Delocalization of random band matrices in high dimensions: spontaneous renormalization

Jun Yin

With Fan Yang and Horng-Tzer Yau

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Matrix in high dimensions

Regular Matrix

Matrix H has entires $H_{i,j}$, with indices $i, j \in \mathbb{N}$, (e.g. $i = 4,\ j = 5$).

Matrix in d- dimension

Matrix H has entires $H_{x,y}$, with indices $x, y \in \mathbb{N}^d$,
(e.g. $d = 2, x = (5, 2), y = (4, 5)$).

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Example.

For fixed L, the transition matrix of simple random walk in $(\mathbb{Z}_L)^d$ is

$$p_{xy} = \frac{1}{2d}\mathbf{1}(\|x - y\| = 1), \quad x, y \in \mathbb{Z}_L^d, \quad \mathbb{Z}_L = \mathbb{Z}/L\mathbb{Z}$$

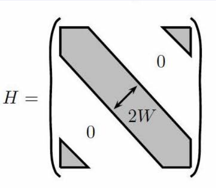
(i.e., A d-dimensional box with side L and periodic boundary condition)

Random band matrices in d-dimension

$H = (H_{xy})_{x,y \in Z_L^d}$, with centered, independent entries up to symmetry $H = H^\dagger$, with band width $W \ll N$:

$$H_{xy} = 0, \qquad |x - y| > W$$

$$H_{xy} \sim W^{-d/2}, \quad |x - y| \le W.$$



The translation-invariant distribution property:

$$H_{xy} \sim H_{x+a,y+a}, \quad \forall x, y, a$$

Eigenvectors of band matrices.

Simple example

Let u_k be one of the eigenvectors of H as $Hu_k = \lambda_k u_k$. $(\lambda_1 \le \lambda_2 \le \cdots)$
Clearly, if H is diagonal matrix, then u_k only has one non-zero entry.

Eigenvectors of band matrices.

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How about general band width W case? For $u_k(\alpha)$, $\alpha = 1, 2, \cdots 2000$:

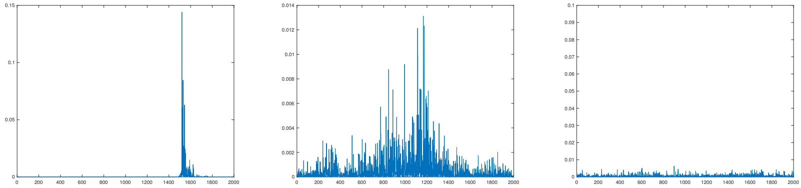

Figure: $|u_k(\alpha)|^2$. $d = 1$, $L = 2000$, $W = L^{1/4}$, $W = L^{1/2}$ and $W = L^{3/4}$.

Eigenvectors of band matrices.

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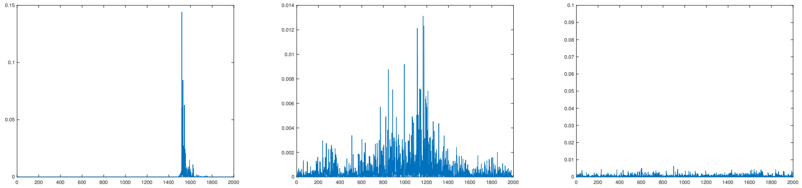


Figure: $|u_k(\alpha)|^2$. $d = 1$, $L = 2000$, $W = L^{1/4}$, $W = L^{1/2}$ and $W = L^{3/4}$.

The key result of this talk: for $d \geq 7$, $W \geq L^{\delta}$, it is the third case.

Eugene Wigner 's gifts

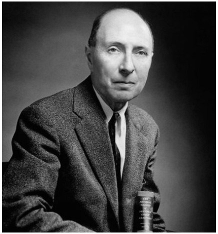
Wigner ensembles:

$$H = (h_{ij})_{i,j=1}^{N}, \quad \mathbb{E}h_{jk} = 0, \quad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}.$$

h_{ij} are independent (not necessarily normal).

Wigner's vision concerning the universality of random matrix statistics

The local spectral statistics of highly correlated quantum systems are given by random matrix statistics.
Random matrix statistics are "universal" laws for highly correlated systems.



Universality (Wigner)

The local eigenvalue / eigenvector statistics of Wigner ensembles are the same as those of the Gaussian ensembles.

Universality phenomenon of Wigner matrices are very successfully proved.

Universality conjectures (theorems) of Wigner matrices

- Wigner-Dyson-Mehta conjecture, i.e., bulk universality.

  Erdős, Bourgade, Yau, Y. (2017) , Schlein, Tao, Vu, Johansson, etc.

  $\mathbb{P}\left(\lambda_i \sim E + \frac{x_1}{N}, \lambda_j \sim E + \frac{x_2}{N}\right)$ are independent of the matrix law.

- Edge universality.

  Lee, Y. (2013), Erdős, Yau, Tao, Vu, Soshnikov, etc.

- Eigenvector universality.

  Bourgade, Yau (2017), Antti, Y., Schlein, Erdős, Tao, Vu, etc.

Universality (mean-field)

The Wigner ensemble is a special class of mean-field models.

Other mean-field models

- sparsity : the adjacency matrices of Erdős-Rényi graphs.
- correlations : d-regular graphs or other models with summable decay of correlation functions.
- models of type $A + B$ with A deterministic.

B. Landon, J.Y. Huang, Bauerschmidt; A. Knowles et al.; Bourgade et al.; Erdős et al. ; Guionnet et al.; Zeitouni et al. .

Wigner's grand vision

Wigner ensemble $\Longrightarrow$ Highly correlated system ?

Highly correlated system, e.g., nucleus is a non-mean-field system.

The fundamental question concerning Wigner's grand vision after the Wigner-Dyson-Mehta conjecture

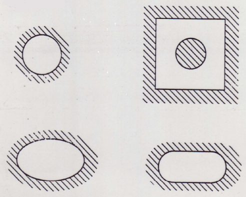
Why can mean-field Wigner ensembles model non-mean field systems?
Hamiltonians in physics are typically short ranged.

Beyond mean-field

- Nucleus
- Random Schrödinger operators (Anderson's models)
- Random band matrices.

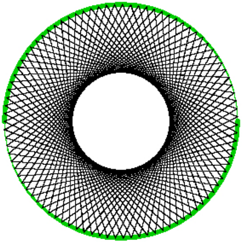
Integrability v.s. Chaos

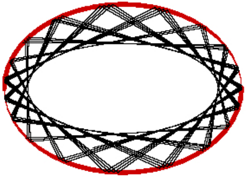
What will the billiard trajectories look like in these cases ?



Integrability v.s. Chaos

For circle case and ellipse case :





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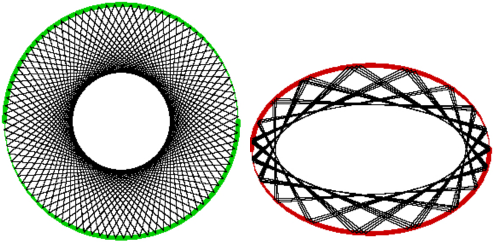
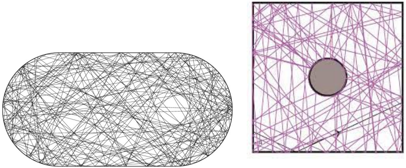


For stadium case and Sinai (hole) case :

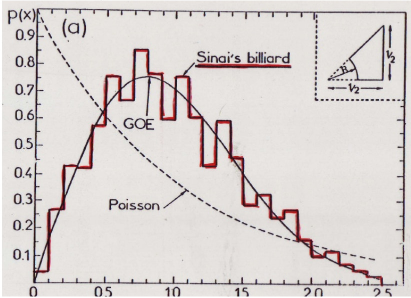


Quantum Chaos

Energy levels in quantum billiards: $H = -\Delta + V$, Δ is Laplace operator

$$H\psi_k := \lambda_k \psi_k$$

For the gaps between eigenvalues:



Integrability v.s. Chaos

Conjectures

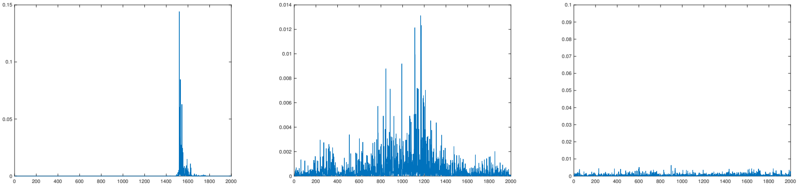
Berry-Tabor conjecture (1977): If the billiard trajectories are integrable, the eigenvalue spacings statistics is given by the Poisson distribution $e^{-x}dx$.

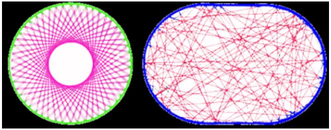
Bohigas-Giannoni-Schmit conjecture (1984): If the billiard is chaotic, the eigenvalue spacings statistics is given by the GOE (Wigner).

Recall Wigner's grand vision

Highly correlated system?
Non mean field model ?
Nucleus?
As long as they are chaotic, their local statistics is given by the GOE.

Localization v.s. Delocalization





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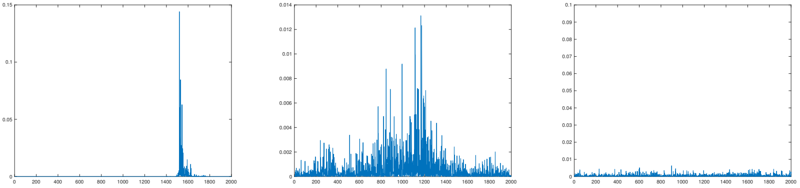

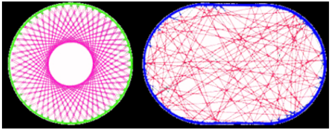

Conjectures:

Integrability: Eigenvectors are localized, and eigenvalues spacing statistics are Poisson.

Chaos: Eigenvectors are delocalized, and eigenvalues spacing statistics are GOE (regular random matrix) type.

Random matrix, Quantum chaos and Non-mean-field

Big picture:

Conjectures

Integrability: Eigenvectors are localized, and eigenvalues spacing statistics are Poisson.

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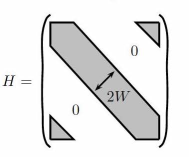
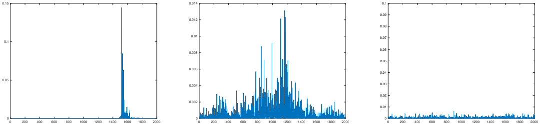
Random band matrix

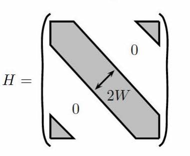


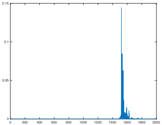


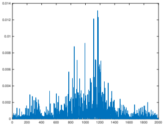
Conjecture: Transition occurs at some critical band width $W_c(L)$

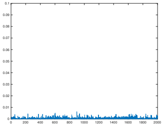
In the bulk, Localization+Poisson, $W \ll W_c$; Delocalization+GOE, $W \gg W_c$.

Random band matrix









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Based on numerics (Casati-Molinari-Izrailev '90; Feingold-Leitner-Wilkinson '91) and non-rigorous supersymmetric arguments (Fyodorov-Mirlin, 1991):

$$W_c(L) = \begin{cases} \sqrt{L} & \text{for } d = 1 \\ \sqrt{\log L} & \text{for } d = 2 \\ O(1) & \text{for } d \geq 3 \end{cases}$$

Results on this conjecture $(d = 1)$

Theorem (Bourgade-Yang-Yau-Y. (2018-2019))

Assume $W \gg L^{\frac{3}{4}}$ *and* $E \in (-2+\kappa, 2-\kappa)$.

- *(Bulk universality)* $\{L(\lambda_k - E)\}_k \to$ *GOE.*
- *(Delocalization) For bulk eigenvector,* $\|\mathbf{u}\|_\infty \leq L^{-\frac{1}{2}+\epsilon}$.
- *(QUE) For any* $|I| > W$, $\left|\sum_{\alpha \in I} |\mathbf{u}_\alpha|^2 - |I|/L\right| \leq L^{-\epsilon}|I|/L$.

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Localization, Gaussian band matrix

For $W \ll L^{\frac{1}{8}}$ Schenker '09.

For $W \ll L^{\frac{1}{7}}$ Peled- Schenker-Shamis-Sodin '17 .

Poisson eigenvalues statistics are still unknown.

A quick review on other results

Delocalization (d=1)

$W \gg L^{6/7}$	Erdős and Knowles
$W \gg L^{4/5}$	Erdős, Knowles, Yau and Y.
$W \gg L^{7/9}$	Y. He and M. Marcozzi
$W \gg L^{3/4}$	Bourgade, Yau, Yang and Y' 2017-18

Eigenvalues at the edge of spectrum (d=1)

$W \sim L^{5/6}$	S. Sodin (2008)

A quick review on some other results

Supersymmetry

When the entries are Gaussian with some specific covariance profiles, e.g. $\mathbb{E}(|H_{ij}|^2) = (-W^2\Delta + 1)^{-1}_{ij}$ supersymmetry techniques can be applied:

$W \sim L$	T. Shcherbina 2014
$W \gg L^{6/7}$	Bao and Erdős 2017
$W \gg L^{1/2}$	T. Shcherbina and M. Shcherbina 2017, 2019
$W \gg L^{1/2}$	M. Disertori, M. Lohmann, S. Sodin 2018

(Expected Density of states)

Large finite W M. Disertori, H. Pinson and T. Spencer 2011

Delocalization of high dimension random band matrix

Previous results

$$W = L^a$$

$a > \frac{6}{d+6}$ L. Erdos and A . Knowles

$a > \frac{d+2}{2d+2}$ L. Erdos, A. Knowles, Yau and Y.

$a > \frac{d+1}{2d+1}$ Y. He and M. Marcozzi

$a > \frac{2}{2d+2}$ Yang and Y.

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Theorem (Yang, Yau and Y. (2021) - main result)

For any $a > 0$ and $d \geq 7$, arXiv:2104.12048, arXiv: 2107.05795

They are the first two parts of this series.

Yang - Yau - Y. (2021)

With high prob, for any $\gamma, \ell, K > 0$, define

$$\mathcal{B}_l^{\gamma,K} := \left\{ \alpha : \inf_{x_0} \sum_x |\mathbf{u}_\alpha(x)|^2 \exp\left[\left(\frac{|x-x_0|}{\ell} \right)^\gamma \right] \le K \right\}$$

$\sim$ eigenvectors exponentially localized in balls of radius ℓ

For any $\ell \le L^{1-c}$ for some $c > 0$,

$$\lim_{N\to\infty} \frac{|\mathcal{B}_l^{\gamma,K}|}{N} = 0,$$

i.e. the density of exponentially localized eigenvectors at scale ℓ vanishes (weak delocalization).

Note:
The first two parts of this proof focused on the complex Gaussian and $d \ge 8$.
With a mild improvement, the third part will prove the general case and $d \ge 7$.

Quantum diffusion conjecture

Basic tool is resolvent: $G(z) = (H - z)^{-1}, \quad z \in \mathbb{C}.$

Quantum diffusion conjecture

For $\eta \gg L^{-d}$, $x \neq y$, $z = E + i\eta$

$$|G_{xy}(z)|^2 \sim \frac{1}{L^d} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^d} \frac{e^{ip\cdot(x-y)}}{\eta + (Wp)^2} \sim \left[\frac{1}{\eta - W^2\Delta}\right]_{xy} := B_\eta(x-y),$$

$B_\eta(x-y)$ is

the Green's function of a random walk with step length W at the time η^{-1}.

Theorem (Yang, Yau, Y' (2021))

Quantum diffusion conjecture holds up to the time $\sim W^2/L^2$, i.e., the time that $|G_{xy}|$ diffuses to the boundary. Particularly,

$$|G_{xy}|^2 \sim \frac{1}{W^2|x-y|^{d-2}}, \quad \operatorname{Im} z \sim W^2/L^2$$

Heuristic explanation

It is one of the heuristic explanation for critical bandwidth W^c for all $d \in \mathbb{N}$.

Thouless theory

Delocalization - localization phase transition occurs at

$$\text{Thouless energy} \sim \text{typical eigenvalue gap} \sim L^{-d}$$

$$\text{Thouless energy} = (\text{Thouless time})^{-1}$$

Thouless time:
The time of the resolvent effectively diffuses to the boundary: L^2/W^2.

$$(\text{typical eigenvalue gap})^{-1} \sim \text{Thouless time}$$
$$\iff L^{2-d} \sim W^2 \implies d > 2 \text{ phase transition occurs at finite large } W$$

For $d = 1$, the phase transition occurs at $W = \sqrt{L}$.

Correction on Quantum diffusion.

Yang-Yau-Y. (2021) Sharp estimate on resolvents

For any positive integer M and for any $x, y \in \mathbb{Z}_L^d$, $S_{xy} = \mathbb{E}|H_{xy}|^2$,

$$\mathbb{E}|G_{xy}|^2 = \left[\frac{|m_{sc}|^2}{1-\left(1+\Sigma^{(M)}\right)|m_{sc}|^2 S}\right]_{xy} + \mathrm{O}(W^{-Md/2}),$$

holds for $\eta \geq W^{2+\varepsilon}/L^2$ with $\varepsilon > 0$ (There are other error terms omitted).

Self-energy Renormalization

The self-energy $\Sigma^{(M)}(z) = \sum_{n\leq M} \mathcal{E}_n(z)$; $\{\mathcal{E}_n\}$ is the $n-th$ order correction. It satisfies the cancellation property (sum zero)

$$\left|\sum_{x\in\mathbb{Z}_L^d} (\mathcal{E}_n)_{0x}(z)\right| \ll \sum_{x\in\mathbb{Z}_L^d} \left|(\mathcal{E}_n)_{0x}(z)\right|$$

Note: the factor is about η.

- In Fourier space,

$$\sum_x \mathbb{E}|G_{0x}(z)|^2 e^{\mathrm{i}p\cdot x} \sim \frac{1}{\eta + p \cdot \mathcal{D}_{eff}^{(M)}(z) p} + \text{ errors.}$$

where the effective diffusion matrix can be computed from $\Sigma^{(M)}$.

- Our result is roughly equivalent to that the unitary evolution $|e^{itH}\psi_0|^2$ induces a random walk up to the Thouless time $t \ll L^2/W^2$.

Renormalization

$$\mathbb{E}|G_{xy}|^2 = \left[\frac{|m_{sc}|^2}{1-\left(1+\Sigma^{(M)}\right)|m_{sc}|^2 S}\right]_{xy} + O(W^{-Md/2}),$$

Renormalization for electron (magnetic moment)

- QED + Expansion with Feynman graph (like Talor's expansion)
- Infrared divergence (Need cancellation)
- Renormalization + "correct" parameters. (Choose better parameters)
- Keep expansion + corrections. (See the cancellation)

Spontaneous renormalization

Renormalization for resolvent

- Expansion with RMT
- Infrared divergence: ($\|x-y\| \gg W$.)
- Renormalization (The parameters are fixed.)
- Spontaneous renormalization (Cancellation property must exists.)
- Keep expansion + corrections. (Obtain sum zero property)

Basic strategy

Step 1: choose decent L, W and η, such that in the expansion of

$$\sum_{xy} \mathbb{E}|G_{xy}|^2 = \sum_k \mathcal{A}_k$$

the term having $\sum_{xy} (\mathcal{E}_1)_{xy}$ is the leading term if it does not have sum zero property.

Step 2: With Ward's identity, $\mathbb{E}|G_{xy}|^2 \sim L^d \eta^{-1}$. It will implies the

$$\sum_{xy} (\mathcal{E}_1)_{xy} \ll \sum_{xy} |(\mathcal{E}_1)_{xy}|$$

Basic strategy

With

$$\sum_{xy} (\mathcal{E}_1)_{xy} \ll \sum_{xy} |(\mathcal{E}_1)_{xy}|$$

Step 3: Since $\mathcal{E}_1$ is the sum of finite graphs, so they must really cancel each other. It likes

$$a, b \in \mathbb{N}, \quad |a+b| \le 0.9 \implies a+b=0.$$

Step 4: Repeat Step 1.
But in this time, since we know the sum zero property of $(\mathcal{E}_1)$, we can choose some decent L, W and η, such that in the expansion of $\sum_{xy} \mathbb{E}|G_{xy}|^2$, the term having $\sum_{xy}(\mathcal{E}_2)_{xy}$ is the leading term.

Step 5 Repeat step 2 and 3 and go on.

Spontaneous renormalization

Interesting fact:

We don't know the exact local structures for this sum zero property, even for the simple case like $n = 6$. We tried matlab for help, but failed.

Thank you!