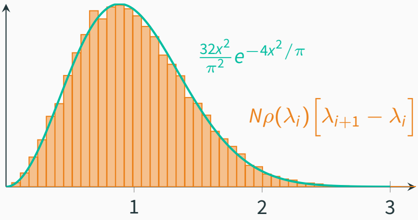

Histogram of rescaled gaps and Wigner surmise

Wigner's revolutionary observation: the gap statistics is very robust, it depends only on the symmetry class (hermitian or symmetric), independent of the distribution.

Formulated as the Wigner-Dyson-Mehta conjecture in 60's, proven around 2010.

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Extension of Wigner's vision to Quantum Chaos: Random matrices model chaotic quantum systems, hence QUE is expected to hold for Wigner matrices with optimal speed.

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Theorem (Cipolloni., E., Schröder 2020)

For the orthonormal eigenvectors $\mathbf{u}_i$ of an $N \times N$ Wigner matrix W and for any bounded deterministic observable (matrix) A

$$\max_{i,j} \left| \langle \mathbf{u}_i, A\mathbf{u}_j \rangle - \delta_{ij} \langle A \rangle \right| \lesssim \frac{N^\epsilon}{\sqrt{N}},$$

with very high probability, where $\langle A \rangle := \frac{1}{N} \operatorname{Tr} A$.

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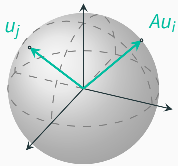
Eigenbasis $\{\mathbf{u}_i\}$ is asymptotically orthogonal to $\{A\mathbf{u}_j\}$ for $\langle A \rangle = 0$

As if $\mathbf{u}_i$ and $A\mathbf{u}_j$ were independently distributed ℓ^2-bounded N-vectors.



Two basic methods in random matrix theory:

Resolvent method and Dyson Brownian Motion (DBM)

Comparison with previous results

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Previous results:

- $A = |\mathbf{q}\rangle\langle\mathbf{q}|$ rank-1 observable = complete delocalization of evectors, $|\langle \mathbf{u_i}, \mathbf{q} \rangle| \lesssim N^{-1/2}$ [Erdős et al. (2009), Knowles-Yin (2013), Bloemendal et al. (2014)][Resolvent method]
- $\langle \mathbf{u}_i, A\mathbf{u}_i \rangle \to \langle A \rangle$ in probability for each $\mathbf{u}_i$ [Bourgade-Yau (2017)] [DBM]
- Simultaneously in i and j [in the bulk] — proven only for Wigner matrices with large (almost $\mathcal{O}(1)$) Gaussian component [Bourgade-Yau-Yin (2020)] [DBM]

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Novelties of our result: [Resolvent method]

- Optimal $N^{-1/2}$ speed of convergence. In physics: Eigenstate Thermalisation Hypothesis
- Limit is controlled in very high probability, and thus simultaneous in i, j.
- Holds uniformly in the entire spectrum (bulk, edge, intermediate regimes)

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These are LLN-type results. Next: What about CLT for $\sqrt{N}\big[\langle \mathbf{u}_i, A\mathbf{u}_i \rangle - \langle A \rangle\big]$?

Averaged CLT for overlaps

CLT (central limit theorem) for $\sqrt{N}\big[\langle \mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle\big]$ can be asked in two senses.

First, we proved CLT after averaging in the index i

Theorem (Cipolloni, E., Schröder 2020)

For any bounded deterministic matrix A, $i_0 \in [\delta N, (1-\delta)N]$ (i.e. bulk) and for any $K \geq N^\epsilon$

$$\frac{1}{\sqrt{2K}} \sum_{|i-i_0|\leq K} \sqrt{N}\Big[\langle \mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle\Big] \overset{m}{=} \mathcal{N}\Big(0, \langle \mathring{A}\mathring{A}^*\rangle\Big) + \mathcal{O}(N^{-\epsilon'})$$

in the sense of moments, where $\mathring{A} := A - \langle A\rangle$ is the traceless part of A.

Similar result holds at the edge with a variance $\frac{\sqrt{2}}{3}\langle \mathring{A}\mathring{A}^*\rangle$.

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$\Longrightarrow$ Indication that $\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle$, $\langle \mathbf{u}_j, \mathring{A}\mathbf{u}_j\rangle$ are asymptotically independent for $i \neq j$.

This CLT is a special case of our general functional CLT: $\langle f(W)A\rangle \approx \mathcal{N}$ for any fn. of the Wigner matrix W; unlike usual tracial CLT in random matrices, this involves eigenvectors as well!

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Second, CLT for each $\langle \mathbf{u}_i, A\mathbf{u}_i\rangle$ without averaging?

CLT for individual overlaps

Using the spectral decomposition $A = \sum_k a_k |\mathbf{q}_k\rangle\langle\mathbf{q}_k|$, our ETH proves

$$\left|\langle \mathbf{u}_i, A\mathbf{u}_j\rangle - \delta_{ij}\langle A\rangle\right| = \left|\frac{1}{N}\sum_k a_k \Big(N\langle \mathbf{u}_i, \mathbf{q}_k\rangle\langle \mathbf{q}_k, \mathbf{u}_j\rangle - \delta_{ij}\Big)\right| \leq \frac{N^\epsilon}{\sqrt{N}}, \qquad \text{w.h.p.}$$

Key message: The fluctuations of $\langle \mathbf{u}_i, \mathbf{q}_k\rangle\langle \mathbf{q}_k, \mathbf{u}_j\rangle$ for different k's are so strongly asymptotically independent that their average follow the CLT-like $1/\sqrt{N}$ scaling. CLT??

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We proved the corresponding CLT for full rank observables:

Theorem (Cipolloni, E., Schröder 2021)

For the bulk eigenvectors $\mathbf{u}_i$ of an $N \times N$ Wigner matrix W and for any bounded deterministic hermitian observable (matrix) A with $\langle \mathring{A}^2\rangle \ge c$ it holds:

$$\sqrt{\frac{N}{2\,\langle \mathring{A}^2\rangle}}\,\big[\langle \mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle\,\big] \to \mathcal{N}(0,1)$$

in the sense of moments, where $\mathring{A} := A - \langle A\rangle$ is the traceless part of A.

To prove this theorem we need DBM methods.

Comparison with previous/related results

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- Finite rank: Joint (squared) Gaussianity for finitely many **u**'s and **q**'s [Marcinek-Yau (2020)] [DBM]

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Related independent result:

Gaussianity of $\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle$ for the special case $A = \sum_{j\in I} |\boldsymbol{q}_j\rangle\langle \boldsymbol{q}_j|$ with $N^{\epsilon} \le |I| \le N^{1-\epsilon}$ and $\boldsymbol{q}_j$ orthonormal, i.e. A is low rank. [Benigni-Lopatto (2021)] [DBM]

Proof of ETH (Resolvent method)

Spectrally averaged overlaps ⇒ Resolvents

For traceless observables, $\langle A \rangle = 0$, define the averaged overlap

$$\Lambda^2 := \max_{i_0, j_0} \frac{1}{N^{2\epsilon}} \sum_{|i-i_0|<N^\epsilon} \sum_{|j-j_0|<N^\epsilon} N|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2$$

Recall: we expect $|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle| \lesssim N^{-1/2}$ for the eigenvector overlaps. (ETH)

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By spectral decomposition with the resolvent $G(z) := (W-z)^{-1}$,

$$\left\langle \Im G(E+\mathrm{i}\eta)\, A\, \Im G(E'+\mathrm{i}\eta')\, A\right\rangle = \frac{1}{N^2}\sum_{ij} N|\langle \mathbf{u}_i, A\mathbf{u}_j\rangle|^2 \frac{\eta}{|\lambda_i - E|^2+\eta^2}\ \frac{\eta'}{|\lambda_j - E'|^2 + (\eta')^2}$$

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By eigenvalue rigidity ($\lambda_i \approx \gamma_i$, the i-th quantile of ρ_{sc})

$$\frac{1}{N}\sum_i \frac{\eta}{|\lambda_i - E|^2+\eta^2} \approx \int \frac{\eta}{|x-E|^2+\eta^2}\rho_{\mathrm{sc}}(x)\,\mathrm{d}x =: \rho(E+\mathrm{i}\eta)$$

in the regime $\eta \gg (N\rho)^{-1} \sim$ level spacing, so we have

$$\Lambda^2 \sim \sup_{E,E'\in[-2,2]} (\rho\rho')^{-1}\left\langle \Im G(E+\mathrm{i}\eta) A \Im G(E'+\mathrm{i}\eta') A\right\rangle,$$

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GOAL: $\langle G A G A\rangle \lesssim 1 \Longrightarrow$ This would give ETH.

What is a local law? Deterministic approximation of random resolvents

For a Wigner matrix W we have the norm bound

$$\|G(z)\| = \left\| \frac{1}{W - E - \mathrm{i}\eta} \right\| \leq \frac{1}{\eta}. \tag{1}$$

The bound in (1) is deterministic. Can we get a better control on $G(z)$ using the randomness?

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Local law for single G:

$$\langle G(z) \rangle = m(z) + \mathcal{O}\left(\frac{1}{N\eta}\right), \qquad m(z) := \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - x^2}}{x - z}\, \mathrm{d}x = O(1)$$

with very high probability.

What is a local law? Deterministic approximation of random resolvents

For a Wigner matrix W we have the norm bound

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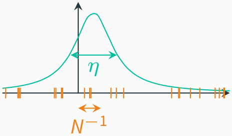
with very high probability. Why are local laws useful?

For $\eta \gg N^{-1}$ the local law gives info about eigenvalues of W on scale η around E:

$$\frac{1}{N} \sum_i \frac{\eta}{(\lambda_i - E)^2 + \eta^2} = \langle \Im G \rangle \approx \Im m$$



$\Longrightarrow$ Rigidity of eigenvalues:

$$|\lambda_i - \gamma_i| \leq \frac{N^{\epsilon}}{N},$$

with high probability, where γ_i is the i-th quantile of the semicircular law (deterministic)

G as a matrix

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Effective gain of size $\sqrt{\eta}$ thanks to $\langle A \rangle = 0$.

Multi-G local laws with traceless observables

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With deterministic matrices in between

$$\langle GAGB \rangle \approx m^2 \langle AB \rangle + \frac{m^4}{1-m^2} \langle A \rangle \langle B \rangle .$$

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Effective gain of size $(\sqrt{\eta})^2$ thanks to twice $\langle A \rangle = 0$. Expect a general pattern

Proof of local law: Resolvent method

Using $WG = zG + I$ and its renormalization $\underline{WG} := WG + \langle G \rangle G$, we have

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$$\mathbf{E} \, |\langle \underline{WGAGA} \rangle|^p = \sum N^{-\text{power}} \langle GAGG^*AA^*GGAG^* \dots \rangle \langle GG^*AA^*GAGAG^* \dots \rangle \dots$$

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Mechanism: Whenever an effective A is lost, we also have fewer G —- they balance out.
Combinatorics needs delicate bookkeeping by Feynman diagrams – main technical work.

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Usually Ward or Schwarz closes it: reducing the number of G's at expense of $1/\eta$. But both procedures lose A, we can't afford it.

We close the hierarchy by the quantity Λ.

Proof of Gaussian fluctuation (via DBM)

DBM for eigenvectors

GOAL: Let $\mathbf{u}_i$ be the eigenvectors of a Wigner matrix W, then

$$\mathbf{E}\left[\sqrt{\frac{N}{2\langle \mathring{A}^2\rangle}}\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle\right]^n \to (n-1)!!\mathbf{1}(n \text{ even}), \qquad \mathring{A} = A - \langle A\rangle .$$

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We do it dynamically:

$$\mathrm{d}W_t = \frac{\mathrm{d}\widehat{B}_t}{\sqrt{N}}, \qquad W_0 = W. \tag{2}$$

The flow (2) adds a Gaussian component of size $\sqrt{t}$ to W_0.

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The flow (2) induces the Dyson Brownian Motion (DBM) for eigenvalues and eigenvectors:

$$\begin{aligned}
\mathrm{d}\lambda_i(t) &= \frac{dB_{ii}(t)}{\sqrt{N}} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}\,\mathrm{d}t \\
\mathrm{d}\mathbf{u}_i(t) &= \frac{1}{\sqrt{N}}\sum_{j\neq i}\frac{\mathrm{d}B_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}\mathbf{u}_j(t) - \frac{1}{2N}\sum_{j\neq i}\frac{\mathbf{u}_i(t)}{(\lambda_i(t)-\lambda_j(t))}\,\mathrm{d}t.
\end{aligned} \tag{3}$$

Very heuristic sketch of the proof I

- Starting from the DBM we write a system of diff. eq. for the n-th moments of the overlaps, e.g.

$$\mathbf{E}\left[|\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle|^2 \; |\langle \mathbf{u}_j, \mathring{A}\mathbf{u}_j\rangle|^4 \middle| \boldsymbol{\lambda}\right], \qquad n = 6.$$

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- Closed equation for a certain lin. combination of overlaps f_t [Bourgade-Yau-Yin (2020)]:

$$\partial_t f_t = \mathcal{L}(t) f_t. \tag{4}$$

Example for $n = 2$:

$$f_t = \mathbf{E}\left[2|\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_j\rangle|^2 + \langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle \, \langle \mathbf{u}_j, \mathring{A}\mathbf{u}_j\rangle \, \middle| \boldsymbol{\lambda} \right].$$

More precisely, $f_t = f_t(i,j)$ is a function of ”two-particle configurations” on $\mathbf{Z}$.

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$$\partial_t f_t = \mathcal{L}(t) f_t. \tag{4}$$

Example for $n = 2$:

$$f_t = \mathbf{E}\left[2|\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_j\rangle|^2 + \langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle \, \langle \mathbf{u}_j, \mathring{A}\mathbf{u}_j\rangle \, \middle| \boldsymbol{\lambda}\right].$$

More precisely, $f_t = f_t(i,j)$ is a function of "two-particle configurations" on $\mathbf{Z}$.

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is a generator of Markov process on n particles.

$\mathcal{L}_r$ acts on the location index of the r-th particle; it has a kernel

$$\frac{1}{N(\lambda_i - \lambda_j)^2} \sim \frac{N}{|i-j|^2}.$$

Very heuristic sketch of the proof I

- Starting from the DBM we write a system of diff. eq. for the n-th moments of the overlaps, e.g.

$$\mathbf{E}\left[|\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle|^2 \, |\langle \mathbf{u}_j, \mathring{A}\mathbf{u}_j\rangle|^4 \middle| \boldsymbol{\lambda}\right], \qquad n = 6.$$

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Note that this is the discretisation of the $\sqrt{-\Delta} = |p|$ operator in $1d$
$\Longrightarrow$ (4) is a (discrete) heat equation with fractional Laplacian

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$$\frac{1}{\eta}\prod_{i=1}^{n}\left(1 - e^{-\eta|p_r|}\right) \qquad \left(\sim \eta^{n-1}\prod_{r=1}^{n}|p_r| \quad \text{morally}\right)$$

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  The replacement is possible on the level of Dirichlet form, $D(f) := \langle f, \mathcal{L} f \rangle$.
- After replacement, technically we rely on
  (i) the energy method for DBM [Marcinek-Yau (2020)] analysing

$$\partial_t \|f_t\|_2^2 = -2 D_t(f_t) \leq 0.$$

  (ii) local laws for $\langle GAGA \ldots \rangle$ with $\langle A \rangle = 0$ [Cipolloni, E., Schröder (2021)].

Summary

We proved:

- **Eigenstate Thermalisation Hypothesis for Wigner matrices:** eigenvector overlaps with deterministic A are $\lesssim N^{-1/2}$.
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Main technical steps:

- Dramatically improved local law for traceless observables.
- Closing the multi-G local law hierarchy with Λ.
- Diagrammatic expansion to extract $\langle A \rangle = 0$ optimally.
- Energy estimates for multi indexed DBM.

THANK YOU VERY MUCH FOR YOUR ATTENTION!