Interacting diffusions on pos. def. matrices

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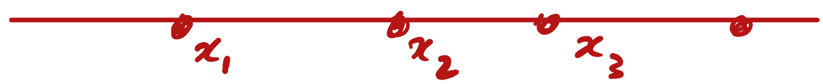
Brownian motions with one-sided reflections



$x_1(t)$ ~ Brownian motion

$x_2(t)$ ~ Brownian motion reflected off $x_1(t)$

etc.

"Step" initial condition: $x_i(0) = 0 \;\; \forall i$

Baryshnikov / Gravner-Tracy-Widom (2000):

Fixed t: $\quad x_n(t) \overset{\text{dist.}}{=} \sqrt{t}\; \lambda_{\max}(GUE_{n \times n})$

Bougerol-Jeulin (2002), O'C-Yor (2002):

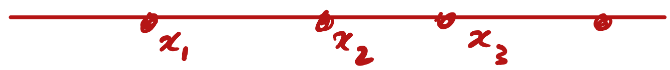
$(x_n(t),\; t \geq 0)$

$\overset{\text{dist.}}{=}$ top line of n-particle Dyson BM

Case $n=2$ equiv. to thm of Pitman (1975)

Exponential interactions :

$$dx_1 = d\beta_1 , \quad dx_2 = d\beta_2 + e^{x_1 - x_2} dt , \ldots$$



n-particle quantum Toda chain :

$$H = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} - \sum_{i=1}^{n-1} e^{y_i - y_{i+1}}$$

Ground state $\quad H\psi_0 = 0 , \quad \psi_0(y) > 0$

Doob transform :

$$H^{\psi_0} = \psi_0(y)^{-1} \circ H \circ \psi_0(y)$$

$$= \sum_{i=1}^{n} \left(\frac{1}{2} \frac{\partial^2}{\partial y_i^2} + \underbrace{\frac{\partial}{\partial y_i} \ln \psi_0(y)}_{=: \mu_i(y)} \frac{\partial}{\partial y_i} \right)$$

$$dy_i = d\beta_i + \mu_i(y)\, dt$$

O'C (2012) : For appropriate initial cond.

$$(x_n(t) , t \geq 0) \overset{\text{dist.}}{=} (y_n(t) , t \geq 0)$$

Case $n=2$ equiv. to thm of Matsumoto-Yor (1999)

The case $n = 2$

$B_t^{(\nu)}, \ t \geq 0$ standard 1-d BM, drift ν

$$A_t^{(\nu)} = \int_0^t e^{-2B_s^{(\nu)}} \, ds$$

Theorem (Matsumoto - Yor 1999)

$$X_t = B_t^{(\nu)} + \ln A_t^{(\nu)} \quad , \ t > 0$$

is a diffusion process with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{d}{dx} \ln K_\nu(e^{-x}) \cdot \frac{d}{dx}$$

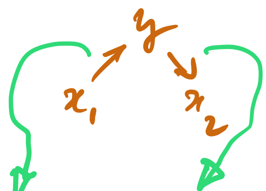
Related Theorem (Dufresne 1990)

If $\nu > 0$, $\quad A_\infty^{(\nu)} \overset{\text{dist.}}{=} \dfrac{1}{2\gamma_\nu}$.

Gamma dist (ν)

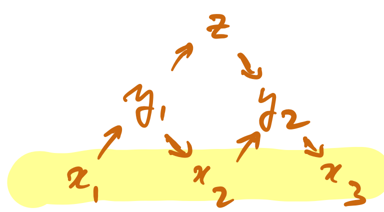
$$\frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x} \, dx$$

Givental's Formula



$$\psi_0(x_1, x_2) = \int_{\mathbb{R}} \exp\left(-e^{x_1 - y} - e^{y - x_2}\right) dy$$

$$= 2\, K_0\left(2\, e^{(x_1 - x_2)/2}\right).$$



$$\psi_0(x_1, x_2, x_3)$$

$$= \int_{\mathbb{R}^3} \exp\left(-e^{x_1 - y_1} - e^{y_1 - x_2} - e^{x_2 - y_2} - e^{y_2 - x_3} - e^{y_1 - z} - e^{z - y_2}\right) dz\, dy_1\, dy_2$$

$GL(n, \mathbb{R})$ - Whittaker fctns.

Non-Abelian Toda lattice

Polyakov (1980), Popowicz (1981, 1983)

$X_1, \dots, X_n \in GL(d, \mathbb{R})$

(*) $\dot{X}_i = P_i \,, \quad \dot{P}_i = P_i X_i^{-1} P_i + X_{i-1} - X_i X_{i+1}^{-1} X_i$

Scalar case: $d = 1$, $X_i = e^{x_i}$

$\ddot{x}_i = e^{x_{i-1} - x_i} - e^{x_i - x_{i+1}}$

$\mathcal{P}_d$ = positive definite $d \times d$ matrices (real)

$\mathcal{S}_d$ = real symmetric $d \times d$ matrices

The space $(\mathcal{P}_d^n, \mathcal{S}_d^n)$ is invariant under the eqns of motion (*).

So a natural quantisation is:

$$H = \sum_{i=1}^{n} \Delta_{X_i} - 2 \sum_{i=1}^{n-1} \operatorname{tr}(X_i X_{i+1}^{-1})$$

Laplace-Beltrami operator on $\mathcal{P}_d$.

Ground State

$$\begin{tikzcd} & Y \arrow[dr] & \\ X_1 \arrow[ur] & & X_2 \end{tikzcd}$$

$$\Psi_0(X_1, X_2) = \int_{\mathcal{P}_d} \operatorname{etr}\left(-X_1 Y^{-1} - Y X_2^{-1}\right) d\mu(Y)$$

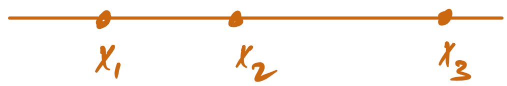
$\operatorname{etr} = \exp \circ \operatorname{tr}$

$$= B_0(X_1 X_2^{-1})$$ Herz (1955)

$$\begin{tikzcd} & & Z \arrow[dr] & & \\ & Y_1 \arrow[ur] \arrow[dr] & & Y_2 \arrow[dr] & \\ X_1 \arrow[ur] & & X_2 \arrow[ur] & & X_3 \end{tikzcd}$$

$$\Psi_0(X_1, X_2, X_3) = \int_{\mathcal{P}_d^3} \operatorname{etr}\left(-X_1 Y_1^{-1} - Y_1 Z^{-1} - \dots \right) d\mu(Y_1)\, d\mu(Y_2)\, d\mu(Z)$$

Main Result



$$\partial X_1 = X_1^{1/2}\, \underbrace{\partial\beta_1}_{BM(S_d)}\, X_1^{1/2} \qquad \equiv BM(P_d)$$

$$\partial X_i = X_i^{1/2}\, \underbrace{\partial\beta_i}_{BM(S_d)}\, X_i^{1/2} + X_{i-1}\, dt \qquad (i \geq 2)$$

Theorem: for appropriate init. cond.,

$$(X_n(t), t \geq 0) \overset{dist.}{=} (Y_n(t), t \geq 0)$$

where $(Y_1, \dots, Y_n)$ is a diffusion in P_d^n with generator H^{Ψ_0}

Brownian motion on P_d

$(Y_t, t \geq 0)$ $\quad$ $\partial Y = Y^{1/2}\, \partial\beta\, Y^{1/2}$ $\leftarrow BM(S_d)$

Evals $\lambda_i(t)$:

$$\lim_{t\to\infty} \frac{1}{t} \ln \lambda_i(t) = \frac{d-2i+1}{2} \quad \text{a.s.}$$

Evectors converge almost surely to random element of $O(d)$.

Simple drift : $\nu \in \mathbb{R}$,

$$\partial Y = Y^{1/2}\, \partial\beta\, Y^{1/2} + \nu Y\, dt \quad .$$

$$\lim_{t\to\infty} \frac{1}{t} \ln \lambda_i(t) = \nu + \frac{d-2i+1}{2} \quad \text{a.s.}$$

This is a Doob transform of $BM(P_d)$ via the eigenfctn $\varphi(X) = (\det X)^{\nu/2}$.

Infinitesimal Generator

$$\partial_X = \frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial x_{ij}}, \quad \vartheta_X = X\partial_X$$

$$\Delta_X = \operatorname{tr} \vartheta_X^2$$

$$\Delta_X^{(\nu)} = \operatorname{tr}\vartheta_X^2 + \nu \operatorname{tr}\vartheta_X$$

Matrix Dufresne Identity (Rider-Valkó 2016)

If Y_t is BM in P_d w/ drift ν,
with $Y_0 = I$ and $\nu < \frac{1-d}{2}$,
then $Z = \int_0^\infty Y_t \, dt$ is inverse Wishart
distributed with

$$\mathbb{E}\, e^{-\operatorname{tr}(XZ)} = \Gamma_d(-\nu)^{-1} \underbrace{\int_{P_d} (\det A)^{\nu} \operatorname{etr}\left(-XA - A^{-1}\right) d\mu(A)}_{B_\nu(X)}$$

Herz Bessel function

$$\Gamma_d(a) = \pi^{d(d-1)/4} \prod_{i=1}^{d} \Gamma\left(a - \frac{i-1}{2}\right)$$

Matsumoto-Yor type theorem

(Rider-Valkó for $|\nu| > (d-1)/2$
O'C 2021 for all $\nu \in \mathbb{R}$)

If Y_t is BM in P_d w/ drift ν with $Y_0 = I$ and $A_t = \int_0^t Y_s \, ds$, then

$$X_t = A_t^{-1} Y_t A_t^{-1}, \quad t > 0$$

is diffusion in P_d with gen.

$$\Delta_X^{(\nu)} + 2 \, tr(\partial_X \ln B_\nu(X) \, \partial_X)$$

More general drifts via spherical fctns:

$\varphi_\mu(X)$ - transform satisfies:

$$\lim_{t\to\infty} \frac{1}{t} \ln \lambda_i(t) = -2\mu_i \quad \text{a.s.}$$

Herz Bessel fctn extends to:

$$B_\mu(X) = \varphi_\mu(X)^{-1} \int \varphi_\mu(AXA)\, \operatorname{etr}(-A^{-1} - AX)\, d\mu(A)$$

"*Theorem*:" If Y is BM "with drift μ" started at X, then (mod technicalities)

$$\mathbb{E} \exp\left(-\int_0^\infty \operatorname{tr} Y_s \, ds\right) = c_\mu^{-1} B_\mu(X)$$

where

$$c_\mu = \int \varphi_\mu(A^2)\, \operatorname{etr}(-A^{-1})\, d\mu(A)$$

$$\underset{\text{Conjecture!}}{=} \prod_i \Gamma(2\mu_i) \prod_{i<j} B\left(\tfrac{1}{2}, \mu_i + \mu_j\right)$$

TRUE for $d \le 3$ and $\forall d$ in complex case with $\frac{1}{2} \to 1$.

Remark

In the complex setting, if $Y_0 = I$,

$$\frac{1}{2}\int_0^\infty \operatorname{tr} Y_s \, ds \overset{\text{dist.}}{=} \sum_i z_i$$

where z has density

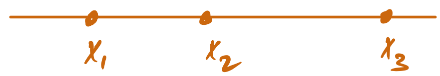
$$b_\mu^{-1} \det\left(z_i^{-2\mu_j}\right) \prod_{i<j} \frac{z_i - z_j}{z_i + z_j} \prod_i e^{-1/z_i} \frac{dz_i}{z_i}$$

"generalised Bures measure"

c.f. Wang-Li (2019)

Pfaffian point process related to type B KP hierarchy.

Invariant measures, etc.



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$$dX_i = X_i^{1/2}\, \underset{\uparrow\, BM(S_d)}{d\beta_i}\, X_i^{1/2} + X_{i-1}\, dt \quad (i \geq 2)$$

Product-form invariant measures ✓

"Burke" output theorem ✓

Hydrodynamic limit ?

eg. Step i.c. $X_0 = I$, $X_i = 0$ $i \geq 1$

What is $\lim_{n\to\infty} \frac{1}{n} \ln \| X_n(nt) \|$?

When $d=1$, essentially free energy of semi-discrete random polymer.