Gibbsian line ensembles and β-corners processes

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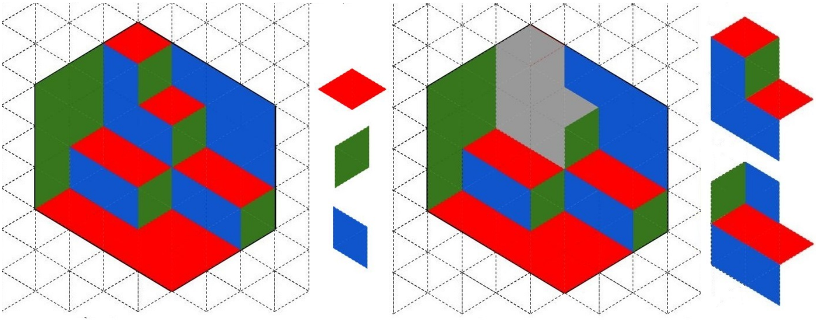
Workshop on Integrable Structures in Random Matrix Theory and Beyond – MSRI, Berkeley

Outline

Gibbs measures

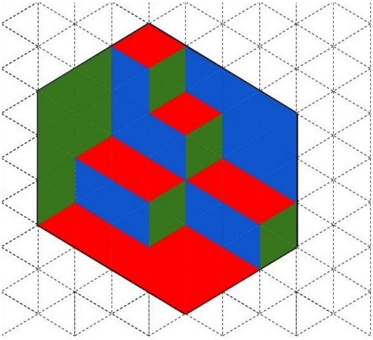
Gibbs property = internal consistency condition of a random model.

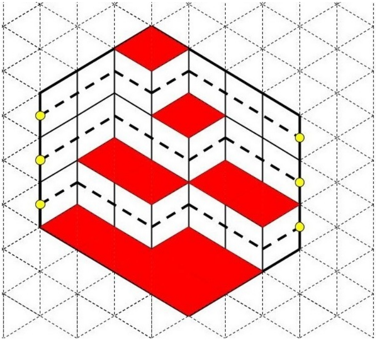
Uniform lozenge tilings of the hexagon



Gibbsian line ensembles

Line ensemble (LE) = finite or countably infinite collection of random continuous curves, defined on the same probability space.





Tiling Gibbs property $\implies$ LE Gibbs property.
LE Gibbs property = locally avoiding Bernoulli random walks (or bridges).

Gibbsian line ensembles

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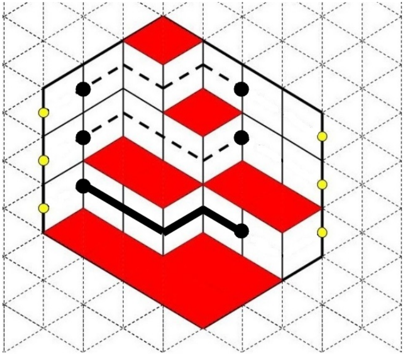
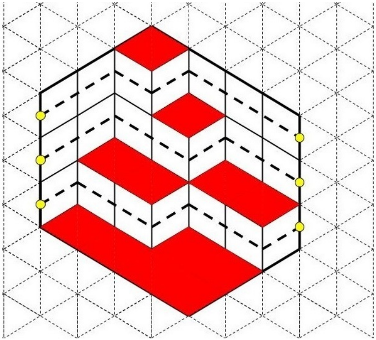
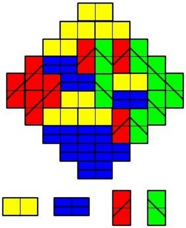




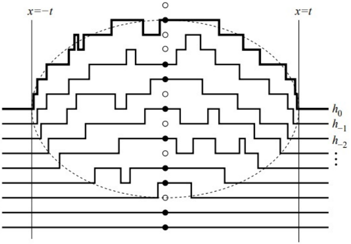
LE Gibbs property = locally avoiding Bernoulli random walks (or bridges).

Gibbsian line ensembles

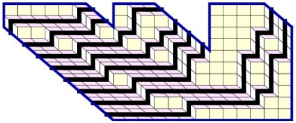
Gibbsian line ensembles = finite or countably infinite collection of random walk trajectories with *local* interactions



Domino tilings of the Aztec diamond [Johansson '02]



Multi-layer PNG model [Prähofer-Spohn '02]



Lozenge tilings of polygons [Petrov '14]

Gibbsian LEs appear in random tilings, last passage percolation and directed random polymers.

Asymptotics of Gibbsian line ensembles

Q: What happens when to a Gibbsian line ensemble $\{\mathcal{L}_i^N\}_{i=1}^N$ as $N \to \infty$?

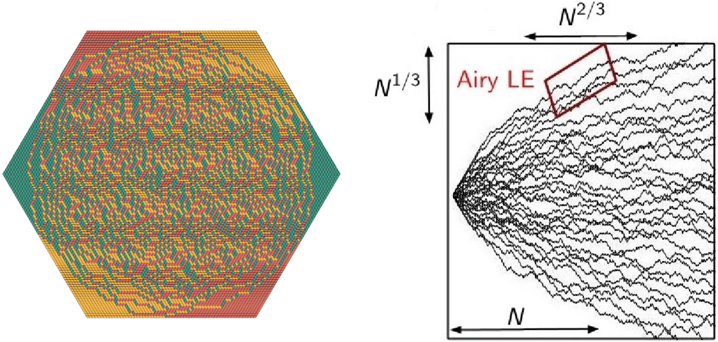
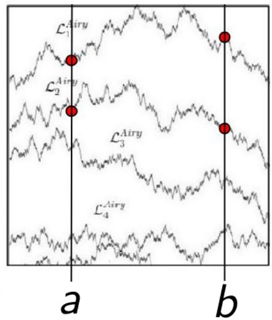
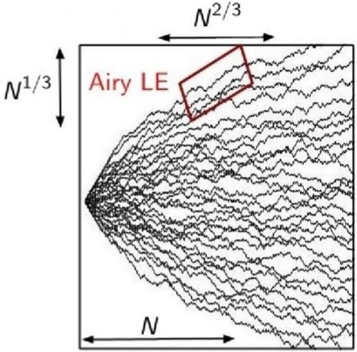


Figure: Simulation due to L. Petrov

We enter the Kardar-Parisi-Zhang (KPZ) universality class
Limiting object: (Parabolic) Airy line ensemble $\{\mathcal{L}_i^{Airy}\}_{i=1}^\infty$

The parabolic Airy line ensemble

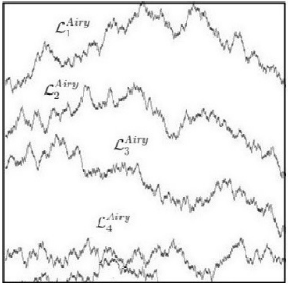


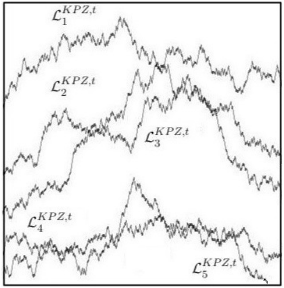


(Parabolic) Airy line ensemble $\{\mathcal{L}_i^{Airy}\}_{i=1}^{\infty}$
$\mathcal{L}_1^{Airy}$ = (parabolic) Airy process $\mathcal{L}_1^{Airy}(0)$ = GUE Tracy-Widom dist.
$\{\mathcal{L}_i^{Airy}\}_{i=1}^{\infty}$ has the Brownian Gibbs property (locally avoiding Brownian bridges)

Key questions for Gibbsian line ensembles

1. **Tightness**
2. **Characterization**
3. **Convergence**
   - Zero temperature: $\{\mathcal{L}_i^{Airy}\}_{i=1}^{\infty}$
   - Positive temperature: $\{\mathcal{L}_i^{KPZ,t}\}_{i=1}^{\infty}$
4. **Properties**
5. **Applications**





Outline

1. Gibbsian line ensembles
2. Convergence of Gibbsian line ensembles
3. Maximal free energy in the log-gamma polymer
4. β-corners processes

Framework for proving convergence to the Airy LE

To prove $\mathcal{L}^N \xrightarrow{\text{u.c.}} \mathcal{L}^{Airy}$ one needs

1. Show that $\mathcal{L}^N \xrightarrow{\text{f.d.}} \mathcal{L}^{Airy}$.
2. Show that $\mathcal{L}^N$ is tight

Zero temperature

- PNG [Prähofer-Spohn '02]
- Domino tilings Aztec diamond [Johansson '02]
- Schur processes [Okounkov-Reshetikhin '03]
- TASEP [Johansson '03]
- Expon. LPP [Borodin-Péché '08]
- Brownian watermellons [Corwin-Hammond '14]

Positive temperature Until 2020 most work was on $\mathcal{L}_1^N \xrightarrow{\text{1.p.}} \mathcal{L}_1^{Airy}$:

- ASEP [Tracy-Widom '08]
- KPZ eqn. [Sasamoto-Spohn '10]
- Macdonald processes [Borodin-Corwin '14]
- Log-gamma polymer [Borodin-Corwin-Remenik '13]
- S6V [Borodin-Corwin-Gorin '16]

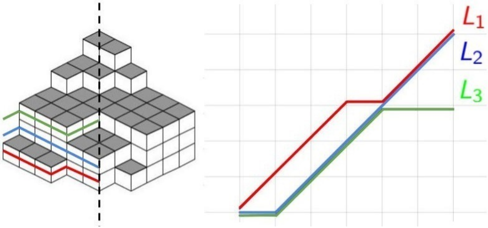
Until 2020: $\mathcal{L}_1^N \xrightarrow{\text{f.d.}} \mathcal{L}_1^{Airy}$

- Physics: replica approach (non-rigorous)
- Math: [Nguyen-Zygouras '16] (incomplete)

Finite dimensional convergence

Theorem (D. '20)

Let $\mathfrak{L}^N = \{L_i^N\}_{i=1}^N$ be the Hall-Littlewood Gibbsian LE (M, N, a, t). The two-point distribution of L_1^N converges to the two-point distribution of $\mathcal{L}_1^{Airy}$.



1. Result is limited to two points and parameter restrictions
2. When reflected L_1^N has the law of the height function of the stochastic six-vertex model [Borodin-Bufetov-Wheeler '16]
3. The first multi-point convergence result for a *non-determinantal (positive temperature) model*. Softer techniques were later developed by [Quastel-Sarkar '20] (ASEP and KPZ) and [Virág '20] (polymer models).

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Use the method of *Macdonald difference operators* [Borodin-Corwin '14]

$$\mathbb{E}\left[\frac{1}{(u_1 t^{-L_1^N(n_1)};t)_\infty}\frac{1}{(u_2 t^{-L_1^N(n_2)};t)_\infty}\right] = \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty} I_M(N_1,N_2)$$

$$I_M(N_1,N_2) = \frac{1}{N_1!N_2!}\int_{\gamma_1^{N_1}}\int_{\gamma_2^{N_1}}\int_{\gamma_3^{N_2}}\int_{\gamma_4^{N_2}} D(\vec{w},\vec{z})G(\vec{w},\vec{z},n_1,u_1)$$

$$D(\vec{\hat{w}},\vec{\hat{z}})G(\vec{\hat{w}},\vec{\hat{z}},n_2,u_2)\cdot CT(\vec{w},\vec{z};\vec{\hat{w}},\vec{\hat{z}})\prod_{i=1}^{N_2}\frac{d\hat{w}_i}{2\pi\iota}\prod_{i=1}^{N_2}\frac{d\hat{z}_i}{2\pi\iota}\prod_{i=1}^{N_1}\frac{dw_i}{2\pi\iota}\prod_{i=1}^{N_1}\frac{dz_i}{2\pi\iota}$$

$$D(\vec{w},\vec{z}) = \det\left[\frac{1}{z_i - w_j}\right]_{i,j=1}^{N_1},\quad CT = \prod_{i=1}^{N_1}\prod_{j=1}^{N_2}\frac{(\hat{z}_j z_i^{-1};t)_\infty}{(\hat{w}_j z_i^{-1};t)_\infty}\frac{(\hat{w}_j w_i^{-1};t)_\infty}{(\hat{z}_j w_i^{-1};t)_\infty}$$

$$G(\vec{w},\vec{z},n_1,u_1) = \prod_{i=1}^{N_1}\left(\frac{1+az_i}{1+aw_i}\right)^M\left(\frac{1+aw_i^{-1}}{1+az_i^{-1}}\right)^{n_1}\frac{S(\log w_i - \log z_i; u_1, t)}{[-\log t]w_i},$$

$(a;t)_\infty = \prod_{m=0}^{\infty}(1-at^m)$ – t-Pochhammer symbol

Framework for proving convergence to the Airy LE

To prove $\mathcal{L}^N \xrightarrow{\text{u.c.}} \mathcal{L}^{Airy}$ one needs

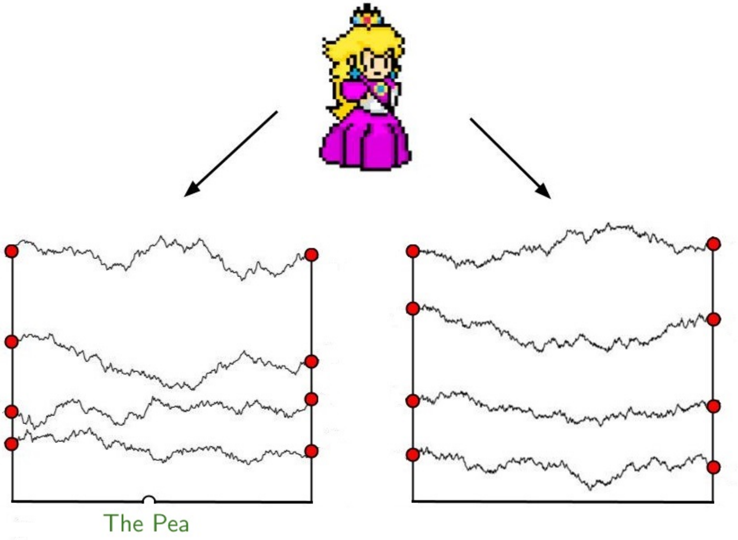
1. Show that $\mathcal{L}^N \xrightarrow{\text{f.d.}} \mathcal{L}^{Airy}$.
2. Show that $\mathcal{L}^N$ is tight

In [D.-Matetski '20] we proposed the following alternate framework.

1. Show that $\mathcal{L}_1^N \xrightarrow{\text{f.d.}} \mathcal{L}_1^{Airy}$ parabolic Airy process.
2. Show that $\mathcal{L}^N$ is tight and that all subsequential limits satisfy the Brownian Gibbs property (locally avoiding Brownian bridges).
3. Show that $\mathcal{L}^{Airy}$ is the unique line ensemble which satisfies the Brownian Gibbs property and has the parabolic Airy process as its top curve.

- The framework reduces the quantitative information we need from $\mathcal{L}^N$ to $\mathcal{L}_1^N$ (useful for models that are non-determinantal)
- $\mathcal{L}_1^N$ is frequently special: KPZ line ensemble $\{\mathcal{L}_i^{KPZ,t}\}_{i=1}^{\infty}$ ($\mathcal{L}_1^{KPZ,t}$ is the solution to the narrow wedge KPZ equation at time t)

The Princess and the Pea



Q: Can the Princess feel the pea? A: Yes, and more!

Classification of Brownian Gibbsian line ensembles

Theorem (D.-Matetski '20)

Suppose that $\mathcal{L}^1$ *and* $\mathcal{L}^2$ *are Brownian Gibbsian line ensembles with laws* $\mathbb{P}_1$ *and* $\mathbb{P}_2$*, respectively. Suppose further that for every* $k \in \mathbb{N}$*,* $t_1 < t_2 < \cdots < t_k$ *and* $x_1, \dots, x_k \in \mathbb{R}$ *we have*

$$\mathbb{P}_1\left(\mathcal{L}_1^1(t_1) \le x_1, \dots, \mathcal{L}_1^1(t_k) \le x_k\right) = \mathbb{P}_2\left(\mathcal{L}_1^2(t_1) \le x_1, \dots, \mathcal{L}_1^2(t_k) \le x_k\right).$$

Then $\mathbb{P}_1 = \mathbb{P}_2$*.*

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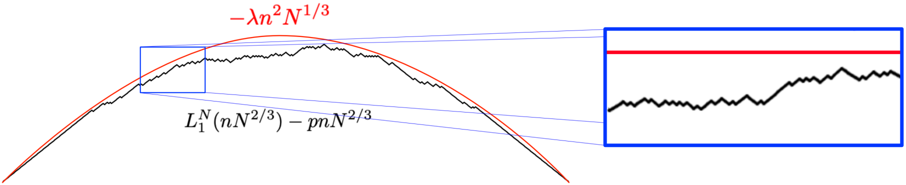
Then $\mathbb{P}_1 = \mathbb{P}_2$.

1. Theorem says that a Brownian Gibbsian line ensemble is completely characterized by its top curve. The (parabolic) Airy line ensemble $\{\mathcal{L}_i^{Airy}\}_{i=1}^{\infty}$ is characterized by (1) the Brownian Gibbs property and (2) $\mathcal{L}_1^{Airy}$ = (parabolic) Airy process
2. Proof is non-constructive (no formulas for $\mathcal{L}_2$ etc.)
3. Positive temperature analogue recently proved in [D. '21]. Specifically, the KPZ_t line ensemble is characterized by its lowest-indexed curve being the narrow wedge KPZ equation and its Gibbs property (called *H*-Brownian Gibbs property)

Tightness of Gibbsian line ensembles

Theorem (Meta-)

Let $\mathfrak{L}^N = \{L_i^N\}_{i=1}^N$ be a sequence of Gibbsian line ensembles. Suppose that for some constants $p \in \mathbb{R}$ and each $n \in \mathbb{Z}$ the r.v.'s $\frac{L_1^N(nN^{2/3}) - pnN^{2/3}}{N^{1/3}}$ are tight and globally parabolic. Then $\frac{L_i^N(xN^{2/3}) - pxN^{2/3}}{N^{1/3}}$ are tight in $C(\mathbb{N} \times \mathbb{R})$ and all subsequential limits satisfy the Brownian Gibbs property.



- 1-point tightness + Gibbs property $\implies$ tightness of whole ensemble
- Avoiding Bernoulli random walkers (proof of concept) [D.-Fang-Fesser-Serio-Teitler-Wang-Zhu '20]
- (H, H^{RW})-Gibbsian line ensembles (log-gamma polymer) [D.-Wu '21]
- Key ingredient: KMT coupling for random walk bridges [D.-Wu '19]

Framework for proving convergence to the Airy LE

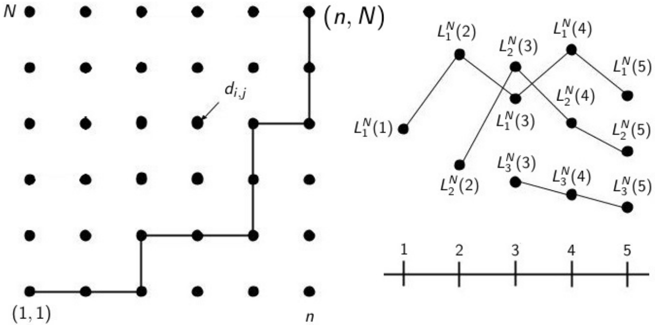
1. Show that $\mathcal{L}_1^N \xrightarrow{\text{f.d.}} \mathcal{L}_1^{Airy}$ parabolic Airy process.
2. Show that $\mathcal{L}^N$ is tight and that all subsequential limits satisfy the Brownian Gibbs property (locally avoiding Brownian bridges).
3. Show that $\mathcal{L}^{Airy}$ is the unique line ensemble which satisfies the Brownian Gibbs property and has the parabolic Airy process on top.

Framework was used to show $\{\mathcal{L}_i^{KPZ,t}\}_{i=1}^{\infty} \implies \{\mathcal{L}_i^{Airy}\}_{i=1}^{\infty}$ as $t \to \infty$.
Step 1 in [Quastel-Sarkar '20], [Virág '20], Step 2 in [Wu '21], and Step 3 in [D.-Matetski '20].
Currently applying it to log-gamma polymer – Step 2 in [D.-Wu '21].

Outline

The log-gamma polymer [Seppäläinen '09]

$d_{i,j}$ are i.i.d. inverse-gamma: $f_\theta(x) = \mathbf{1}\{x > 0\}\Gamma(\theta)^{-1}x^{-\theta-1}\exp(-x^{-1})$.
Partition functions: $Z^N(n) = \sum_{\pi:(1,1)\to(n,N)} w(\pi)$, $w(\pi) = \prod_{(i,j)\in\pi} d_{i,j}$

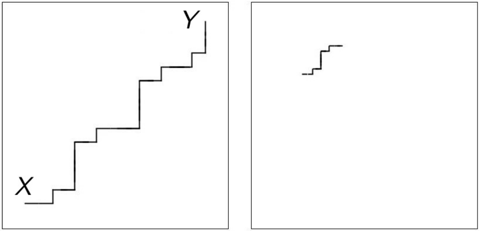


Free energies: $L_1^N(n) = \log Z^N(n)$
Using *geometric RSK* [Corwin-O'Connell-Seppäläinen-Zygouras '14] showed that L_1^N embeds as the lowest indexed curve in a line ensemble with a nice Gibbs property (uses inverse-gamma weights - very special)

Maximal free energy F_N

$$Z(X;Y) = \sum_{\pi:X\to Y} w(\pi),\ w(\pi) = \prod_{(i,j)\in\pi} d_{i,j}$$

Maximal free energy: $\mathsf{F}_N = \max_{(1,1)\leq X\leq Y\leq(N,N)} \log Z(X;Y)$.



$\infty \longleftarrow \mathbb{E}[d_{i,j}] \longrightarrow 0$

$$\mathbb{P}_d(\pi) = \frac{w(\pi)}{Z_d}, Z_d = \sum_{\pi\in\square} w(\pi)$$

$$\mathbb{P}_d(\pi : X \to Y) = \frac{Z(X;Y)}{Z_d}$$

Why study F_N?

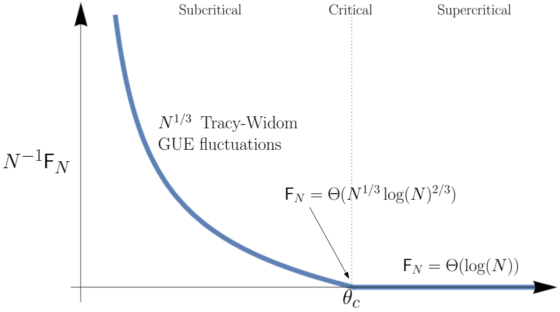
1. Interesting phase transition for F_N from $\theta < \theta_c$ to $\theta > \theta_c$ (next slide).
2. F_N is related to the smallest singular value of a random operator on the honeycomb lattice [Kotowski-Virág '19].
3. F_N is a proxy for studying a free directed polymer path measure.

Maximal free energy F_N

Theorem (Barraquand-Corwin-D. '21abc)

If $\theta_c = 2\Psi^{-1}(0) > 0$ (Ψ is the digamma function) then

- *For $\theta < \theta_c$, $F_N + 2\Psi(\theta/2)N$ has order $N^{1/3}$ GUE Tracy-Widom fluctuations;*
- *For $\theta = \theta_c$, $F_N = \Theta(N^{1/3}(\log N)^{2/3})$;*
- *For $\theta > \theta_c$, $F_N = \Theta(\log N)$.*



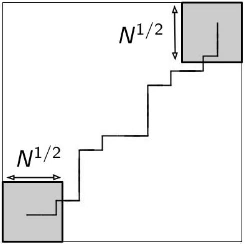
Theorem extends [Kotowski-Virág '19], which considers $\theta < \theta_c/2$.

Ideas behind the proof: subcritical case $\theta < \theta_c$

Maximal free energy: $\mathsf{F}_N = \max_{(1,1) \le X \le Y \le (N,N)} \log Z(X;Y)$.

Lower bound: $\mathsf{F}_N \ge \log Z(1,1;N,N)$ and $\log Z(1,1;N,N) + 2\Psi(\theta/2)N$ has order $N^{1/3}$ GUE Tracy-Widom fluctuations [Borodin-Corwin-Remenik '13], [Krishnan-Quastel '18] and [Barraquand-Corwin-D. '21a].

Upper bound: $N^{-1/3}(\mathsf{F}_N + 2\Psi(\theta/2)N)$.



Key ingredients:

- Moderate deviation estimates for $\log Z(X;Y)$ from [Barraquand-Corwin-D. '21a]
- Tightness of $\log Z(X;\cdots)$ and $\log Z(X;\vdots)$ [Barraquand-Corwin-D. '21b] and [D.-Wu '21]
- Polymer structure

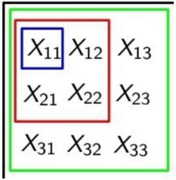
Outline

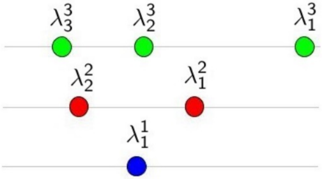
1. Gibbsian line ensembles
2. Convergence of Gibbsian line ensembles
3. Maximal free energy in the log-gamma polymer
4. β-corners processes

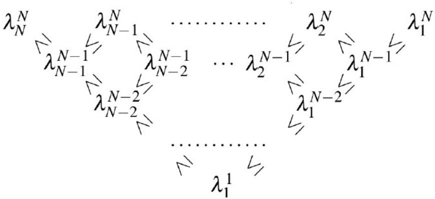
β-corners processes

Gaussian Unitary Ensemble. $\{\xi_{ij}, \eta_{ij}\}_{i,j=1}^{\infty}$ are i.i.d. $N(0,1)$ variables.

$$X_{ij} = \begin{cases} \xi_{ij} & \text{if } i = j \\ 2^{-1/2}(\xi_{ij} + \sqrt{-1}\eta_{ij}) & \text{if } i < j \\ 2^{-1/2}(\xi_{ij} - \sqrt{-1}\eta_{ij}) & \text{if } i > j \end{cases}$$







GUE-corners process: ($\beta = 2$ corners process). Gibbs property:
$\mathbb{P}(\lambda^1, \dots, \lambda^{k-1} | \lambda^k) = \prod_{i=1}^{k-1} I(\lambda^i, \lambda^{i+1})$ with $I(\lambda^i, \lambda^{i+1}) = \mathbf{1}\{\lambda^i \preceq \lambda^{i+1}\}$

β-corners processes

Gaussian Orthogonal/Unitary/Symplectic Ensemble X_{ij} are real ($\beta = 1$), complex ($\beta = 2$) and quaternion ($\beta = 4$).

Continuous β-corners process:

$$\mathbb{P}(\lambda^1, \ldots, \lambda^{k-1} | \lambda^k) = \prod_{i=1}^{k-1} I(\lambda^i, \lambda^{i+1})$$

$I(\lambda^j, \lambda^{j+1}) = \mathbf{1}\{\lambda^j \preceq \lambda^{j+1}\} \prod_{1 \le b < a \le j} (\lambda_b^j - \lambda_a^j)^{2-\beta} \prod_{a=1}^{j} \prod_{b=1}^{j+1} |\lambda_a^j - \lambda_b^{j+1}|^{\frac{\beta}{2}-1}$

Discrete β-corners process:

$$\mathbb{P}(\lambda^1, \ldots, \lambda^{k-1} | \lambda^k) = \prod_{i=1}^{k-1} I(\lambda^i, \lambda^{i+1})$$

$I(\lambda^j, \lambda^{j+1}) = J_{\lambda^{j+1}/\lambda^j}(1)$ (skew) Jack symmetric function with $\theta = \beta/2$. When $\beta = 2$ then $J_{\lambda^{j+1}/\lambda^j}(1) = S_{\lambda^{j+1}/\lambda^j}(1) = \mathbf{1}\{\lambda^j \preceq \lambda^{j+1}\}$.

Discrete β-corners process are *integrable discretizations* of continuous β-corners processes. Special cases of ascending Macdonald processes [Borodin-Corwin '14]. Appear in distributions on irreducible representations [Bufetov-Gorin '18].

Discrete β-ensembles

Projecting β-corners processes to their top level we get
Continuous: $\mathbb{P}(\lambda_1, \ldots, \lambda_N) \propto \prod_{1\le i<j\le N}(\lambda_i - \lambda_j)^{2\beta} \prod_{i=1}^N e^{-V_N(\lambda_i)}$.
Discrete: $\mathbb{P}(\ell_1, \ldots, \ell_N) \propto \prod_{1\le i<j\le N} Q_\theta(\ell_i - \ell_j) \prod_{i=1}^N e^{-V_N(\ell_i)}$, $Q_\theta(x) := \frac{\Gamma(x+1)\Gamma(x+\theta)}{\Gamma(x)\Gamma(x+1-\theta)}$.

β-log gas and *Discrete β-ensembles* - [Borodin-Gorin-Guionnet '17]

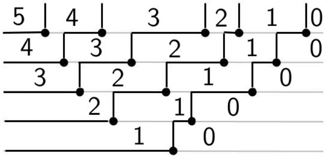
- Law of large numbers: $\mu_N = N^{-1} \sum_{i=1}^N \delta_{\ell_i/N}$ concentrate near μ_{eq} (Wigner's Semicircle Law)
- Global Central Limit Theorem: $\sum_{i=1}^N f(\ell_i/N) - \mathbb{E}\left[\sum_{i=1}^N f(\ell_i/N)\right]$ is asymptotically Gaussian [Borodin-Gorin-Guionnet '17]
- Edge universality: [Bourgade-Erdős-Yau '14],[Guionnet-Huang '19]
- Edge large deviations: [Johansson '98, '00], [Féral '08]

Theorem (Das-D. '21)

For general V_N's the random variables ℓ_1/N satisfy a large deviation principle.

Upper tail rate is N and lower tail rate is N^2. Rate functions are explicit but *different* from continuous β-log gases (due to discreteness of the model).

Global asymptotics for β-corners processes



β-corners process $\Longrightarrow$ *height function $H(x,y)$*
Conjecture: $H(x,y)$ converges to a suitable pullback of the *Gaussian Free field on $\mathbb{H}$*.

Known for Wigner matrices [Borodin '10].

Stieltjes transform: $G(z,s) = \sum_{i=1}^{s} \frac{1}{z-\ell_i^s}$.
[D.-Knizel '21]: *multi-level loop equations* or *Nekrasov equations* [Nekrasov '16] = functional equations relating joint cumulants of $G(z_1,s_1),\dots,G(z_k,s_k)$.

1. Generalize single level loop equations in [Borot-Guionnet '13] and Nekrasov equations in [Borodin-Gorin-Guionnet '17]
2. [D.-Knizel '19]: $G(z_1,N), G(z_2,N-1), N^{1/2}[G(z_3,N)-G(z_3,N-1)]$ have joint Gaussian limits.

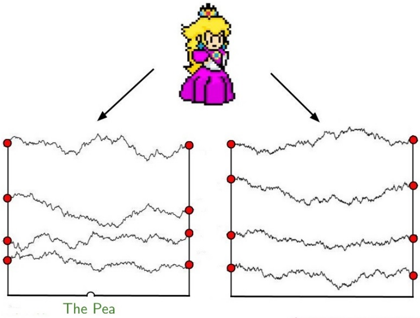
Explanation: Top two levels converge to the same $1D$ slice of the $2D$ GFF, and their difference to a certain directional derivative of the GFF. The analogue for Wigner matrices is proved in [Erdős-Schröder '18].

Summary

Gibbs properties + integrable (algebraic) input + analytic tools

- KPZ universality for Gibbsian line ensembles
- Asymptotics for polymer models
- Asymptotics for β-corners processes
- (Did not discuss) Convergence of six-vertex models to the GUE cornerss process [D. '18], [D.-Rychnovsky '20]

Thank you!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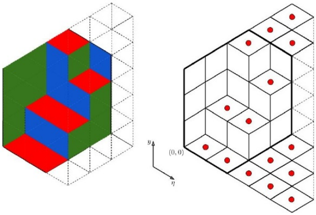

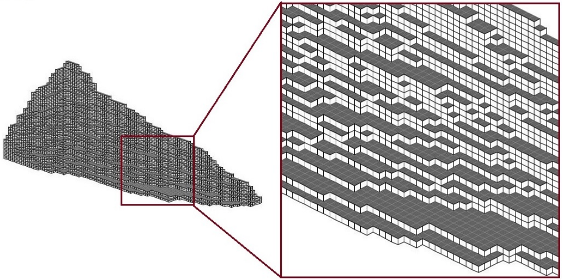

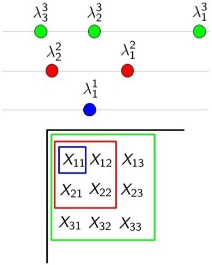
λ_3^3 λ_2^3 λ_1^3
λ_2^2 λ_1^2
λ_1^1
X_{11} X_{12} X_{13}
X_{21} X_{22} X_{23}
X_{31} X_{32} X_{33}