Towards KPZ Universality

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with
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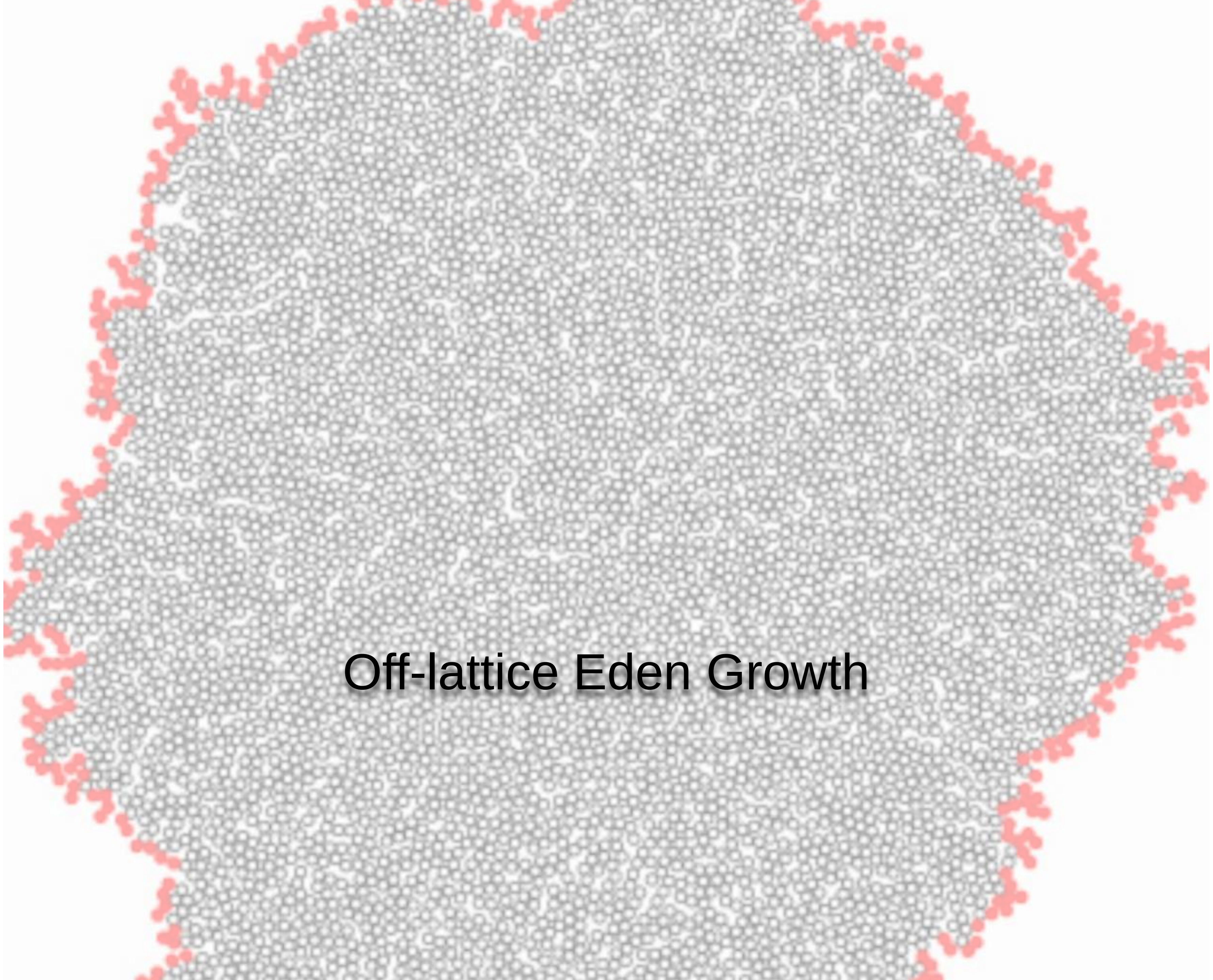
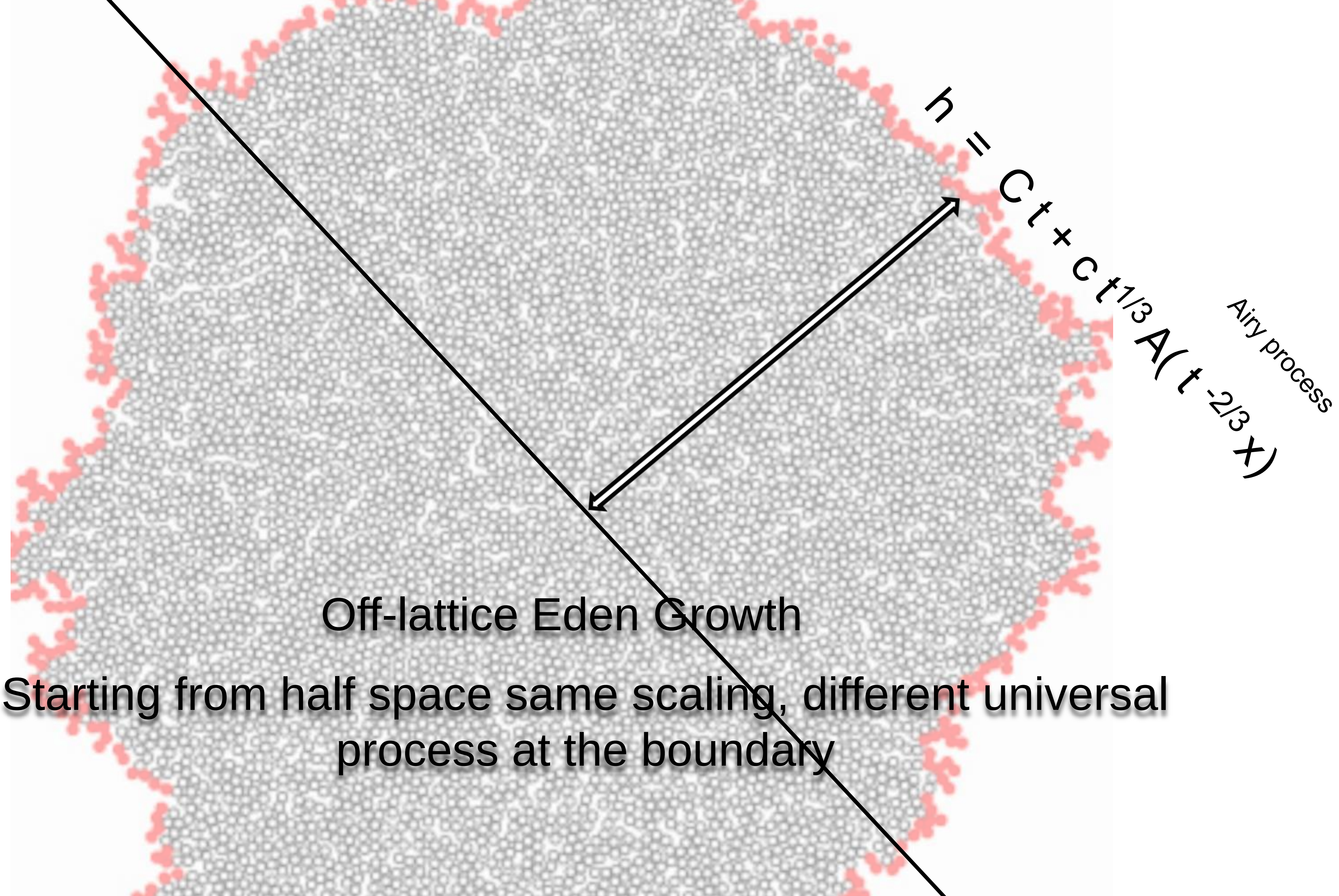
Plan of the talk

1. KPZ universality class
2. Solvability of TASEP & the KPZ fixed point
3. Convergence of KPZ eq/exclusion processes to fixed pt

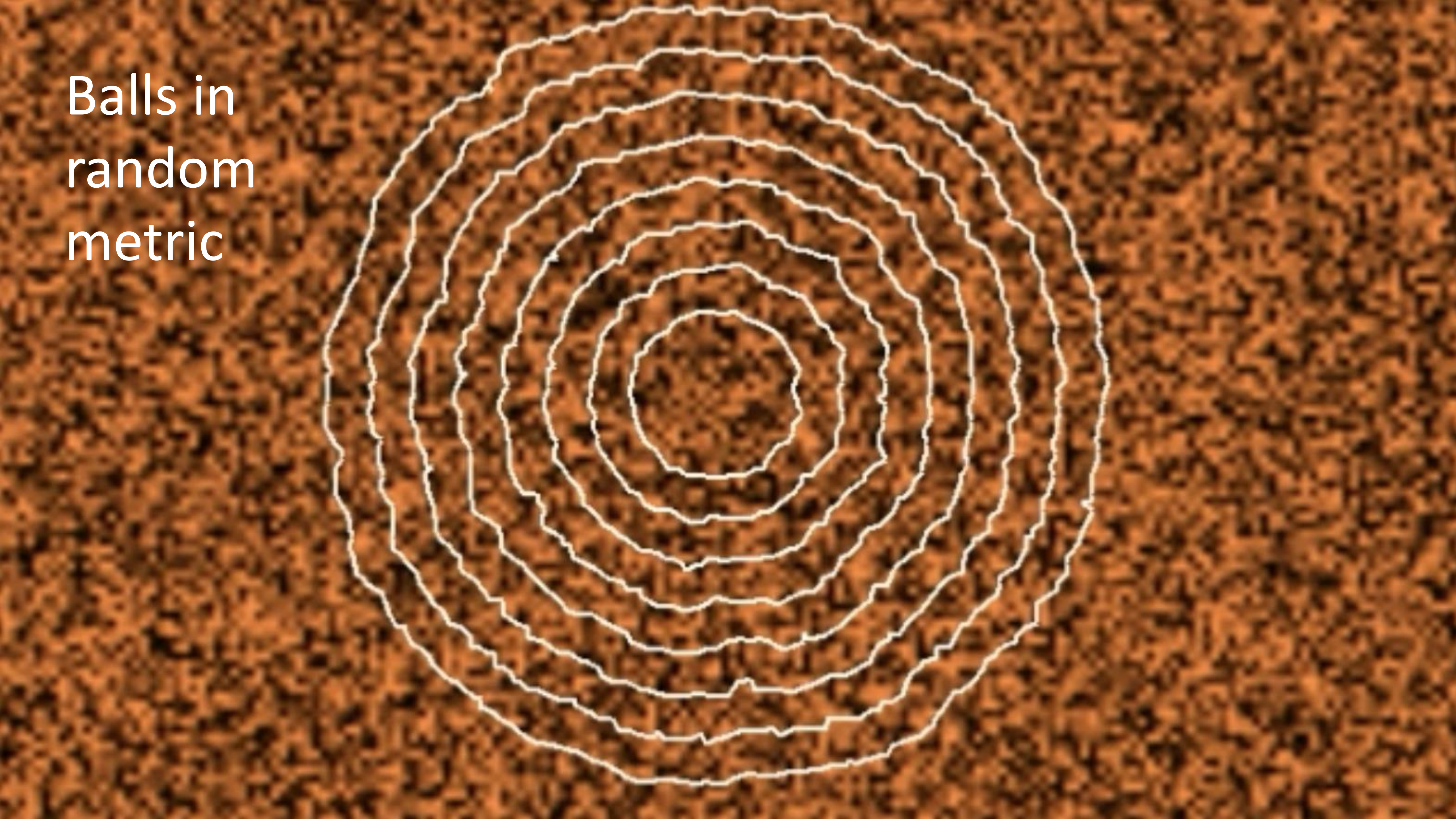

Off-lattice Eden Growth



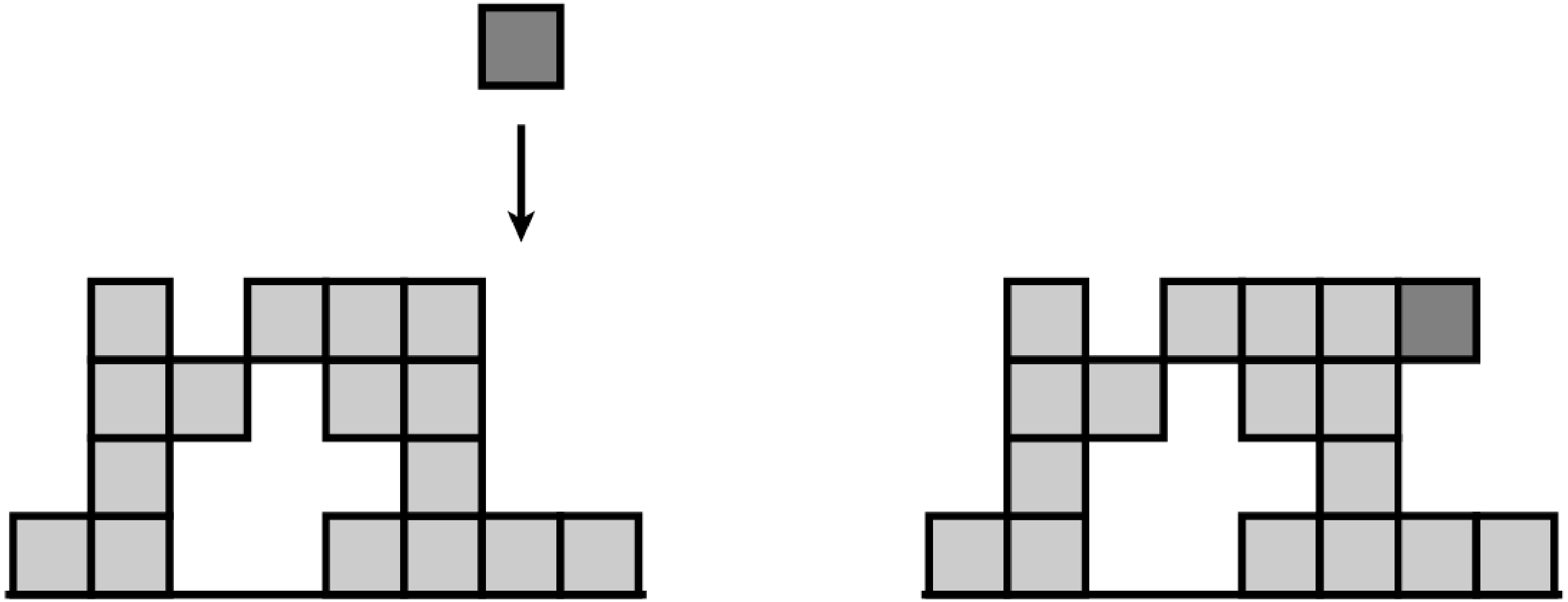
Off-lattice Eden Growth

Starting from half space same scaling, different universal process at the boundary

Balls in random metric



Directed Models: Ballistic aggregation



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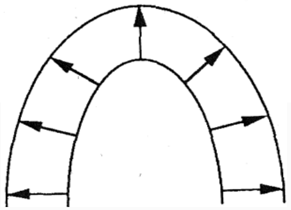
Kardar-Parisi-Zhang equation (d=1)

$$\partial_t h = \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\partial_x^2 h}_{\text{relaxation}} + \underbrace{\xi}_{\text{space-time white noise}}$$



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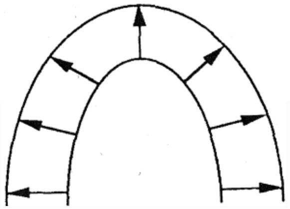
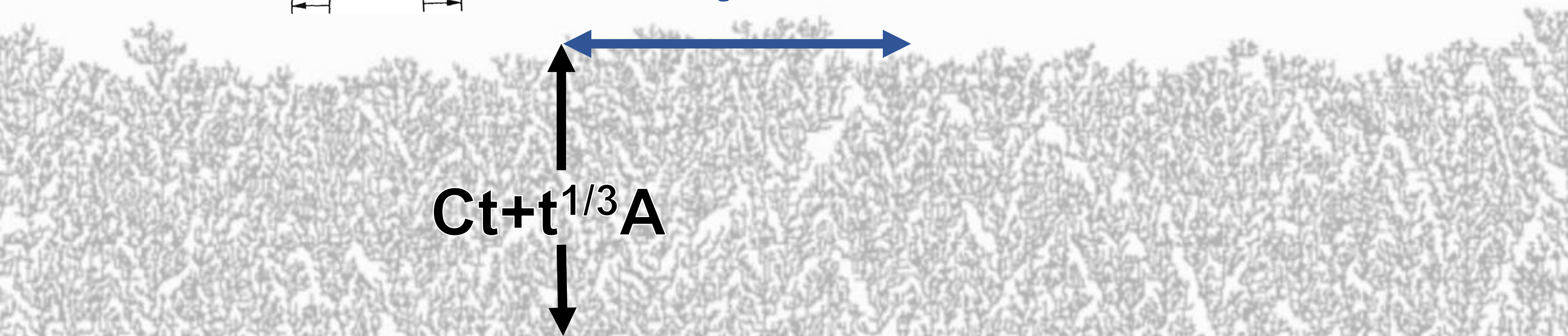
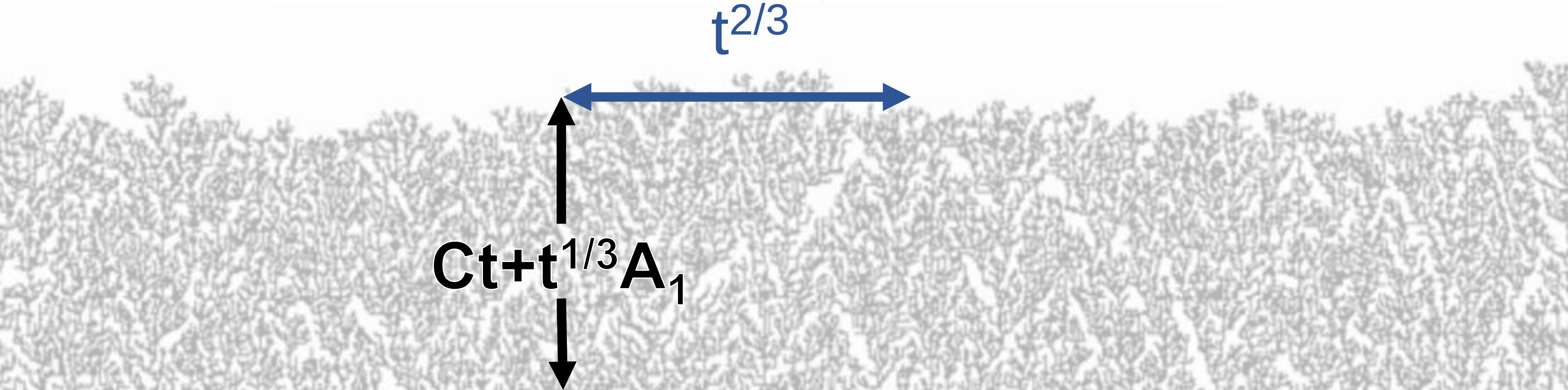




Kardar-Parisi-Zhang equation (d=1)

$$\partial_t h = (\partial_x h)^2 + \partial_x^2 h + \xi$$

1:2:3 Scaling $\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - c_\varepsilon t$



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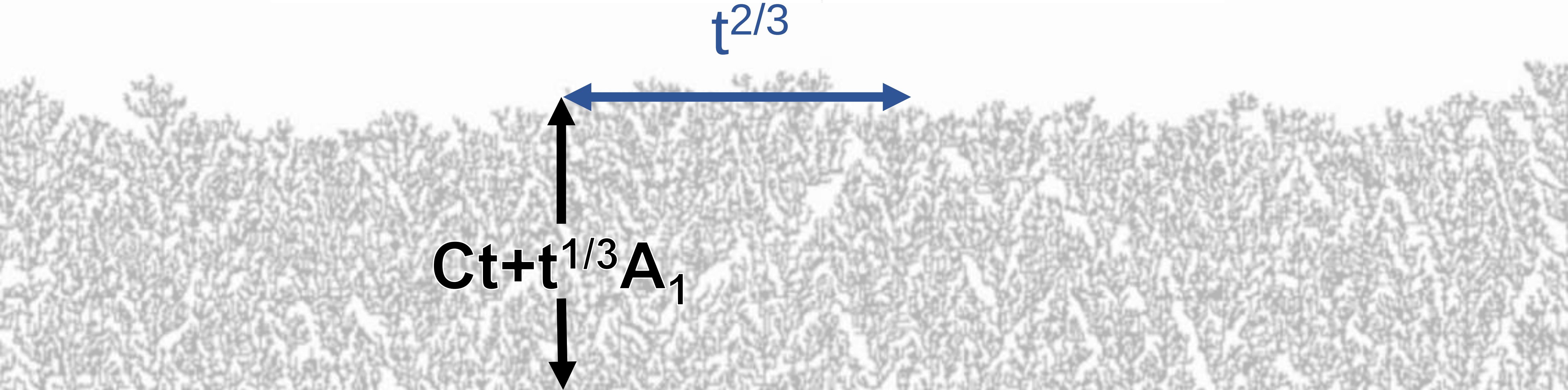


Kardar-Parisi-Zhang equation (d=1)

$$\partial_t h = (\partial_x h)^2 + \partial_x^2 h + \xi$$

$$F(t,r) = \lim_{\varepsilon \to 0} P(\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - c_\varepsilon t \leq r)$$

$\phi = \partial_r^2 \log F$ satisfies KP (!) $\partial_r(\partial_t \phi + \frac{1}{2}\partial_r \phi^2 + \frac{1}{12}\partial_r^3 \phi) + \frac{1}{4}\partial_x^2 \phi = 0$

Tracy-Widom GUE (GOE) distributions are simply similarity solutions of KP (KdV)

$$F_{\mathrm{GUE}}(s) = e^{-\int_s^\infty dx\,(x-s)^2 q^2(x)} \qquad F_{\mathrm{GOE}}(s) = e^{-\frac{1}{2}\int_s^\infty dx\, q(x)} \sqrt{F_{\mathrm{GUE}}(s)}$$

$$\phi(t,x,r) = \partial_r^2 \log \mathbb{P}(h(t,x) \le r)$$

<u>Narrow wedge initial</u> $\phi(t,x,r) = t^{-2/3}\psi(t^{-1/3}(r + \frac{x^2}{t}))$

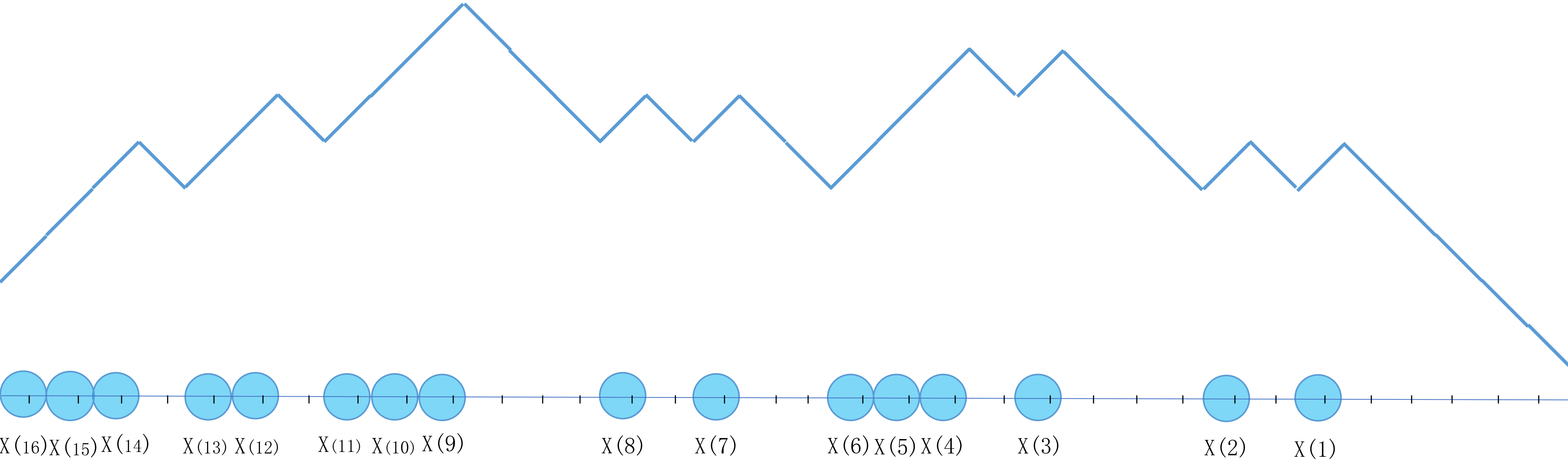
$\psi''' + 12\psi\psi' - 4r\psi' - 2\psi = 0$ $\quad \psi = -q^2$ gives Painleve II $q'' = rq + 2q^3$

<u>Flat initial data</u> $\phi(t,r) = (t/4)^{-2/3}\psi((t/4)^{-1/3} r)$

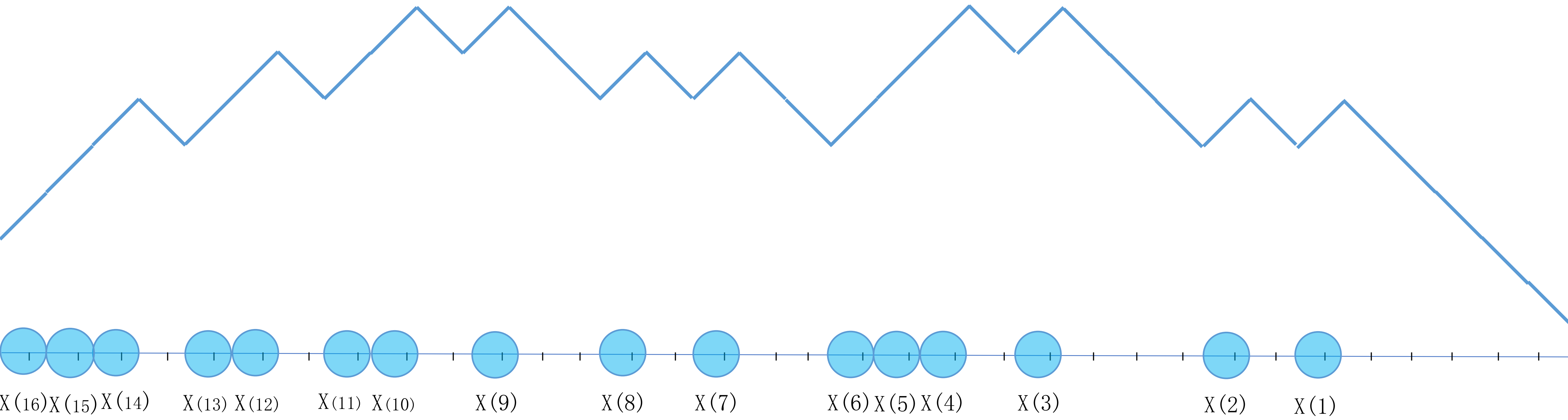
$$\partial_r\left(\partial_t\phi + \phi\partial_r\phi + \partial_r^3\phi\right) + \partial_x^2\phi = 0 \quad \Longrightarrow \quad \partial_t\phi + \phi\partial_r\phi + \partial_r^3\phi = 0$$

$\psi = \frac{1}{2}(q' - q^2)$ again gives Painleve II

TASEP



TASEP



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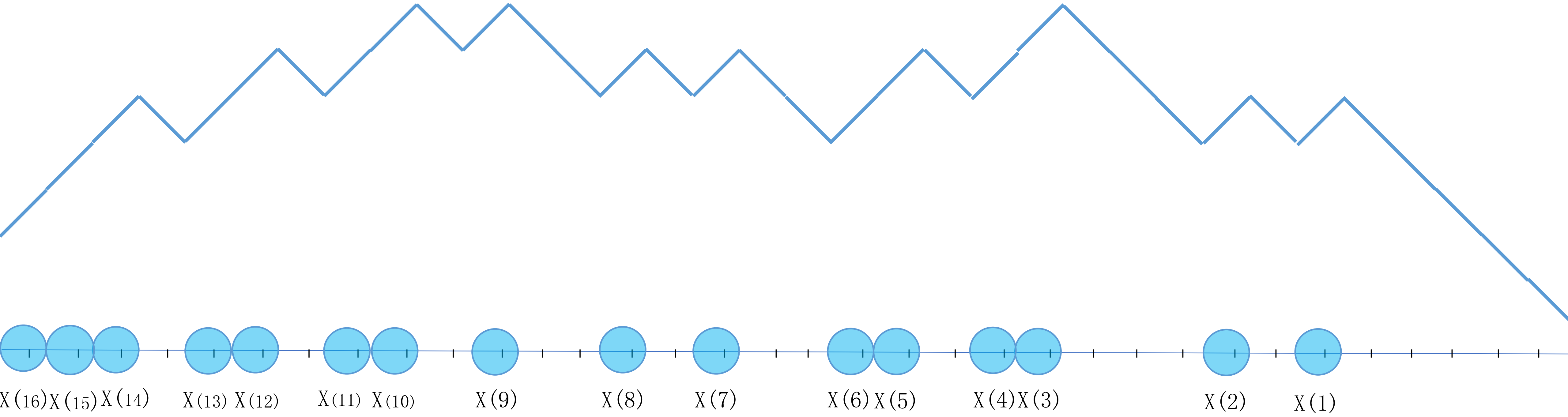
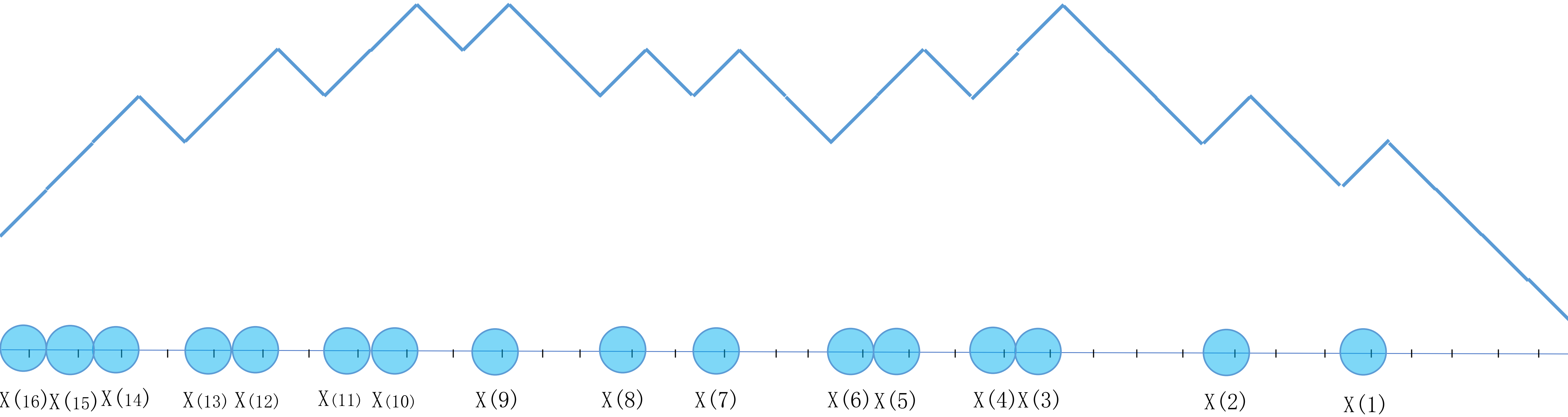


TASEP



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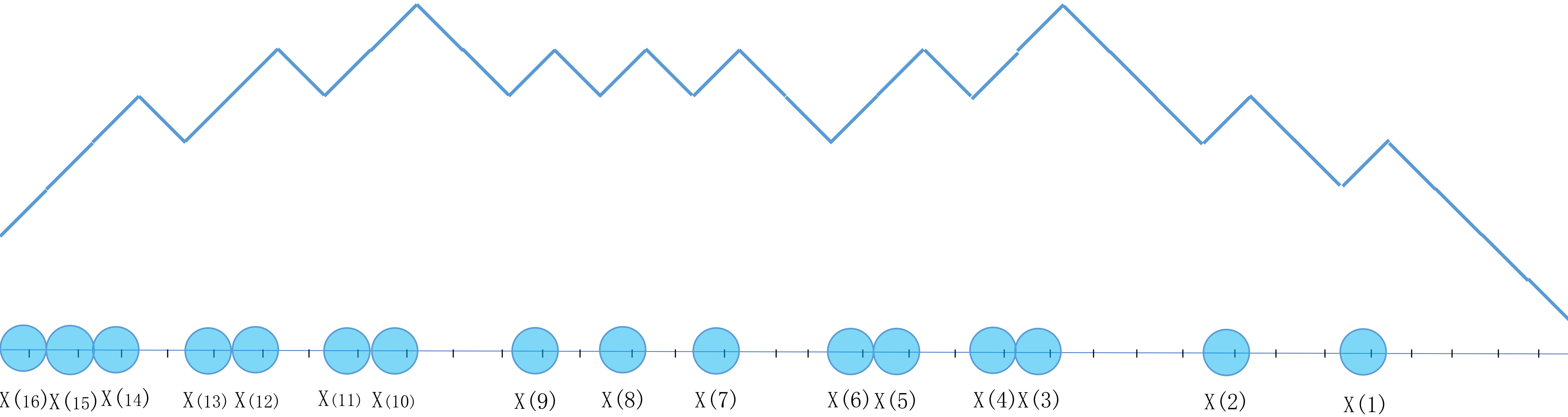
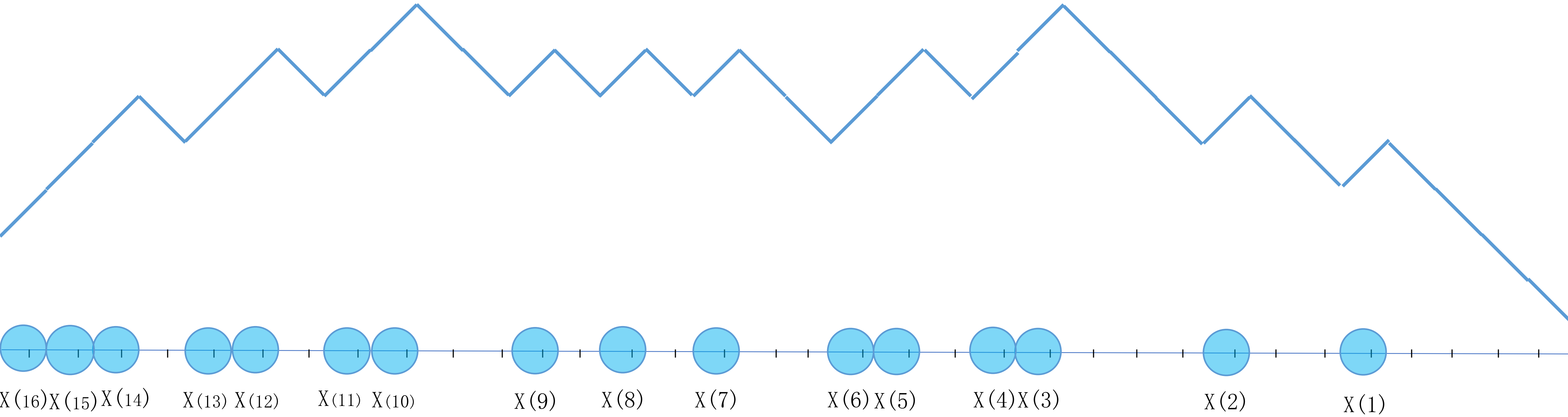
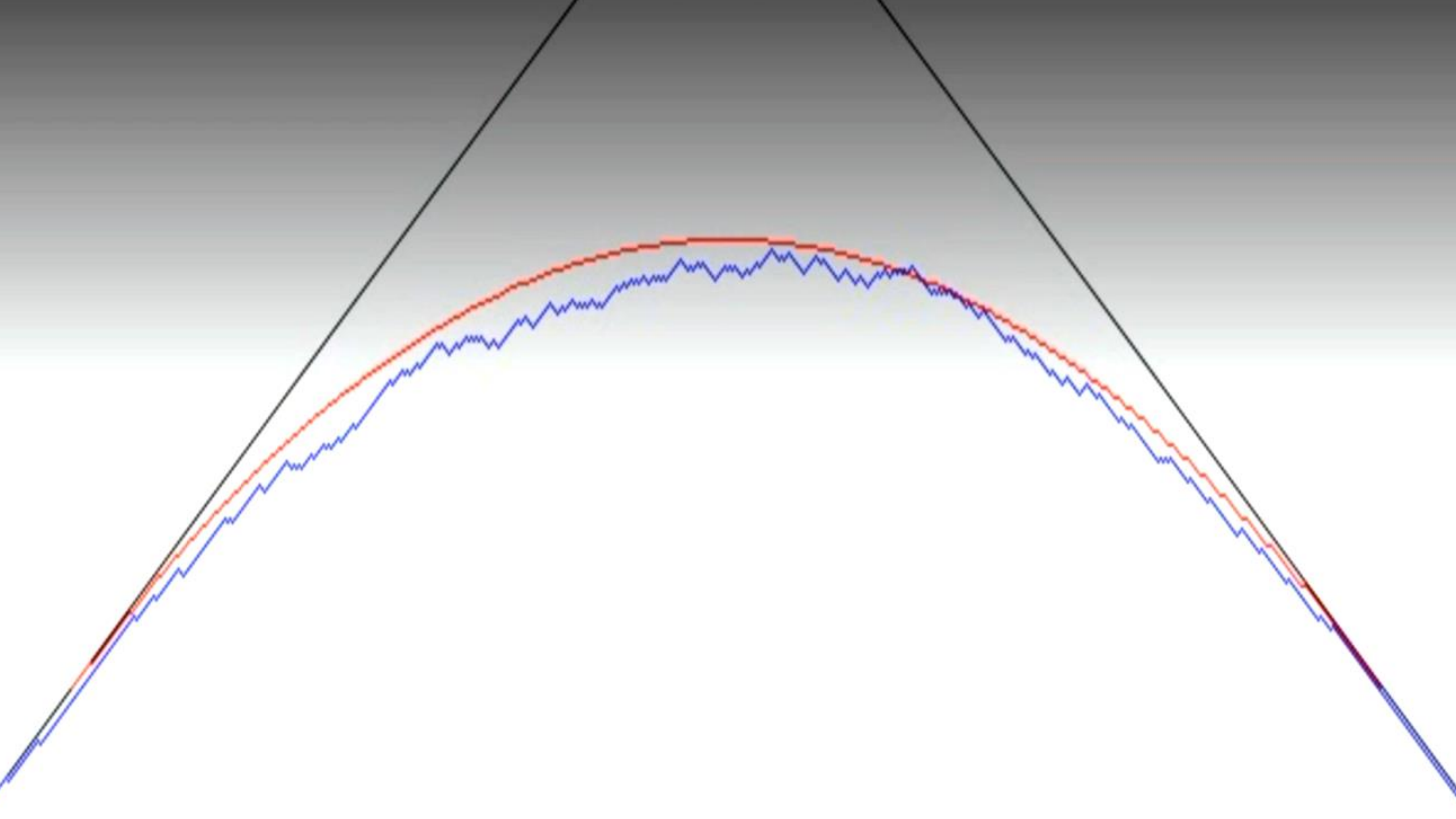


TASEP

$$-2\mathbf{1}_{\wedge} = \frac{1}{2}\left[(\nabla^{-}h)(\nabla^{+}h) - 1 + \Delta h\right]$$





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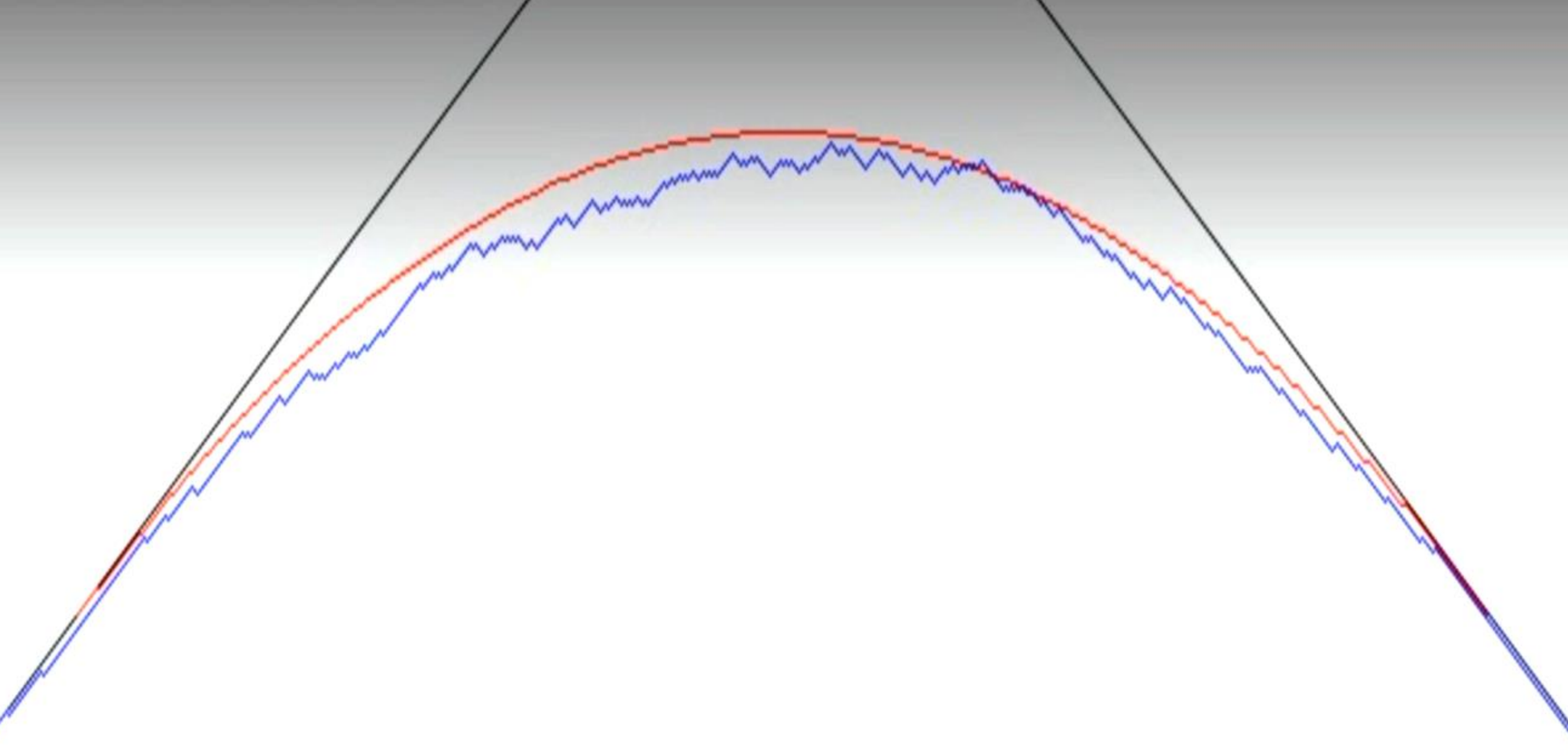

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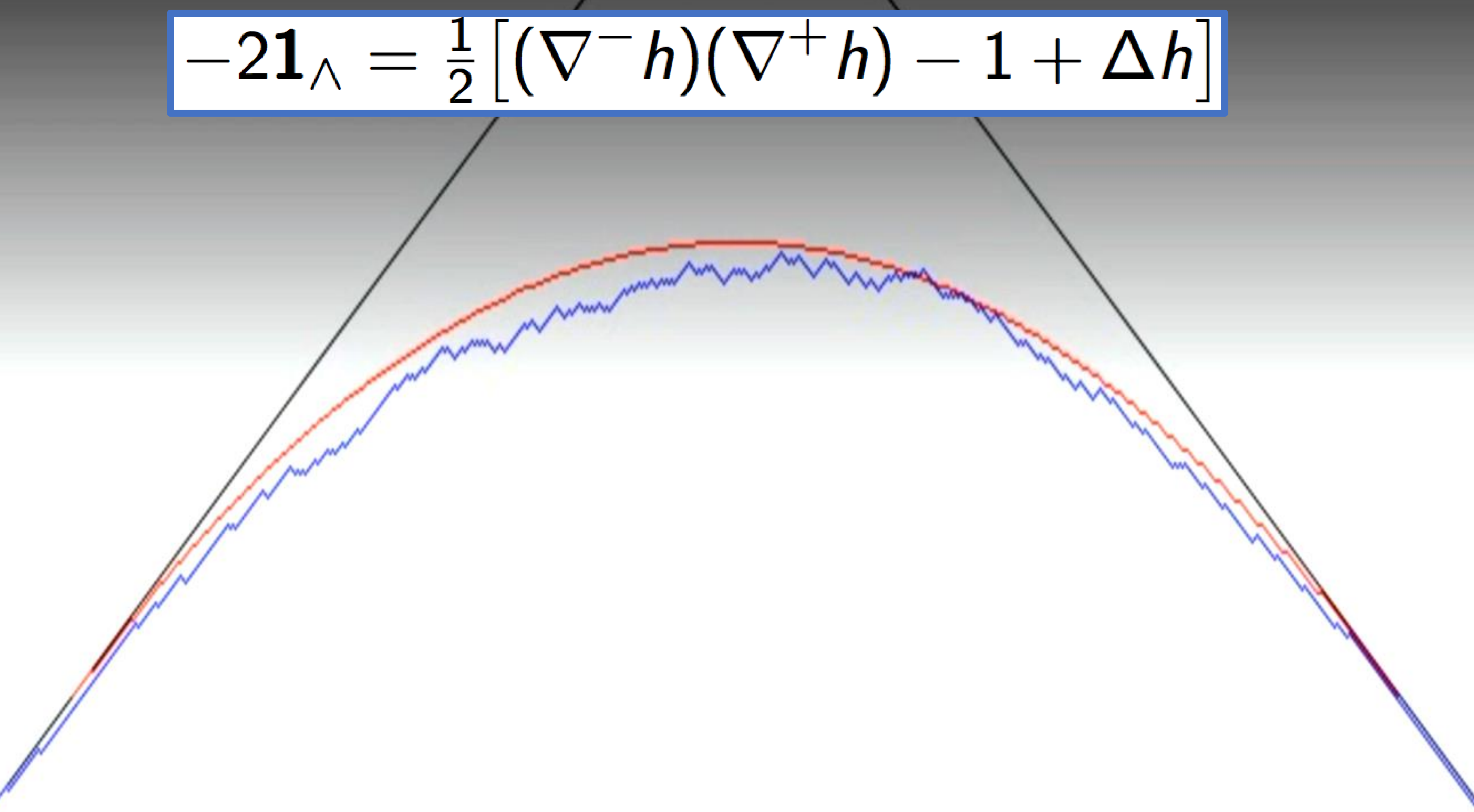
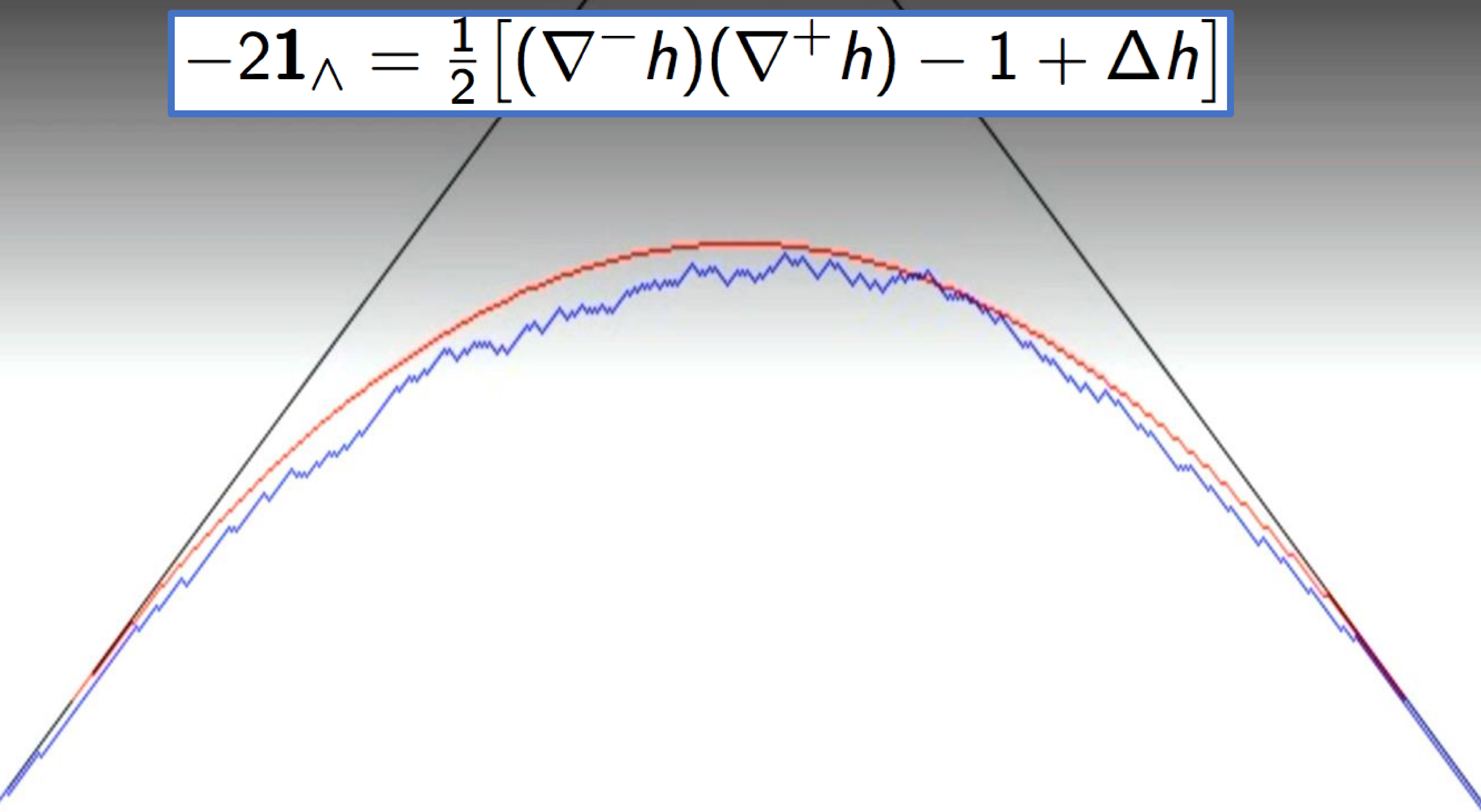
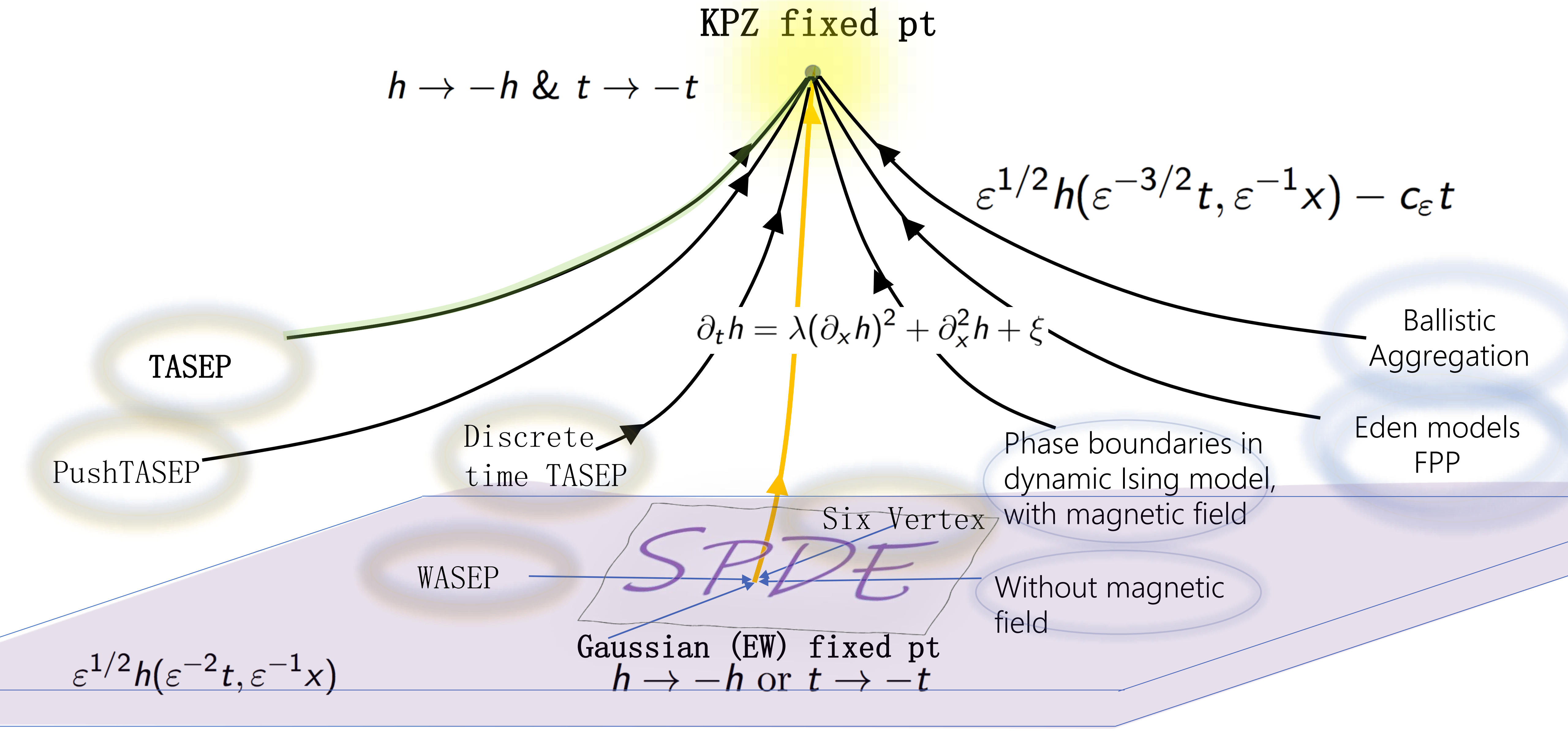


$$-2\mathbf{1}_{\Lambda} = \frac{1}{2}\left[(\nabla^{-}h)(\nabla^{+}h) - 1 + \Delta h\right]$$

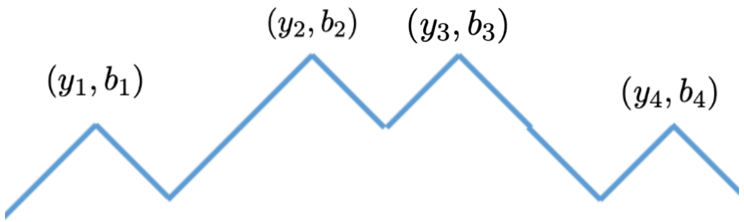




KPZ fixed pt
$h \to -h$ & $t \to -t$
$\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - c_\varepsilon t$
$\partial_t h = \lambda(\partial_x h)^2 + \partial_x^2 h + \xi$
TASEP
PushTASEP
Discrete time TASEP
Ballistic Aggregation
Eden models FPP
Phase boundaries in dynamic Ising model, with magnetic field
Six Vertex
SPDE
WASEP
Without magnetic field
Gaussian (EW) fixed pt
$h \to -h$ or $t \to -t$
$\varepsilon^{1/2} h(\varepsilon^{-2} t, \varepsilon^{-1} x)$

TASEP Transition probabilities (Matetski, Q, Remenik ‘17)

ic = initial condition



$$P\big(h(t, x_i) \le a_i,\ i = 1, \ldots, m\big) = \det\Big(\mathbf{I} - \mathbf{R}^{\mathrm{ic}}\, \bar{\mathbf{Q}}^{d_n - s_1} e^{\frac{t}{2}\nabla^-}\, \mathbf{R}^{\mathrm{fc}}\, e^{-\frac{t}{2}\nabla^-} \mathbf{Q}^{s_1 - d_n}\Big)$$

$$s_i = \tfrac{1}{2}(x_i + a_i) \qquad d_i = \tfrac{1}{2}(y_i + b_i) \qquad \mathbf{Q}(x, y) = \frac{1}{2^{x-y}} \mathbf{1}_{x>y}$$

$$\mathbf{R}^{\mathrm{ic}} = \big(\mathbf{I} - \mathbf{1}_{\le y_n} \mathbf{Q}^{d_n - d_{n-1}} \mathbf{1}_{\le y_{n-1}} \cdots \mathbf{Q}^{d_2 - d_1} \mathbf{1}_{\le y_1} \mathbf{Q}^{d_1 - d_n}\big)$$

$$\mathbf{R}^{\mathrm{fc}} = \big(\mathbf{I} - \mathbf{Q}^{s_1 - s_m} \underbrace{\mathbf{1}_{>x_m} \mathbf{Q}^{s_m - s_{m-1}} \mathbf{1}_{>x_{m-1}} \cdots \mathbf{Q}^{s_2 - s_1} \mathbf{1}_{>x_1}}\big)$$

Trans. Prob.s of geometric walk
staying above final condition

Where does this formula come from?

(97 Schutz) N particle trans prob’s given by NXN determinant full of contour integrals

(00 Johansson) Large N limit for wedge i.d. (Toeplitz)

(05 Sasamoto,Borodin,..) Biorthogonalization problem for Charlier polynomials shifted by i.c.

(17 Matetski,Q,Remenik) Solve biorthog problem by rw hitting times

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What can you do with such a complicated exact formula?

(21 Q, Li-Cheng Tsai) Large deviations for TASEP with Jensen-Varadhan rate function

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In 1:2:3 limit, "generator" becomes ∂^3

For 1:2:3 KPZ asymptotics use $\mathbf{Q}^{-1} = I + 2\nabla^+$ to write

$$e^{-\epsilon^{-3/2}\boldsymbol{t}\nabla^-}\mathbf{Q}^{-\epsilon^{-3/2}\boldsymbol{t}/2} = e^{\epsilon^{-3/2}\boldsymbol{t}[\underbrace{-\nabla^- + \frac{1}{2}\log(I+2\nabla^+)}_{\sim \frac{1}{3}\partial^3}]}$$

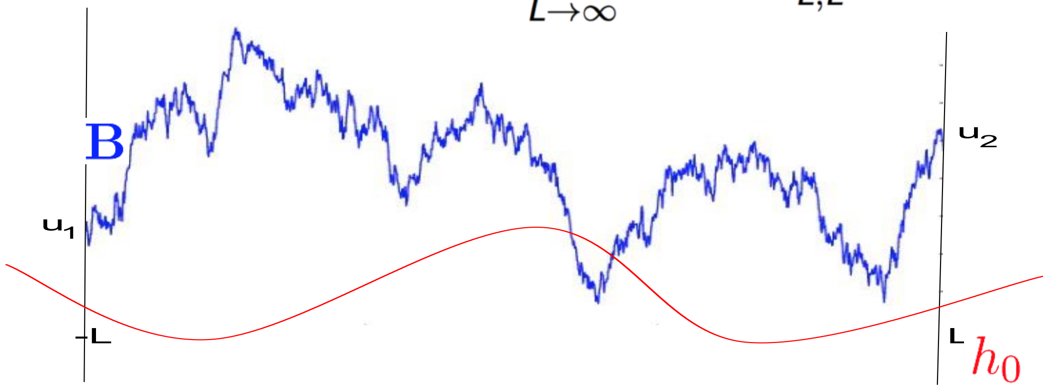
$$\mathbf{R}^{\mathrm{fc}} = \big(\mathbf{I} - \underbrace{\mathbf{Q}^{s_1 - s_m}\mathbf{1}_{>x_m}\mathbf{Q}^{s_m - s_{m-1}}\mathbf{1}_{>x_{m-1}} \cdots \mathbf{Q}^{s_2 - s_1}\mathbf{1}_{>x_1}}\big)$$

Becomes Trans. Prob.s of Brownian motion staying above final condition

KPZ fixed point

$$\mathbb{P}_{h_0}\big(h(t,x_1)\le a_1,\dots,h(t,x_M)\le a_M\big)=\det\big(I-K_{h_0,t,\vec{x},\vec{a}}\big)_{\prod_{i=1}^M L^2[0,\infty)}$$

$$\big(K_{h_0,t,\vec{x},\vec{a}}\big)_{ij}=e^{\frac{t}{3}\partial^3-x_i\partial^2+a_i\partial}\underbrace{K_{h_0}}_{\lim_{L\to\infty} e^{-L\partial^2}\mathbf{P}^{\mathrm{hit}\,h_0}_{-L,L}e^{-L\partial^2}}e^{-\frac{t}{3}\partial^3+x_j\partial^2-a_j\partial}$$



$$\mathbf{P}^{\mathrm{hit}\,h_0}_{-L,L}(u_1,u_2)=\mathbb{P}_{\mathbf{B}(-L)=u_1}\big(\mathbf{B}\text{ hits hypo}(h_0)\text{ on }[-L,L],\ \mathbf{B}(L)\in du_2\big)\frac{1}{du_2}$$

$h_0\mapsto K_{h_0,t,\vec{x},\vec{a}}$ continuous bijection [upper semi-cont fns] ⟺ [trace class]

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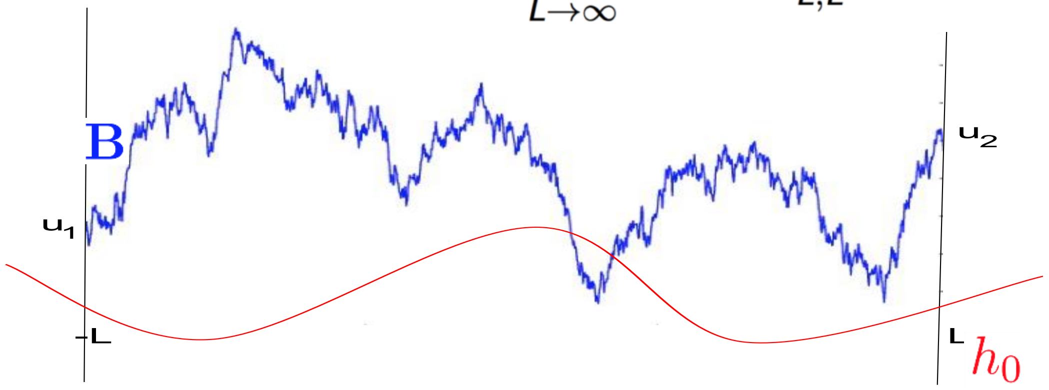

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KPZ fixed point (Kadomtsev-Petviashvilli version) (Remenik-Q 2019)

$$F(t, x_1, \ldots, x_n, r_1, \ldots, r_n) = \mathbb{P}_{\mathfrak{h}_0}(\mathfrak{h}(t, x_1) \leq r_1, \ldots, \mathfrak{h}(t, x_n) \leq r_n)$$

$q = \partial_r Q$ $\quad \partial_r \log F = \mathrm{tr}\, Q$ satisfy the matrix KP Equation in $x = x_1 + \cdots x_n \quad r = r_1 + \cdots r_n$

$$\partial_t q + \tfrac{1}{2}(q\partial_r q + \partial_r q\, q) + \tfrac{1}{12}\partial_r^3 q + \tfrac{1}{4}\partial_x^2 Q + \tfrac{1}{2}(q\partial_x Q - \partial_x Q\, q) = 0$$

If n =1 $\phi = \partial_r^2 \log F$ satisfies KP $\partial_r\big(\partial_t \phi + \phi\partial_r\phi + \tfrac{1}{12}\partial_r^3\phi\big) + \tfrac{1}{4}\partial_x^2\phi = 0$

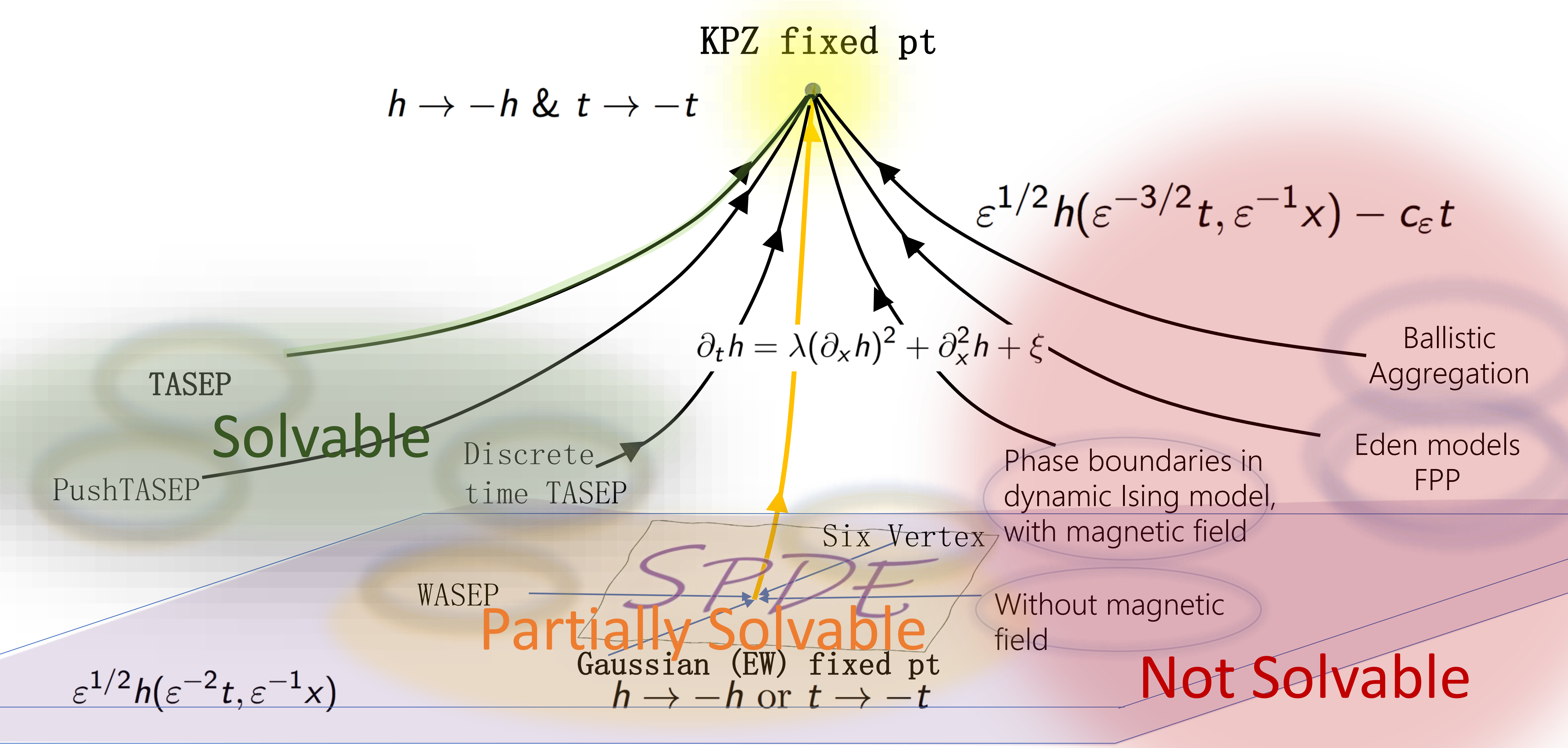
This, and the previous description using det's show KPZ fixed point is integrable Markov process (= trans prob's are a completely integrable system)

Third description, as a Variational Formula

$$\mathfrak{h}(\mathbf{t}, \mathbf{x}; \mathfrak{h}_0) \stackrel{\text{dist}}{=} \sup_{\mathbf{y}\in\mathbb{R}} \left\{\mathbf{t}^{1/3}\mathcal{A}(\mathbf{t}^{-2/3}\mathbf{x}, \mathbf{t}^{-2/3}\mathbf{y}) - \tfrac{1}{\mathbf{t}}(\mathbf{x}-\mathbf{y})^2 + \mathfrak{h}_0(\mathbf{y})\right\}$$

$\mathcal{A}(\mathbf{x}, \mathbf{y})$ = Airy sheet

Constructed by Dauvergne-Virag as a strong limit of a special last passage percolation model



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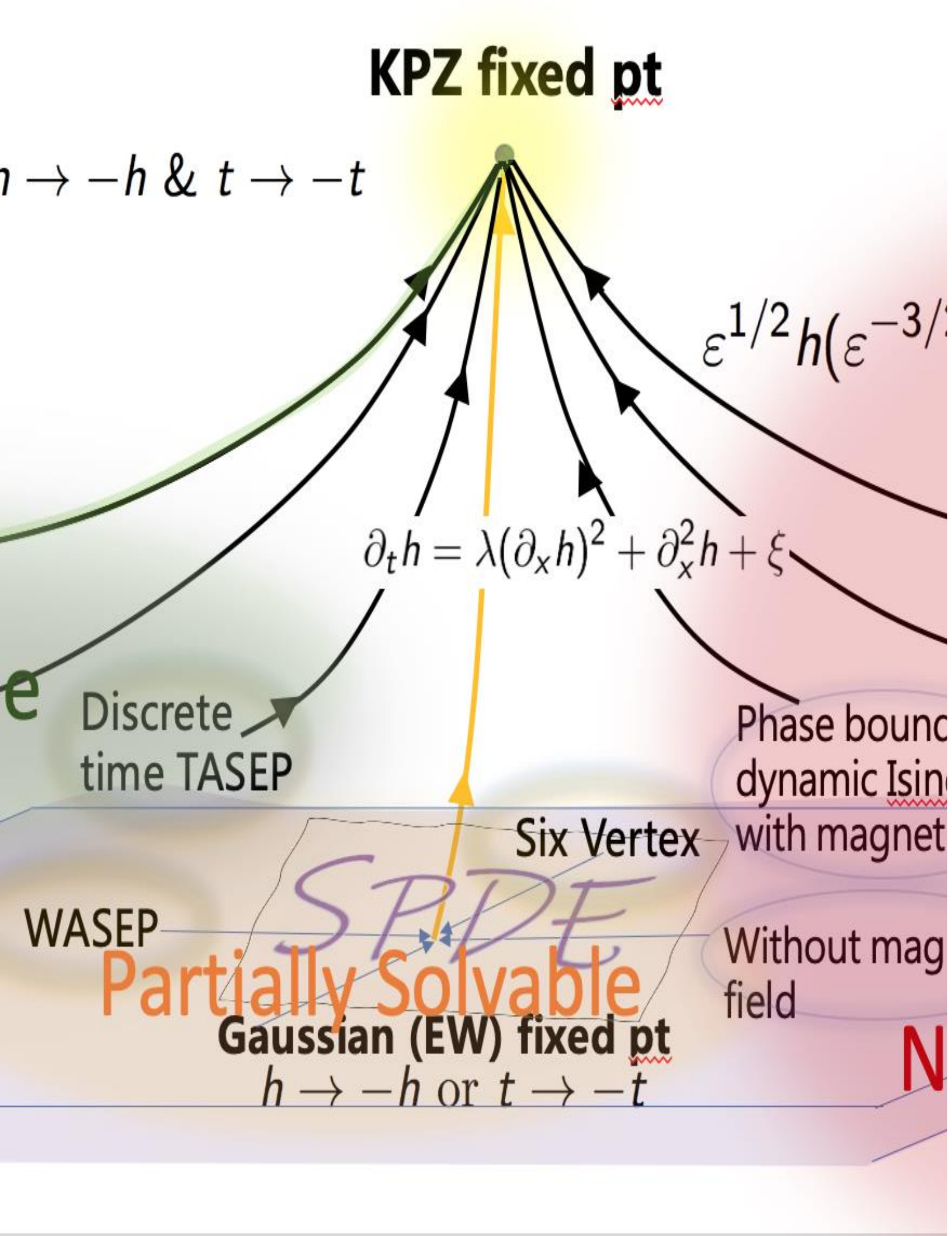
Not Solvable



How to prove convergence to the KPZ fixed point?

Industrial scale effort on solvable models (Schur processes, etc. 2000-2010) and partially solvable models (MacDonald processes, ASEP, Six vertex, etc. 2010-present). Find a formula, take its limit

Virag and Sarkar-Q independently found proofs that KPZ equation converges to fixed point

Both start with the fact that one only needs convergence on a dense class of initial data

Virag uses a fantastic symmetry of certain last passage models + approximate version of variational formula. Requires at least partial solvability

Ours based on energy estimates. Works for at least one class of not solvable models, finite range exclusion processes. But initial data needs to be randomized a little

Comparing two Markov processes

$$p_i(h,t,B) = \mathbb{P}(h_i(t) \in B \mid h(0) = h),\ i = 1,2$$

$\partial_t p_i = \mathscr{L}_i p_i$ $\qquad \mathscr{L}_i =$ generator $\qquad \nu =$invariant measure

$$\int_B f_i(t,h)d\nu(h) = \int \mathbb{P}(h_i(t) \in B \mid h(0) = h) f(0,h) d\nu = p_i(f(0)d\nu, t, B)$$

$$p_1(f(0)d\nu, t, B) - p_2(f(0)d\nu, t, B) = \int_0^t \int f_2(t-s,h)(\mathscr{L}_1 - \mathscr{L}_2)p_1(h,s,B)d\nu(h)ds.$$

Proof. $\partial_s f_2(t-s,h)p_1(h,s,B) = -\mathscr{L}_2^* f_2(t-s,h)p_1(h,s,B) + f_2(t-s,h)\mathscr{L}_1 p_1(h,s,B)$

Take p_1 to be TASEP. Difference of generators is generator of symmetric simple exclusion

$$p^{\text{TASEP}}(f(0)d\nu, t, B) - p^{\text{ASEP}}(f(0)d\nu, t, B) = \varepsilon^{-3/2} \int_0^t \int \sum_x \nabla_x f^{\text{ASEP}}(t-s,h) \nabla_x p^{\text{TASEP}}(h,s,B) d\nu(h) ds.$$

$\nu =$ two sided random walk $\times$ Lebesgue measure in $h(0)$

Energy method (Q-Sarkar 20)

$$p^{\text{TASEP}}(f(0)d\nu, t, B) - p^{\text{ASEP}}(f(0)d\nu, t, B) = \varepsilon^{-3/2} \int_0^t \int \sum_x \nabla_x f^{\text{ASEP}}(t-s, h) \nabla_x p^{\text{TASEP}}(h, s, B) d\nu(h) ds.$$

$$\int f^{\text{ASEP}}(t)^2 d\nu - \int f(0)^2 d\nu = -\varepsilon^{-3/2} \int_0^t \int \sum_x \left(\nabla_x f^{\text{ASEP}}(t-s)\right)^2 d\nu ds$$

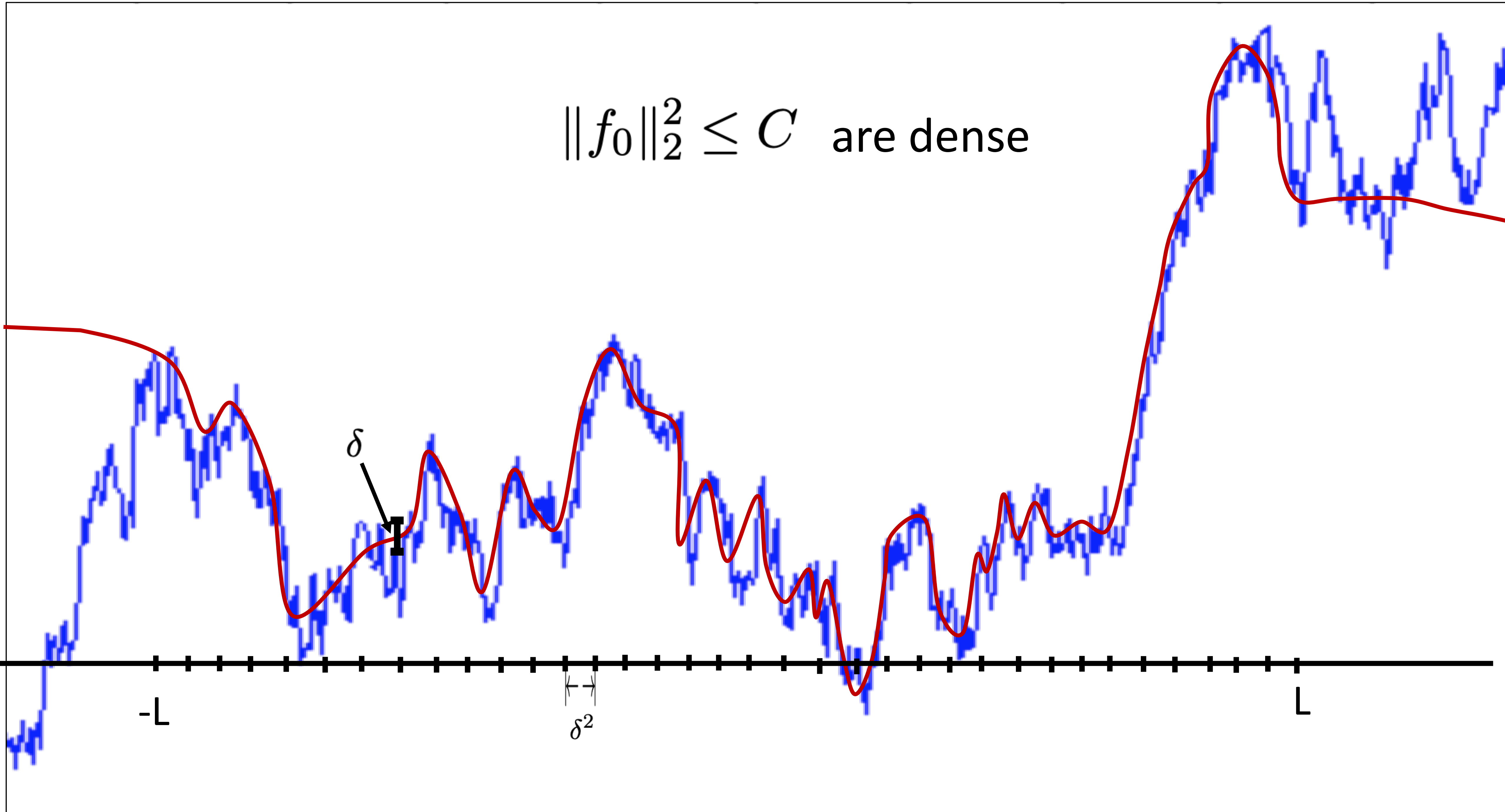
$$\left|p^{\text{TASEP}}(f(0)d\nu, t, B) - p^{\text{ASEP}}(f(0)d\nu, t, B)\right|^2 \le \|f_0\|_2^2 \, \varepsilon^{-3/2} \int_0^t \int \sum_x \left(\nabla_x p^{\text{TASEP}}\right)^2 d\nu ds$$

$B = \{h : h(x) \le g(x), x \in \mathbb{R}\}$

$\nabla_x p^{\text{TASEP}} = \mathbb{P}^{\text{TASEP}}\Big(\arg\max\{h_t(y; -g) + h(y)\} = x, \max\{h_t(y; -g) + h(y)\} = 0\Big)$

If you do a little average over shifts of B then $\int \sum_x \left(\nabla_x \bar{p}^{\text{TASEP}}\right)^2 d\nu \le C\varepsilon^2 a^{-2}$

$$\left|\bar{p}^{\text{TASEP}}(f(0)d\nu, t, B) - \bar{p}^{\text{ASEP}}(f(0)d\nu, t, B)\right|^2 \le \varepsilon^{1/2} \|f_0\|_2^2 \, a^{-2} \qquad a = \text{range of little average}$$



$\|f_0\|_2^2 \leq C$ are dense
δ
δ^2
-L
L

For KPZ eqn

Same computation gives difference between the (little averaged) TASEP probs and the WASEP (weakly asymmetric simple exclusion) probs. Approximating KPZ equation by WASEP gives

$$\left|\bar{p}^{\mathrm{FP}}(f(0)d\nu, t, B) - \bar{p}^{\mathrm{KPZ}}(f(0)d\nu, t, B)\right|^2 \le \delta^{1/2}\|f_0\|_2^2\, a^{-2} \qquad \partial_t h = \tfrac{1}{4}(\partial_x h)^2 + \delta\partial_x^2 h + \delta^{1/2}\xi$$

For finite range exclusions

The key fact we used was ★ $\int f_2(\mathscr{L}_1 - \mathscr{L}_2)p_1 d\nu \le \varepsilon^{-3/2}\sqrt{\sum_x \int (\nabla_x f_2)^2 d\nu}\sqrt{\sum_x \int (\nabla_x p_1)^2 d\nu}$

$\mathscr{L}_2$ generator of finite range exclusion $\qquad$ $\mathscr{L}_1$ generator of TASEP, same mean

Then $\mathscr{L}_1 - \mathscr{L}_2$ generator of mean zero exclusion

Xu, Varadhan (93) Mean zero exclusions satisfy the strong sector condition ★

arXiv:1701.00018 [pdf, other] math.PR math-ph

The KPZ fixed point

Authors: Konstantin Matetski, Jeremy Quastel, Daniel Remenik

arXiv:1908.10353 [pdf, other] math.PR math-ph

KP governs random growth off a one dimensional substrate

Authors: Jeremy Quastel, Daniel Remenik

arXiv:2008.06584 [pdf, other] math.PR math-ph

Convergence of exclusion processes and KPZ equation to the KPZ fixed point

Authors: Jeremy Quastel, Sourav Sarkar