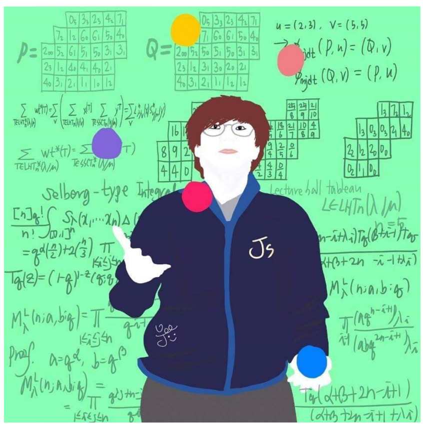
Lecture Hall tableaux, non-intersecting paths, tilings

Sylvie Corteel



Jang Soo Kim

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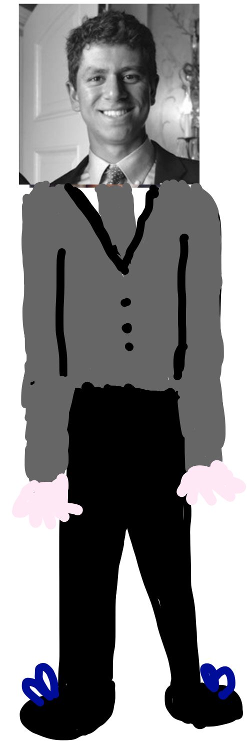
David Keating



Matthew Nicoletti

Zhongyang Li

Tableaux, non intersecting paths and tilings

Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$$

Ferrers diagram

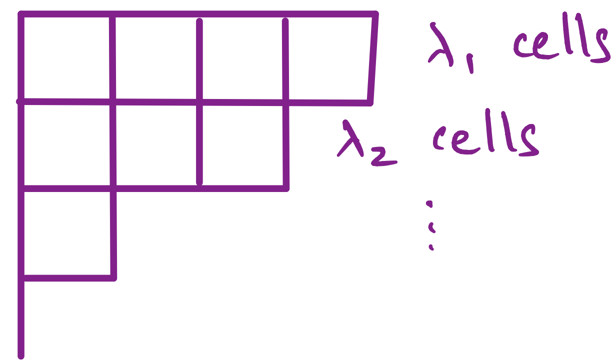
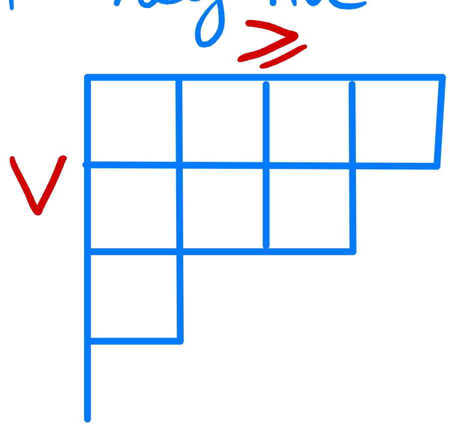


Tableau : filling of the diagram with non negative integers $< n$



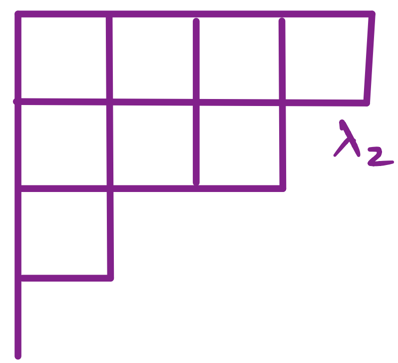
Tableaux, non intersecting paths and tilings

Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0$$

Ferrers diagram

$\lambda = (4,3,1,0)$



λ_1 cells

λ_2 cells

$\vdots$

Tableau : filling of the diagram with non negative integers $< n$

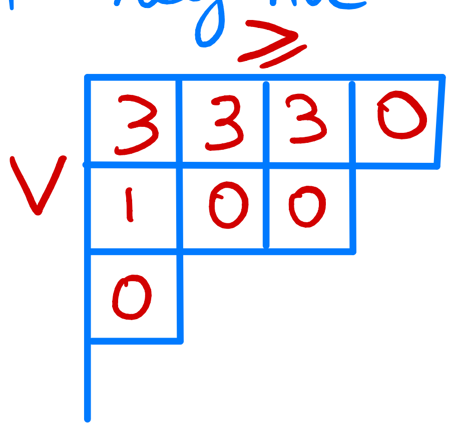


__Tableau__ : filling of the diagram with non negative integers $< n$

3	3	3	0
1	0	0	
0			

__Non intersecting paths__ $\lambda = (4,3,1,0)$

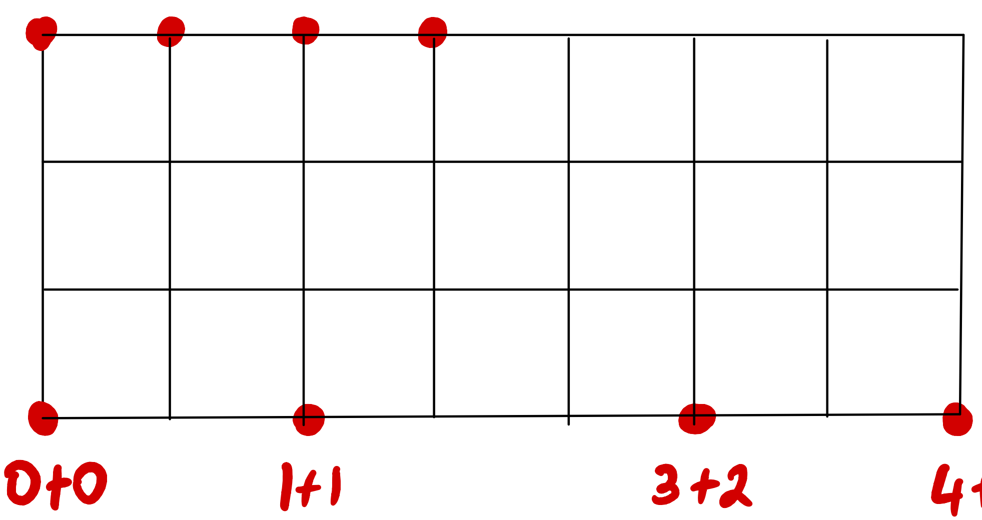
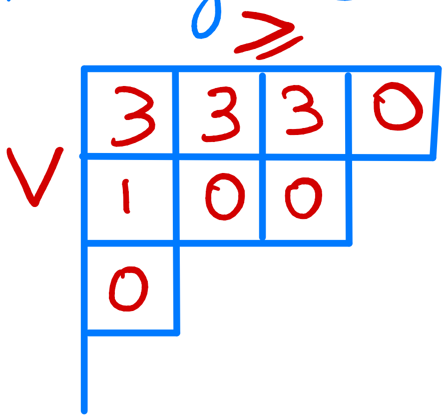


Tableau : filling of the diagram with non negative integers < n



Non intersecting paths

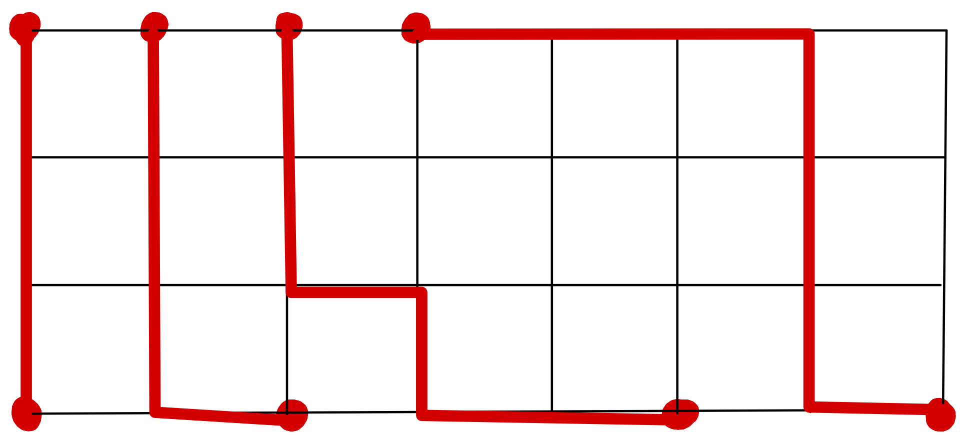
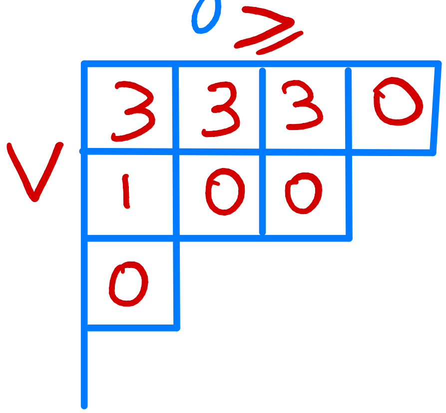
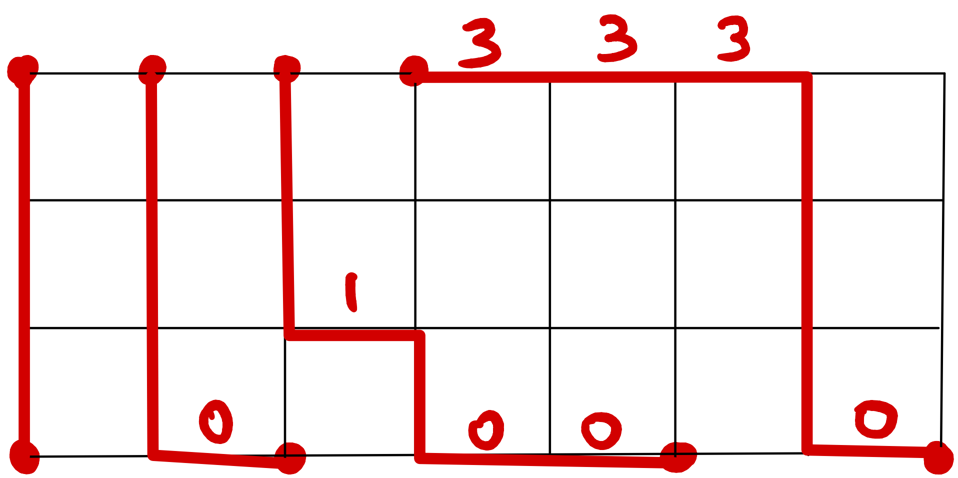


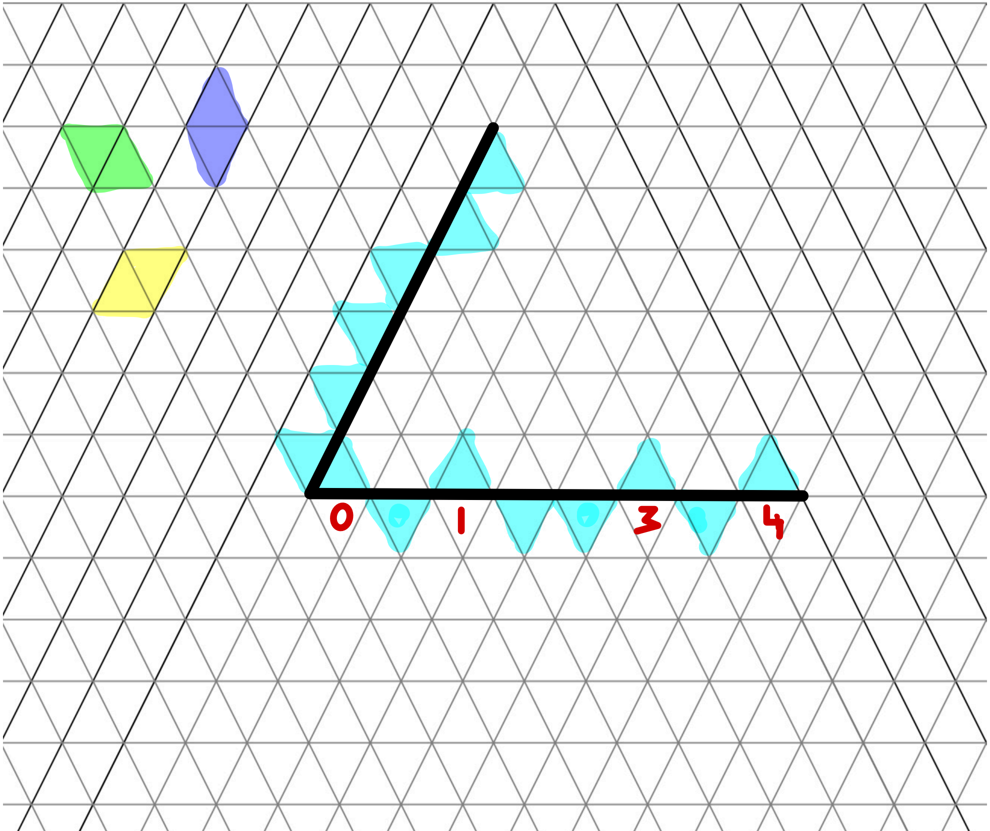
Tableau : filling of the diagram with non negative integers $< n$

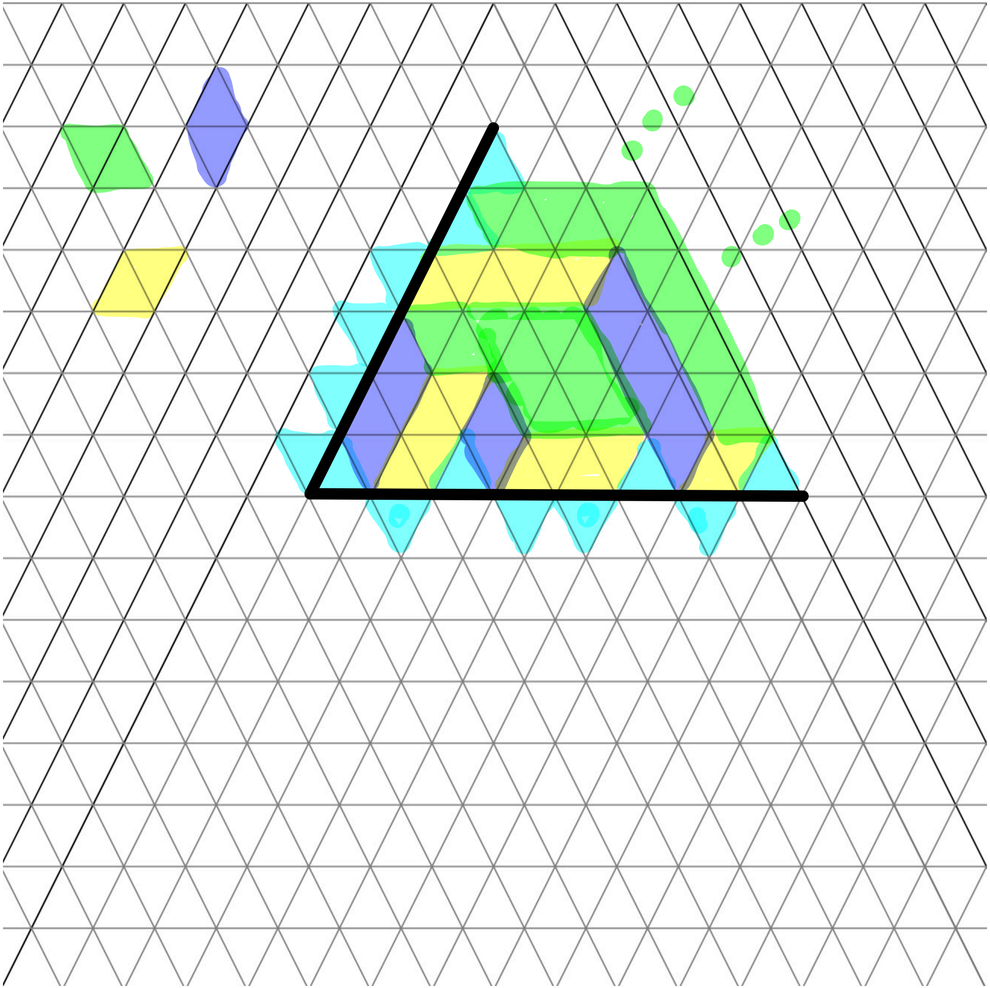


Non intersecting paths

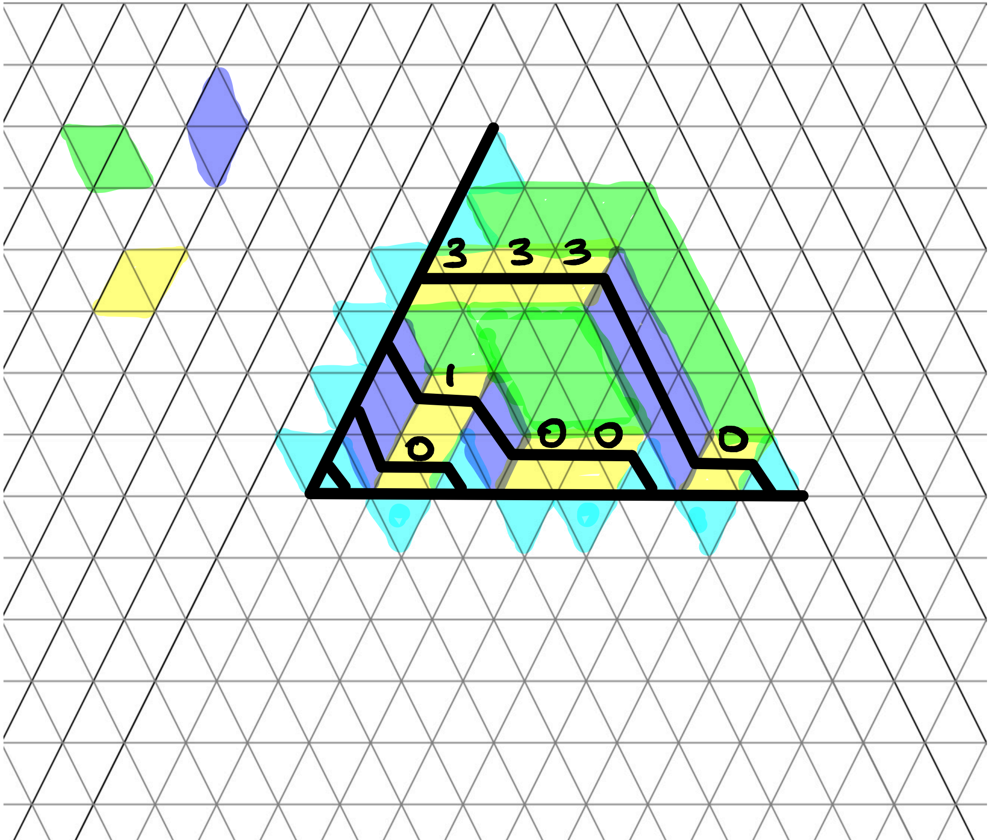


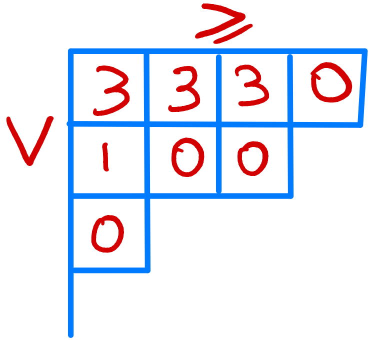
Tilings



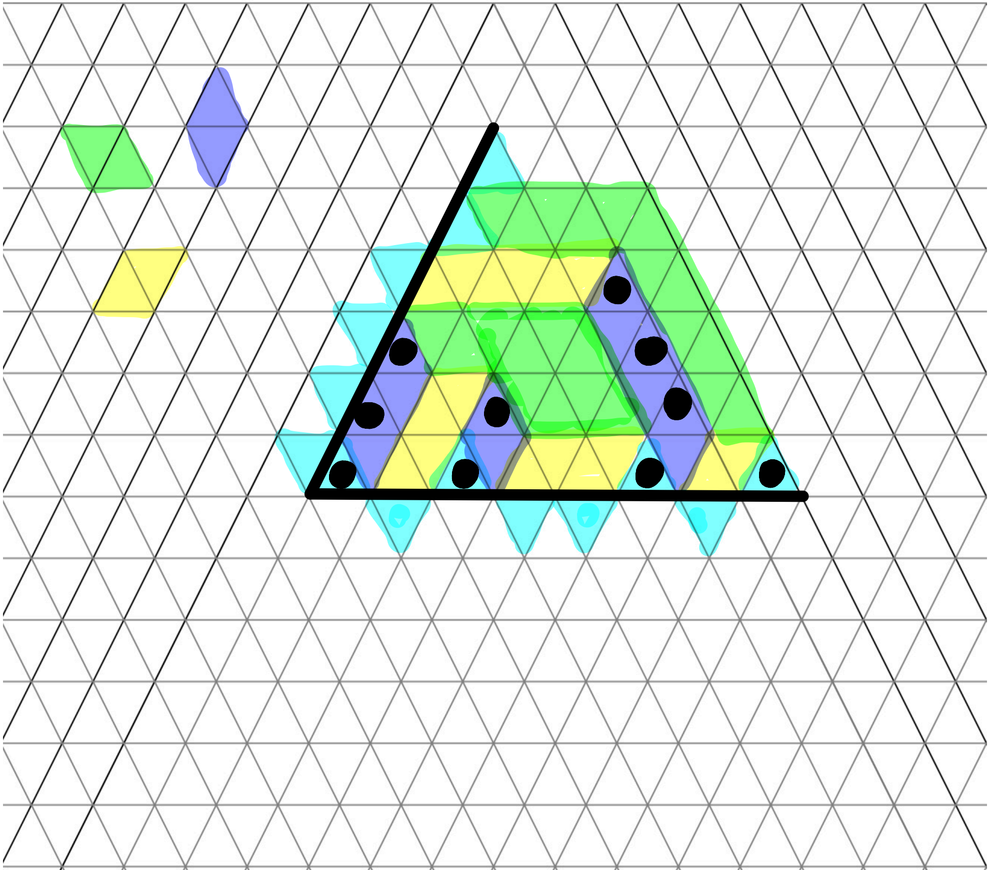


Tilings <=> Paths <=> Tableau





Tilings <=> Particles



3

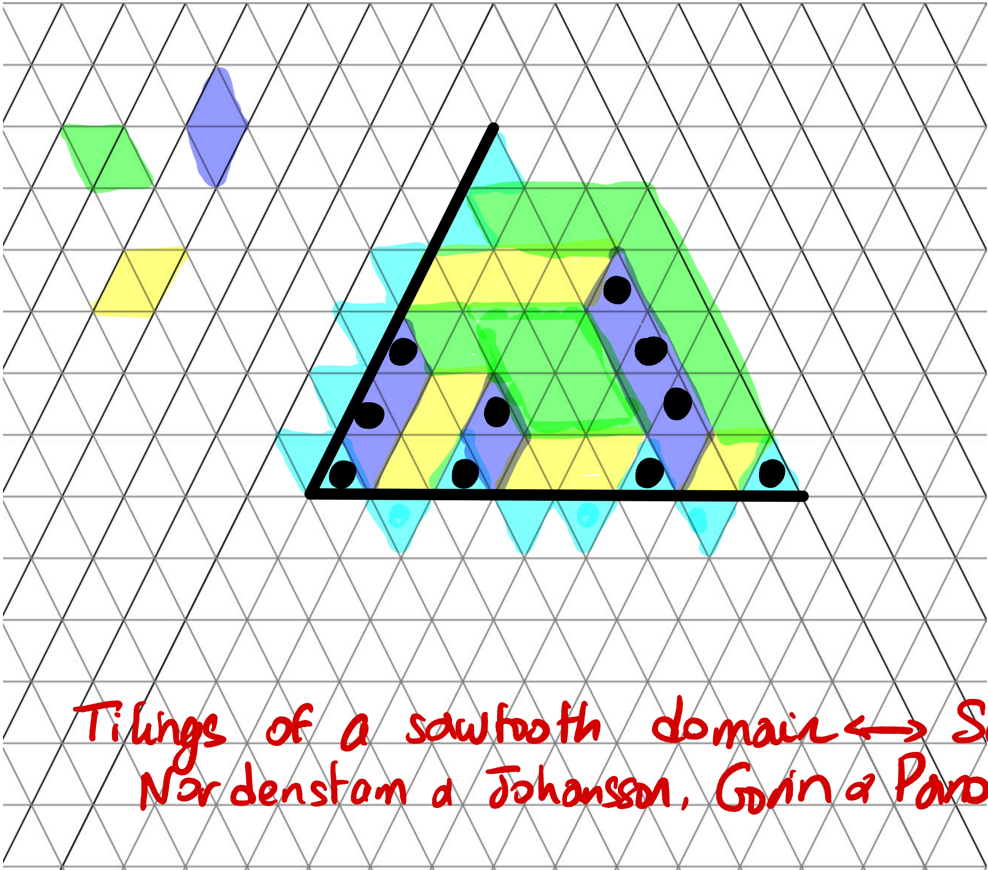
3 0

3 1 0

4 3 1 0

Theorem The vertical tiles are distributed like the eigenvalues of a random GUE matrix

Tilings ⟺ Particles



$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 3 & & 0 & & \\ & 3 & & 1 & & 0 & \\ 4 & & 3 & & 1 & & 0 \end{array}$$

Theorem The vertical tiles are distributed like the eigenvalues of a random GUE matrix

Tilings of a sawtooth domain ⟷ Schur polynomials
Nordenstam & Johansson, Gorin & Panova, Novak, Aggarwal
...

Lecture Hall

Origin: Combinatorics of Coxeter groups

90s: Eriksson & Eriksson, Bousquet-Mélou

Type A

$$\sum_{w \in \tilde{A}_n/A_n} q^{\ell(w)} = \sum_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0} q^{|\lambda|}$$

Type C

$$\sum_{w \in \tilde{C}_n/C_n} q^{\ell(w)} = \sum_{\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \dots \geq \frac{\lambda_n}{1} \geq 0} q^{|\lambda|}$$

Lecture-Hall partitions

Lecture Hall "all over"

Coxeter Combinatorics

q-series
Computer algebra
Bijection
Little q-Jacobi pols

Polytopes
Ehrhart theory
Geometric combinatorics

P-partitions

Lecture Hall "all over"

Coxeter
Combinatorics

q-series
Omega operator
Bijection
$\boxed{\text{Little } q\text{-Jacobi pols}}$

Polytopes
Ehrhart theory
Geometric combinatorics

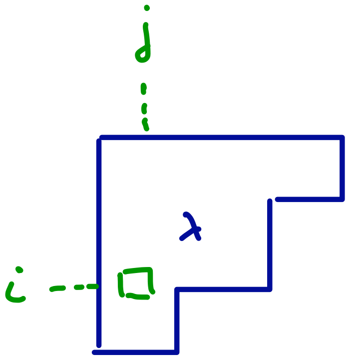
P-partitions Brändén & Leander
Tableaux, paths & Tilings

TODAY

Lecture Hall tableaux

Two partitions λ

Two integers n, t



Fill the cell (i,j) with $T_{i,j}$

$$\begin{cases} \dfrac{T_{ij}}{n-i+j} \geq \dfrac{T_{ij+1}}{n-i+j+1} \\ \dfrac{T_{ij}}{n-i+j} > \dfrac{T_{i+1,j}}{n-1-i+j} \end{cases}$$

$$T(i,j) < t\,(n-i+j)$$

Example

$n = 5$ $\lambda = (4,3,1,0,0)$

$t = 4$

T

16	16	9	4
12	13	6	
2			

$\geqslant$

V

$\frac{16}{5}$	$\frac{16}{6}$	$\frac{9}{7}$	$\frac{4}{8}$
$\frac{12}{4}$	$\frac{13}{5}$	$\frac{6}{6}$	
$\frac{2}{3}$			

Associate a monomial to each tableau

Given T_{ij} $\nearrow$ $b_{ij} = \left\lfloor \frac{T_{ij}}{n-i+j} \right\rfloor$

$\searrow$ $e_{ij} = T_{ij} - (n-i+j) \left\lfloor \frac{T_{ij}}{n-i+j} \right\rfloor$

$$wt(T) = \prod_{(i,j) \in \lambda} y_{b_{ij}} \prod_{\substack{(i,j) \in \lambda \\ b_{ij} = t-1}} x_{e_{ij}}$$

16	16	9	4
12	13	6	
2			

$$wt(T) = y_3^2 \, y_2^2 \, y_1^2 \, y_0^2 \, x_1 x_0$$

$$wt(T) = \prod_{(i,j)\in\lambda} y_{b_{ij}} \prod_{\substack{(i,j)\in\lambda \\ b_{ij}=t-1}} x_{e_{ij}}$$

$$\begin{array}{|c|c|c|c|} \hline 16 & 16 & 9 & 4 \\ \hline 12 & 13 & 6 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array}$$

b

$$\begin{array}{|c|c|c|c|} \hline 3 & 2 & 1 & 0 \\ \hline 3 & 2 & 1 \\ \cline{1-3} 0 \\ \cline{1-1} \end{array}$$

e

$$\begin{array}{|c|c|c|c|} \hline 1 & 4 & 2 & 4 \\ \hline 0 & 3 & 0 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array}$$

$$wt(T) = y_3^2\, y_2^2\, y_1^2\, y_0^2\, x_1 x_0$$

Theorem (C., Kim 19)

The generating polynomial

$$Z_\lambda(x_0,\ldots,x_{n-1},y_0,\ldots,y_{t-1}) = \sum_{T \text{ shape } \lambda} wt(T) =$$

$$S_\lambda(w_0, w_1, \ldots, w_{n-1})$$

where $w_i = x_i y_{t-1} + y_{t-2} + \ldots + y_0$

Theorem (C., Kim 19)

The generating polynomial

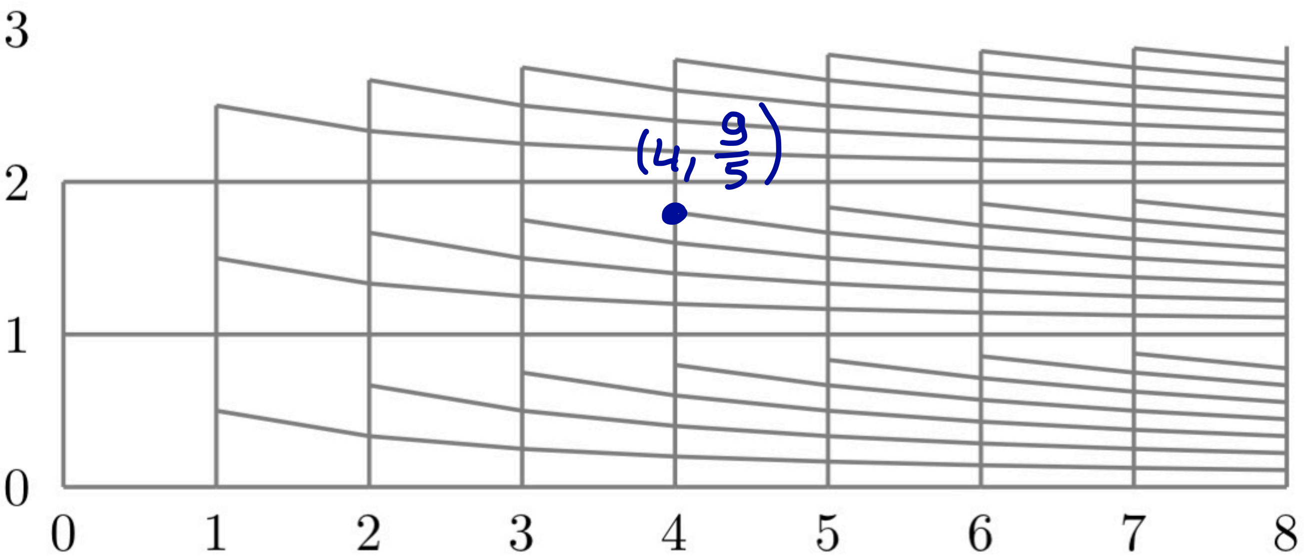
$$Z_\lambda(x_0,\dots,x_{n-1},y_0,\dots,y_{t-1}) = \sum_{T \text{ shape } \lambda} wt(T) =$$

$$S_\lambda(w_0, w_1, \dots, w_{n-1})$$

where $w_i = x_i y_{t-1} + y_{t-2} + \dots + y_0$

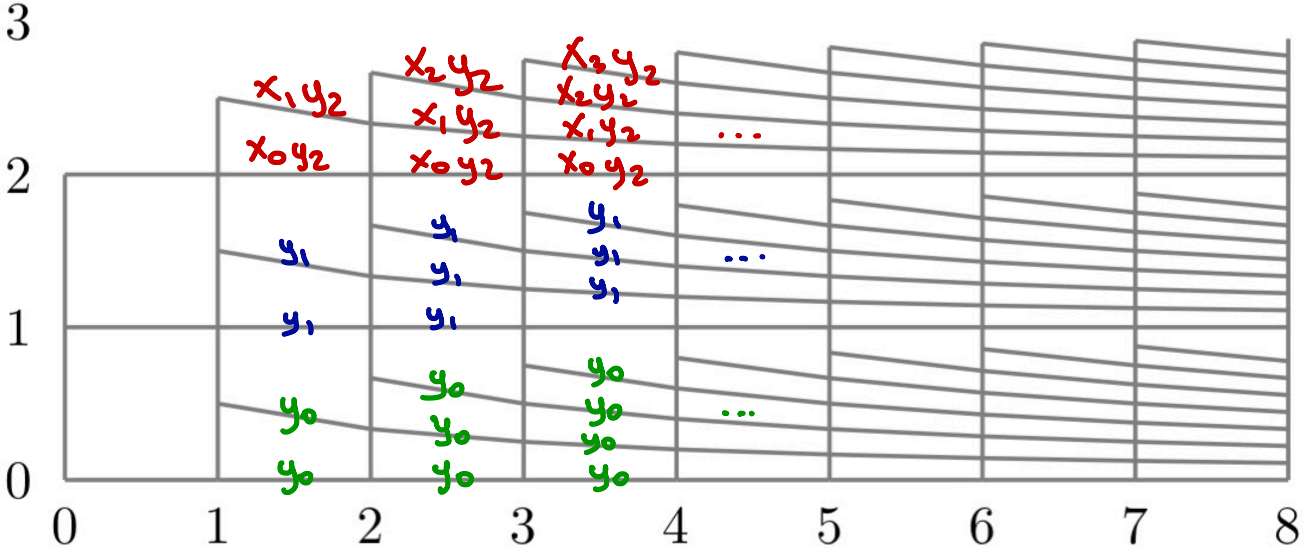
$n=1 \Rightarrow$ Classical case

Lecture Hall graph

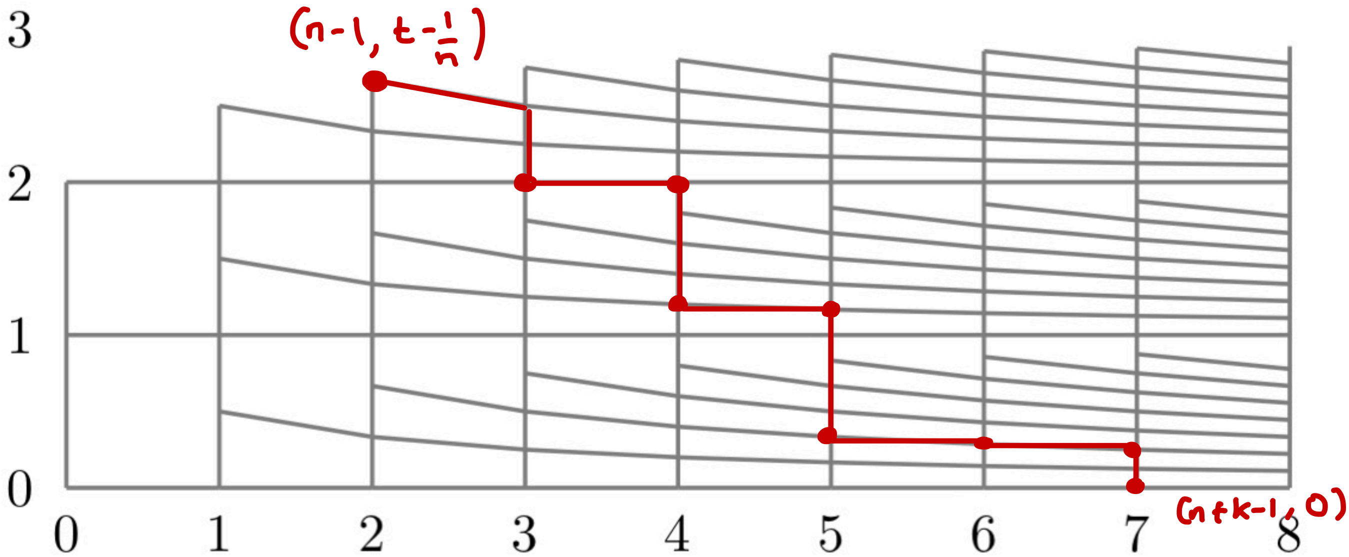


Vertices $\left(x, \frac{y}{x+1}\right)$ $x \in \mathbb{Z}^+$, $y < t(x+1)$ $y \in \mathbb{Z}^+$

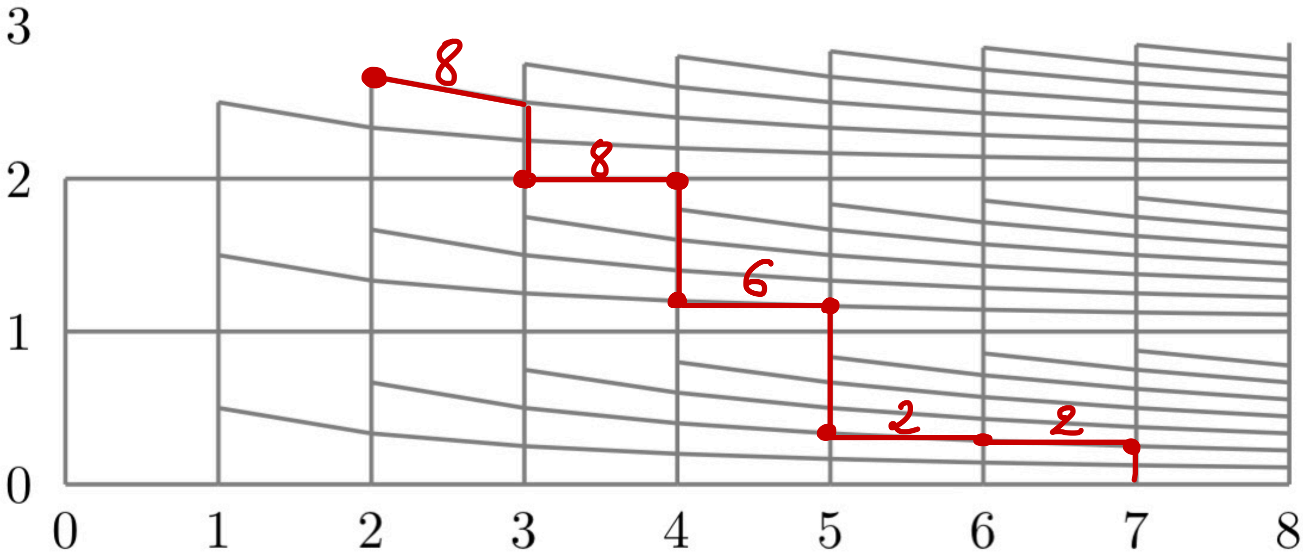
Lecture Hall graph



Path on the lecture hall graph

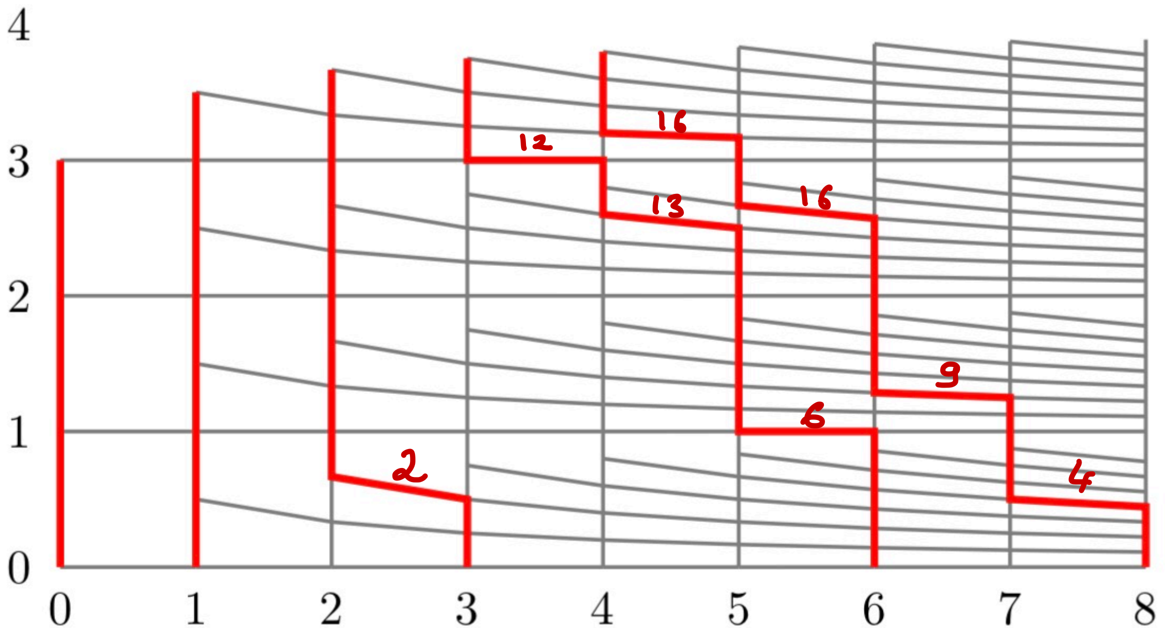


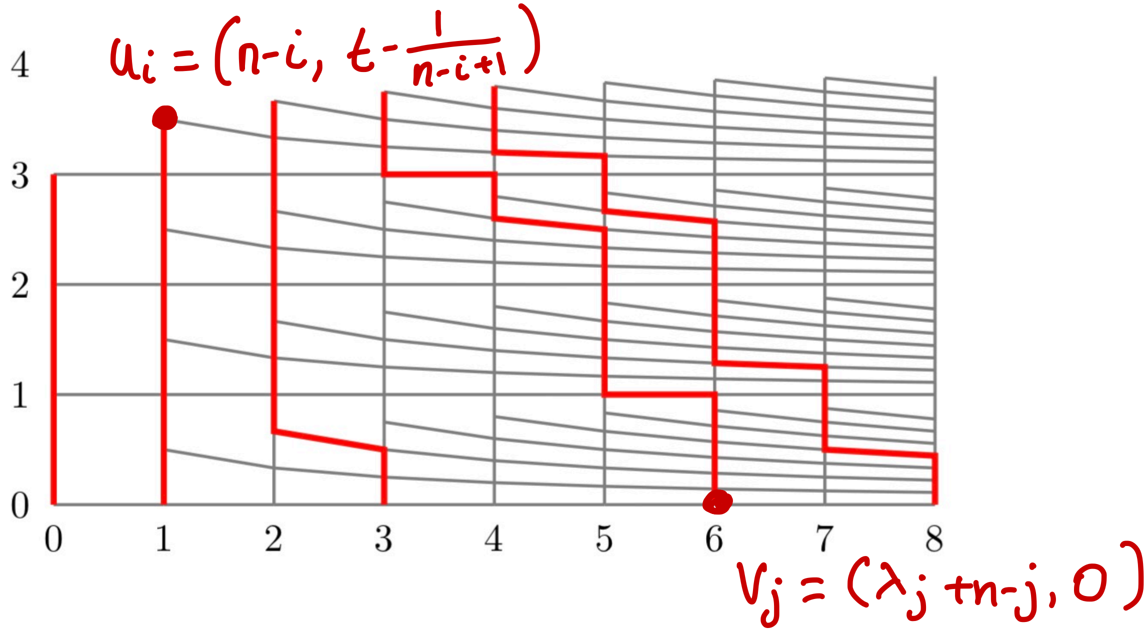
ex $n = 3$ $k = 5$ $t = 3$



$$3 > \frac{8}{3} \geqslant \frac{8}{4} \geqslant \frac{6}{5} \geqslant \frac{2}{6} \geqslant \frac{2}{7}$$

16	16	9	4
12	13	6	
2			





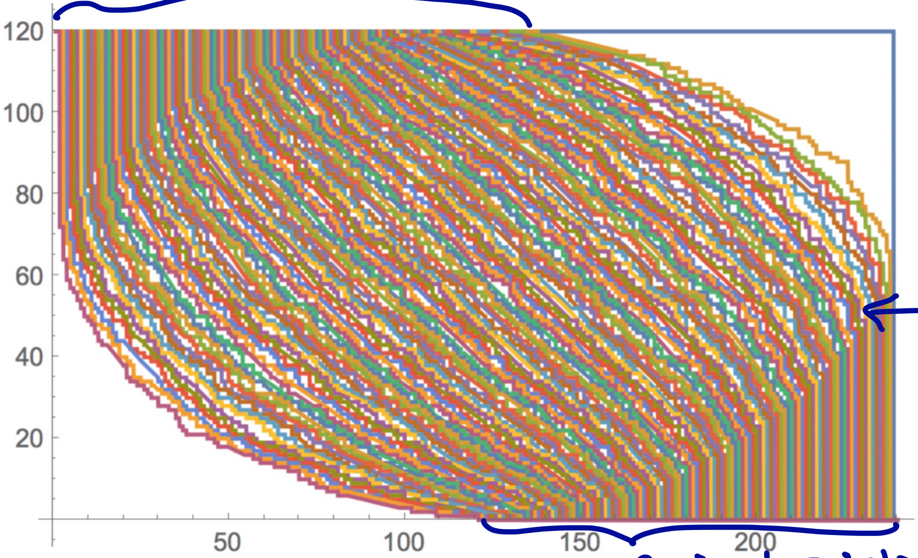
$$P(u_i, v_j) = h_{\lambda_i - i + j}(w_0, \dots, w_{n-1})$$

$$z_\lambda = \det\left(P(u_i, v_j)_{1 \le i,j \le n}\right)$$

How does a random tableau look like?

Fix λ, n, t, pick a random tableau

starting points



arrival points

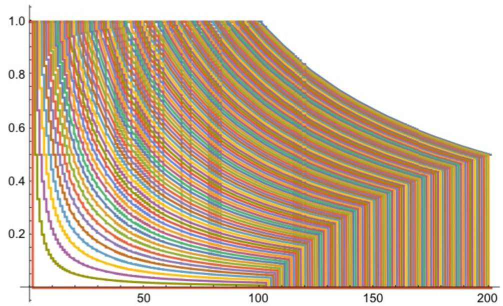
$n = 120$

$\lambda = (n, \ldots, n)$

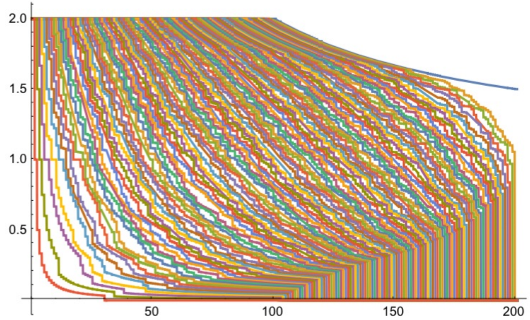
$t = n$

non intersecting paths

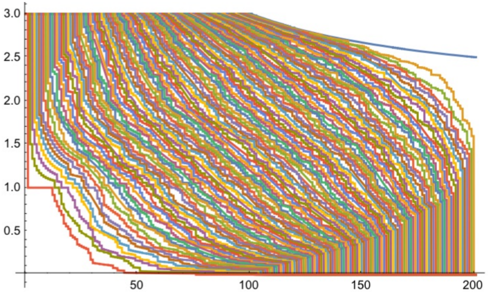
$\lambda = (n, \dots, n) \qquad n = 100$



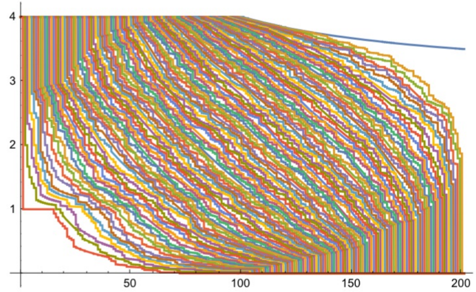
$t = 1$



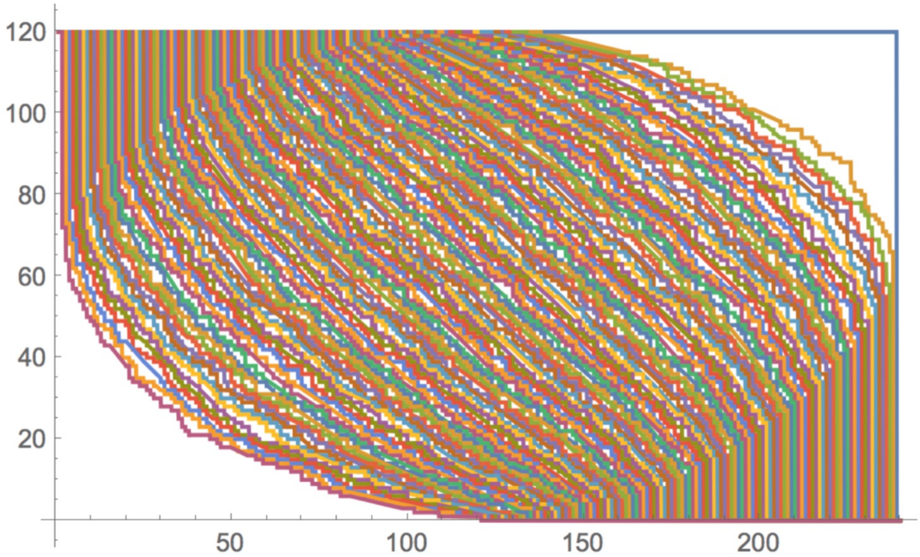
$t = 2$



$t = 3$



$t = 4$

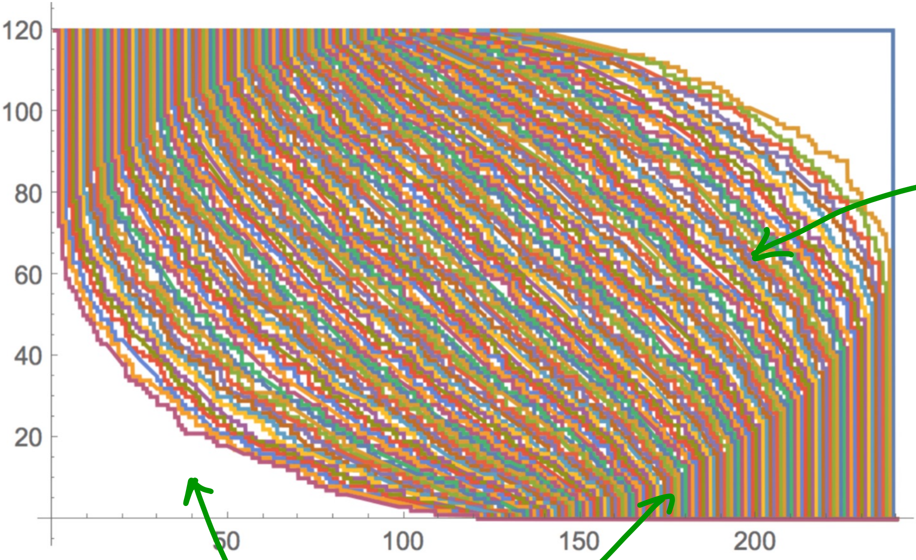


$n = 120$

$t = 120$

Arctic curve phenomenon

Sharp phase separation

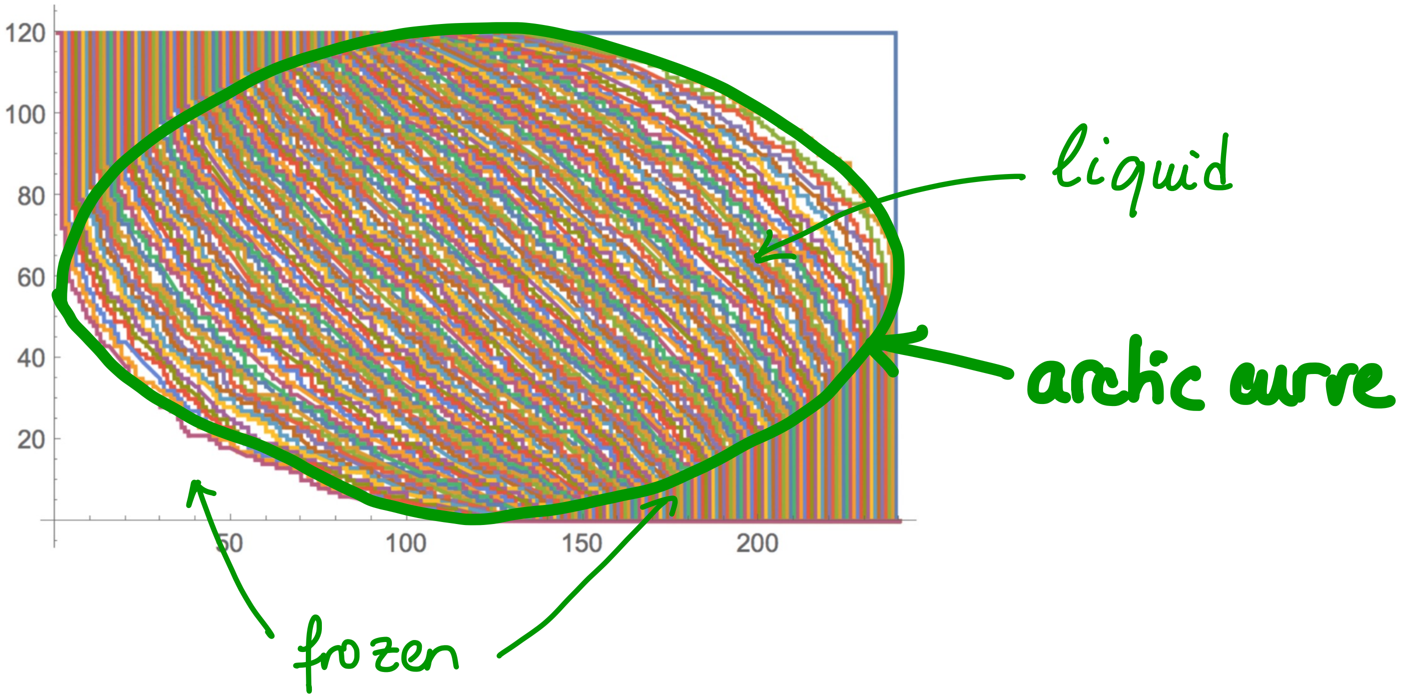


liquid

frozen

Arctic curve phenomenon

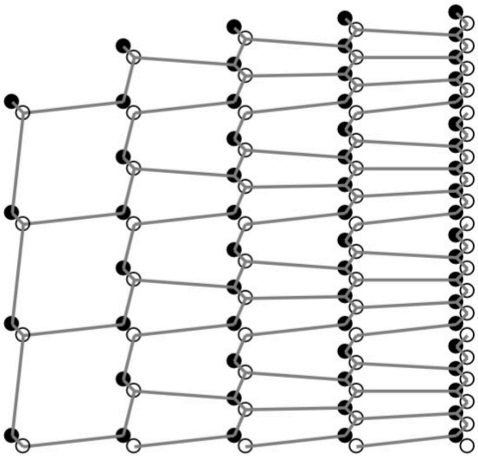
Sharp phase separation



How to compute the arctic curve? First technique

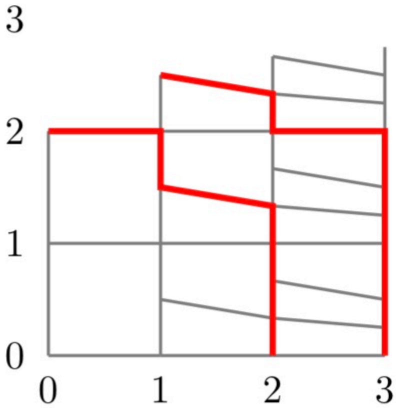
Non intersecting paths $\Longrightarrow$ Dimer model

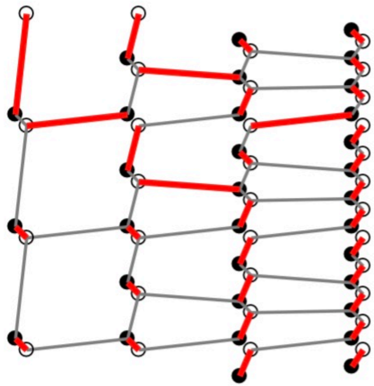
Each vertex $\bullet \rightsquigarrow$ 

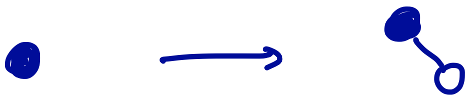


How to compute the arctic curve? First technique

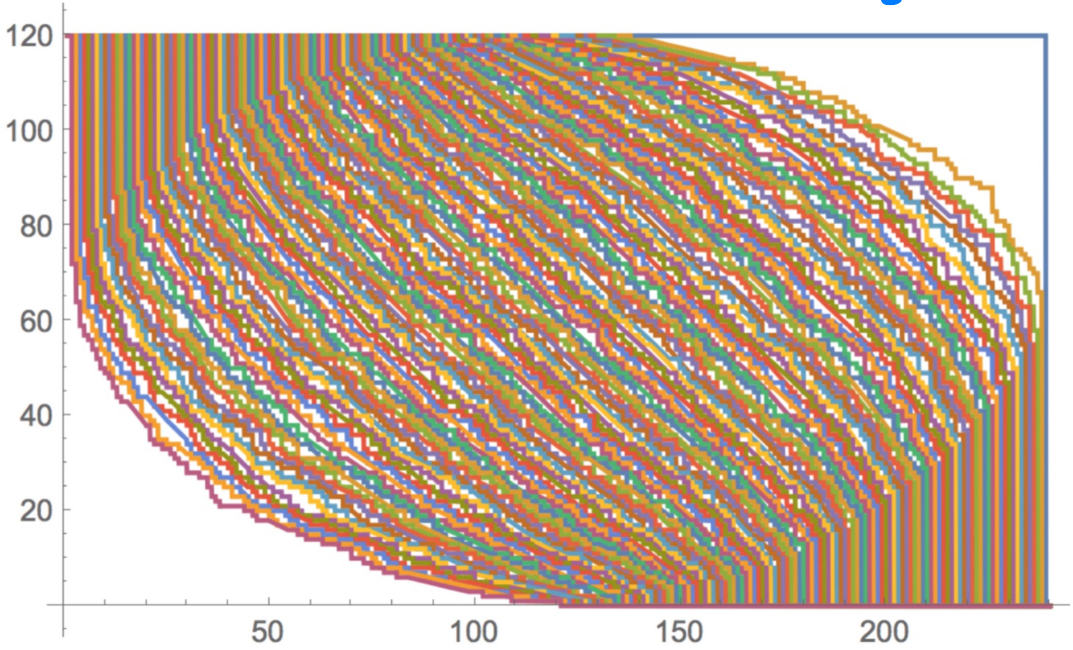
Non intersecting paths $\Rightarrow$ Dimer model







Ansatz $\rightsquigarrow$ guess the inverse of the Kasteleyn matrix (Keating, Reshetikhin, Sridhar)

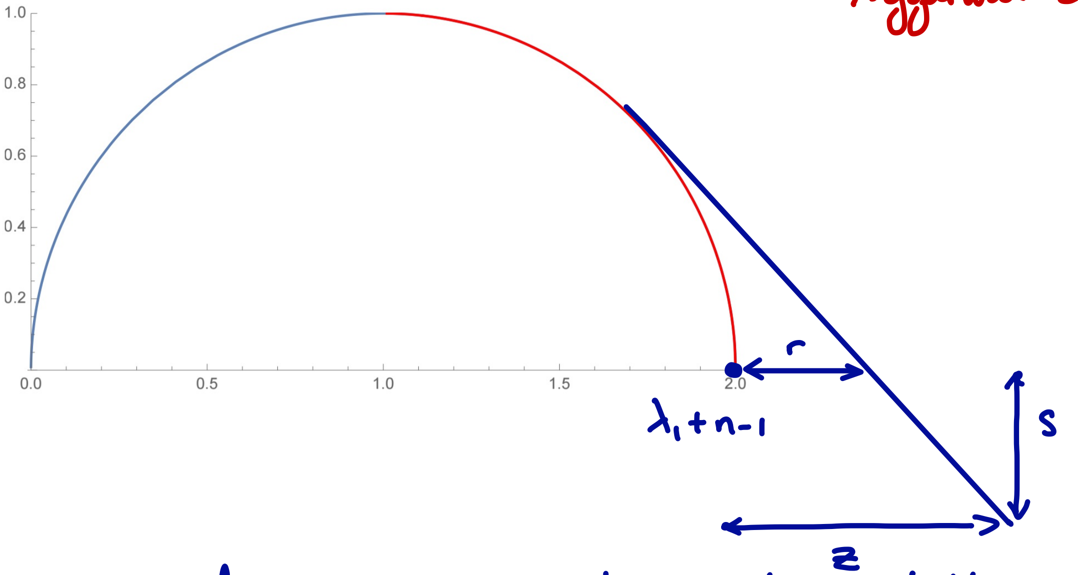


$\lambda = (n, \dots, n)$ $\qquad$ $t = n$

Arctic curve
$x^2 - 4y + 4y^2 = 0$

Second technique: Tangent method (Colomo & Sportiello 2016)

Aggarwal 2019

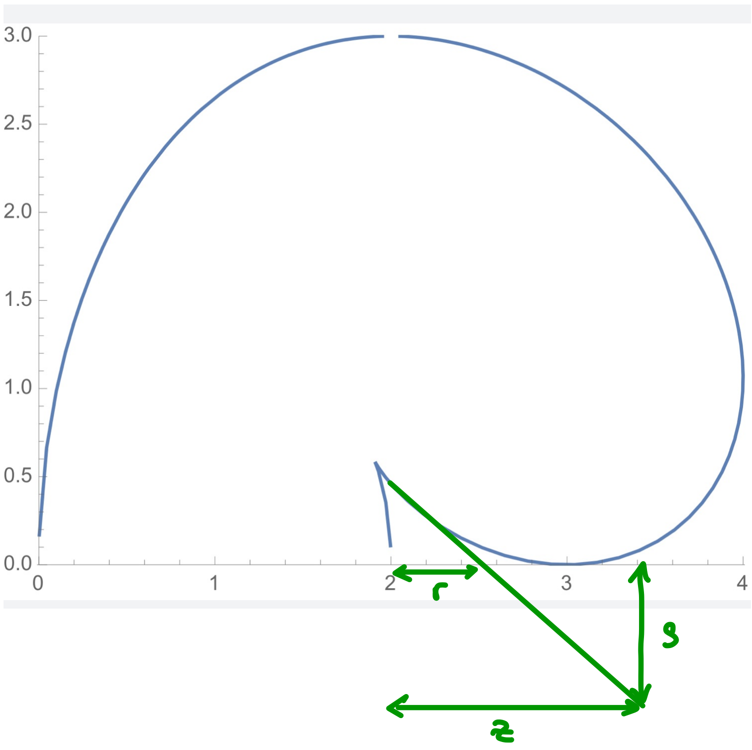


Philosophy: fix z, s, compute most probable r

$\Rightarrow$ gives a straight line tangent to the arctic curve

Vary $z \Longrightarrow$ give a parametrization of the curve

Cusps



General result

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $a_i = \lambda_i + n - i$

Suppose $n = N \to \infty$ and $a_i = N\,\alpha\left(\frac{i}{N}\right)$ and $t = N\tau$

$\alpha : [0,1] \to \mathbb{R}$

"Theorem" (C., Keating, Nicoletti 2019)

The arctic curve is parametrized by

$$\begin{cases} X(x) = \dfrac{x^2 I'(x)}{I(x) + x\,I'(x)} \\ Y(x) = \dfrac{1}{I(x) + x\,I'(x)} \end{cases}$$

$$I(x) = \frac{1}{\tau}\, e^{-\int_0^1 \frac{1}{x - \alpha(u)}\,du}$$

General result

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$$\begin{cases} X(x) = \dfrac{x^2 I'(x)}{I(x) + x\,I'(x)} \\[2ex] Y(x) = \dfrac{1}{I(x) + x\,I'(x)} \end{cases}$$

$$I(x) = \frac{1}{\tau}\, e^{-\int_0^1 \frac{1}{x - \alpha(u)}\,du}$$

General result

Let $\lambda = (\lambda_1, \dots, \lambda\ )$ and $a_i = \lambda_i + t - i$

Suppose $n \to \infty$ and $a_i = n\ \alpha\left(\frac{i}{n}\right)$ and $t = n\tau$

$\alpha : [0,1] \to \mathbb{R}$

"**Theorem**" (C., Keating, Nicoletti 2019)

The arctic curve is parametrized by

$$\begin{cases} X(x) = \dfrac{x^2 I'(x)}{I(x) + x\, I'(x)} \\[2ex] Y(x) = \dfrac{1}{I(x) + x\, I'(x)} \end{cases}$$

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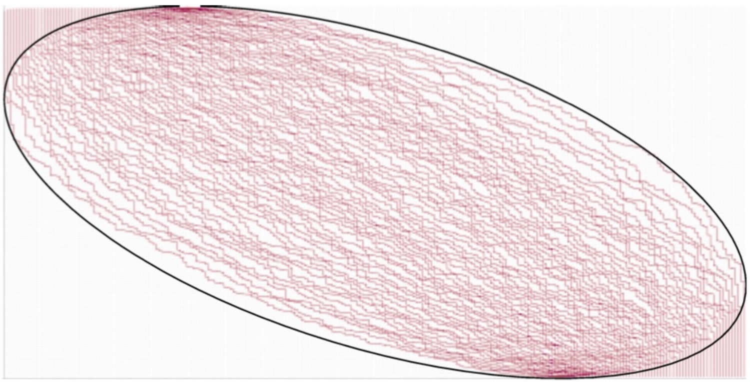
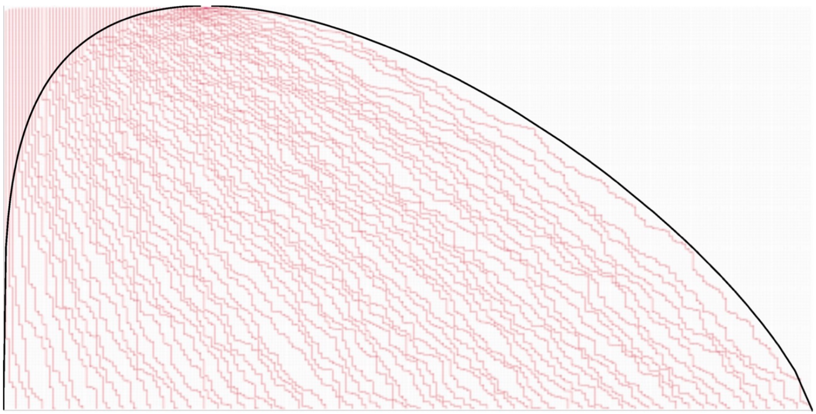


$p \geq 1$

$\lambda = (pn, pn, \ldots, pn)$

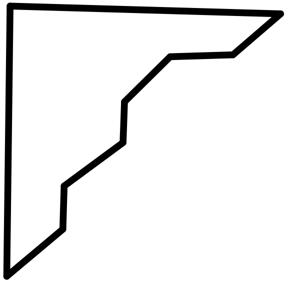
Algebraic curve of degree 2

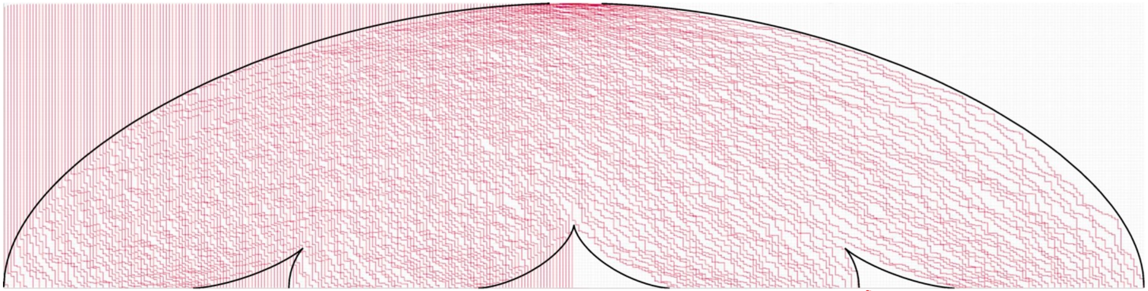


$\lambda = (pn, p(n-1), \ldots, 2p, p)$

Algebraic curve of degree p.

λ





cusps

Theorem (Li 21)

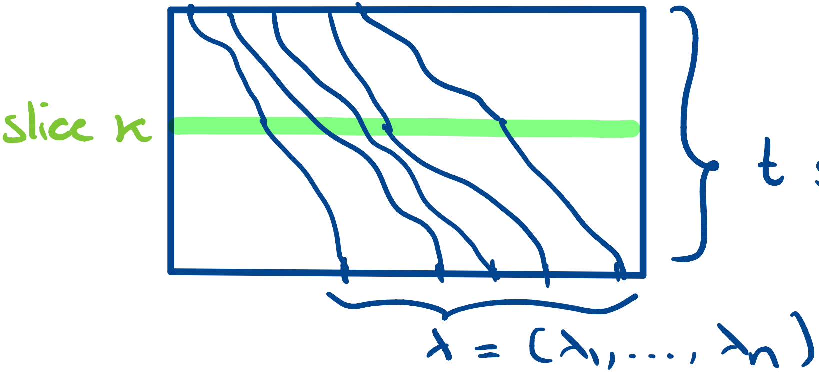
The "theorem" holds

Main tool asymptotic of Schur polynomials

Lemma A.1. *If $(\lambda(N))$ is a regular sequence of signatures, then the sequence of counting measures $m(\lambda(N))$ converges weakly to a measure $\mathbf{m}$ with compact support. When the β_is are equal to 1, there exists an explicit function $H_{\mathbf{m}}$, analytic in a neighborhood of 1, depending on the weak limit $\mathbf{m}$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \log\left(\frac{s_{\lambda(N)}(u_1,\dots,u_k,1,\dots,1)}{s_{\lambda(N)}(1,\dots,1)}\right) = H_{\mathbf{m}}(u_1) + \cdots + H_{\mathbf{m}}(u_k), \tag{A.1}$$

Theorem (Li 21) Weight y_i on slice i

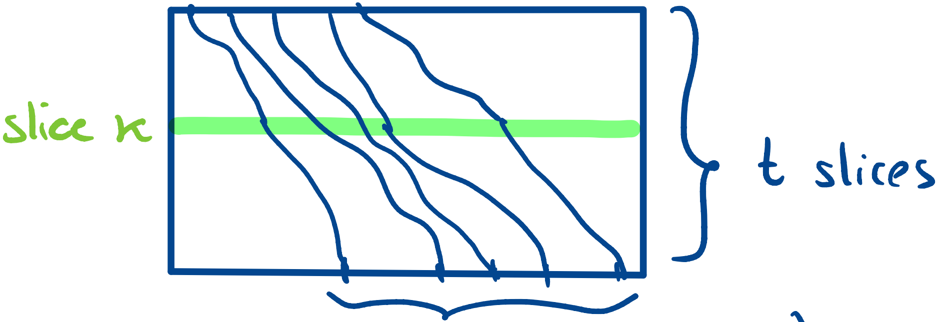


As $n \to \infty$ $\frac{t}{n} \to \tau$ $\frac{y_0 + \dots + y_{k-1}}{y_0 + \dots + y_{t-1}} \to s$

the measure on slice k converges in probability to a deterministic measure m_y whose moments are:

$$\int_{\mathbb{R}} x^j \mathbf{m}_y(dx) = \frac{1}{2(j+1)\pi\mathbf{i}} \oint_1 \frac{dz}{z-1+s} \left((z-1+s) H'_{\mathbf{m}_0}(z) + \frac{z-1+s}{z-1} \right)^{j+1}$$

Theorem (Li 21) Weight y_i on slice i



$\lambda = (\lambda_1, \dots, \lambda_n)$

As $n \to \infty$ $\frac{t}{n} \to \tau$ $\frac{y_0 + \dots + y_{k-1}}{y_0 + \dots + y_{t-1}} \to s$

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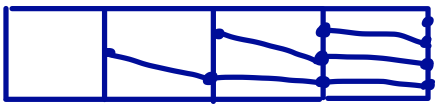
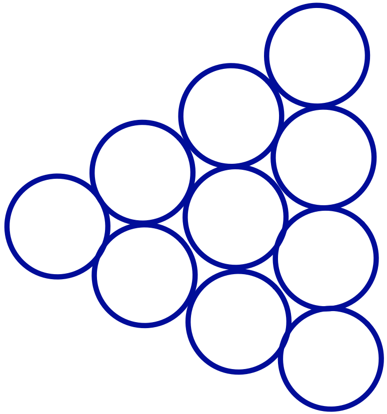
$\Rightarrow$ Can also compute the height function

What will happen if we change the geometry of the lattice?

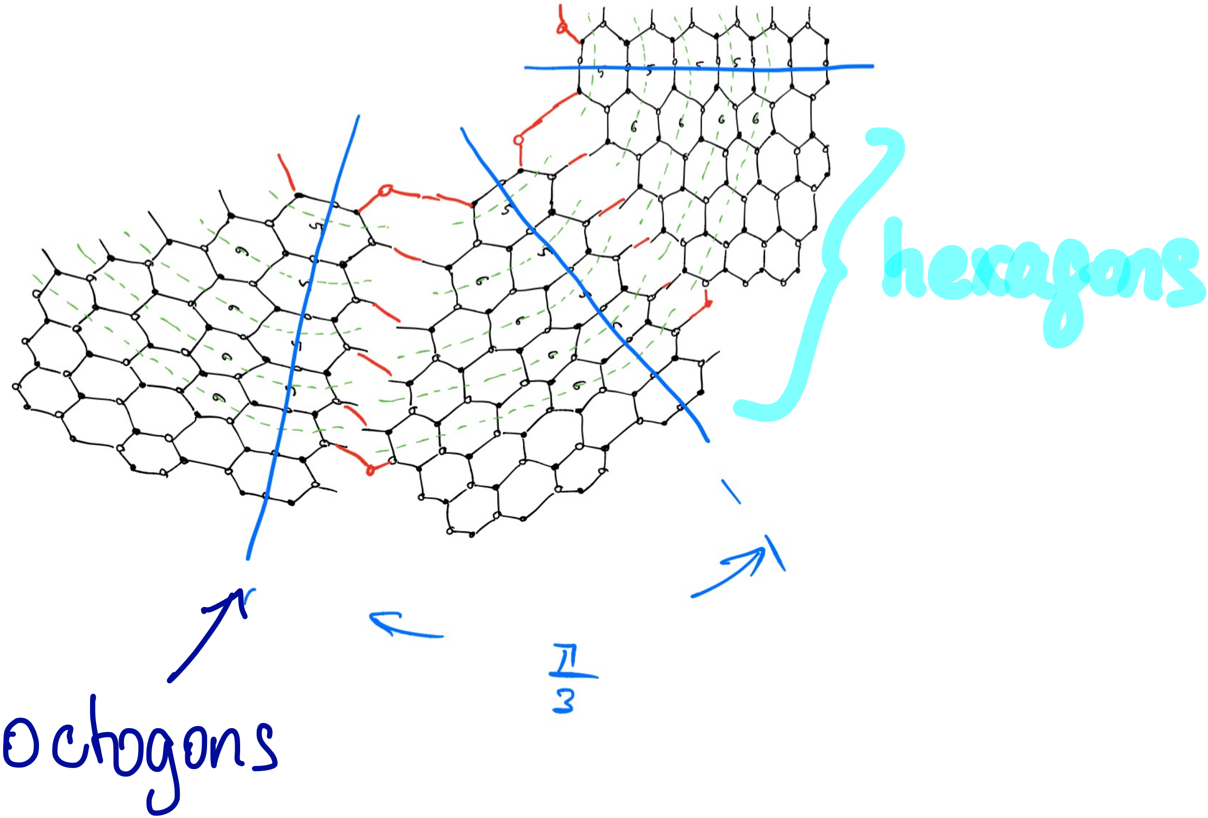
(Reshetikhin)



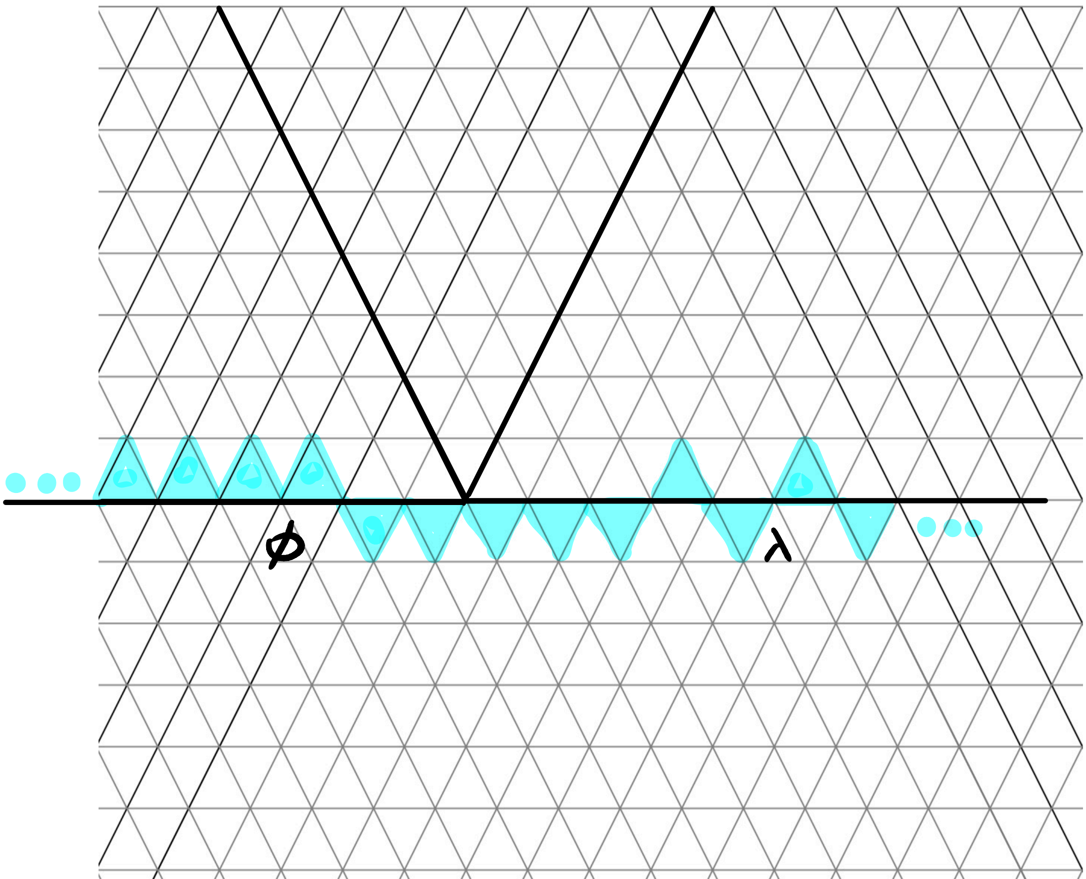
Lecture Hall graph

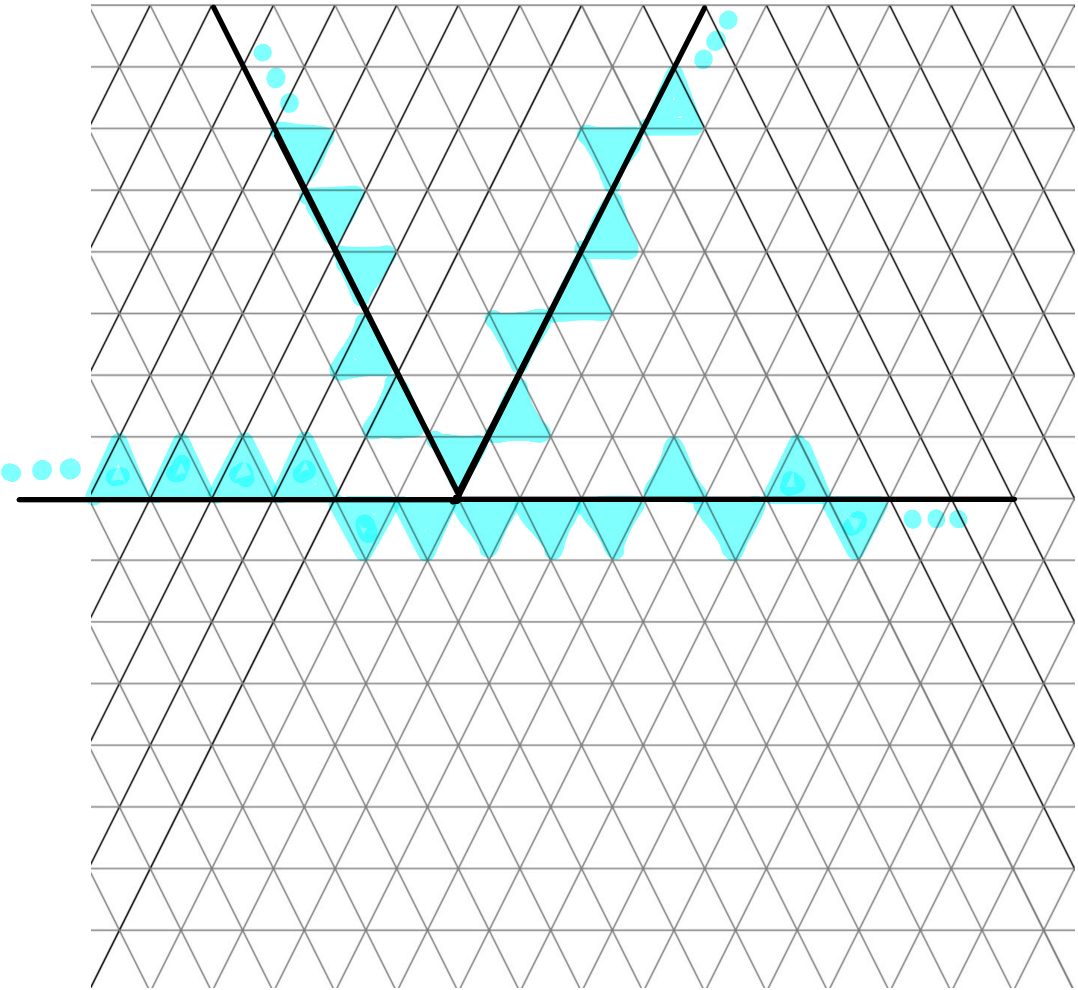


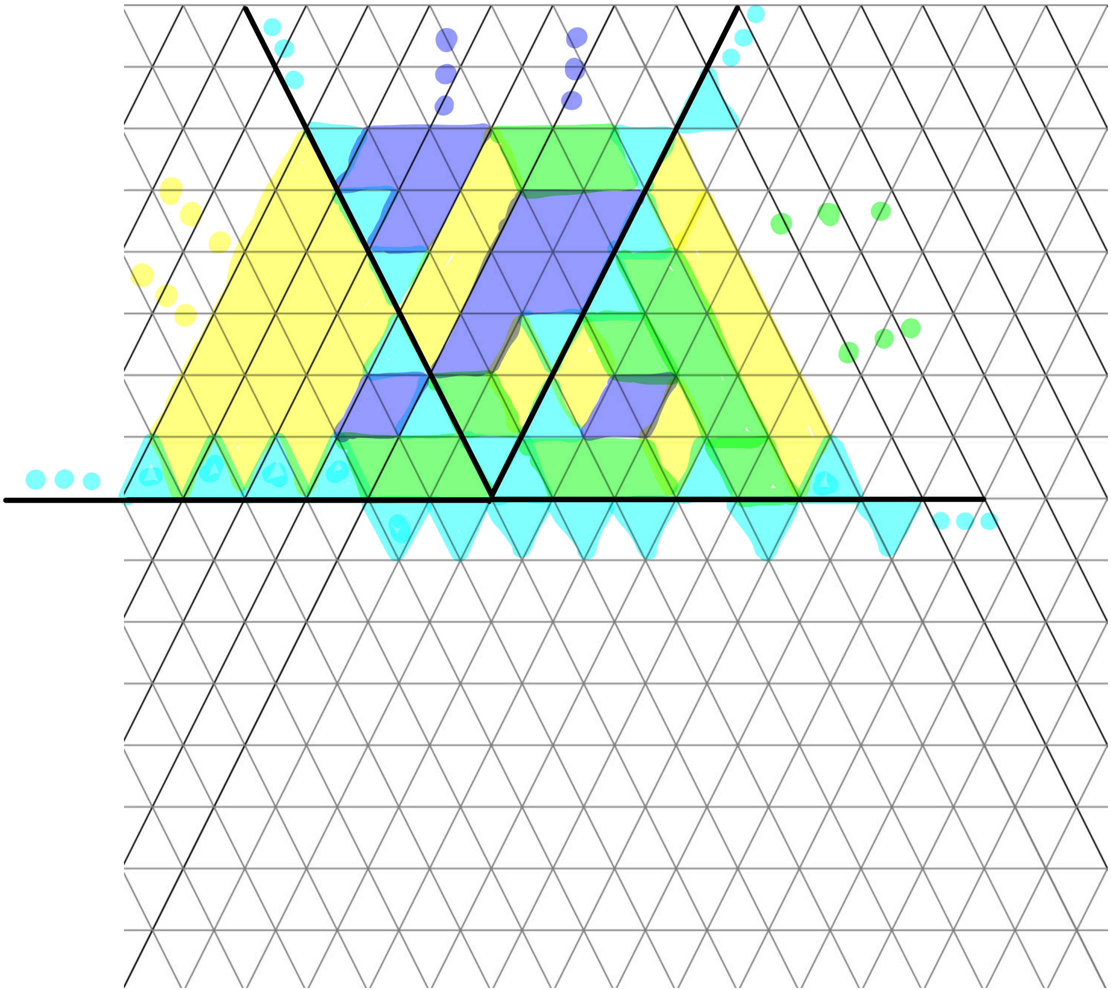
Dimers

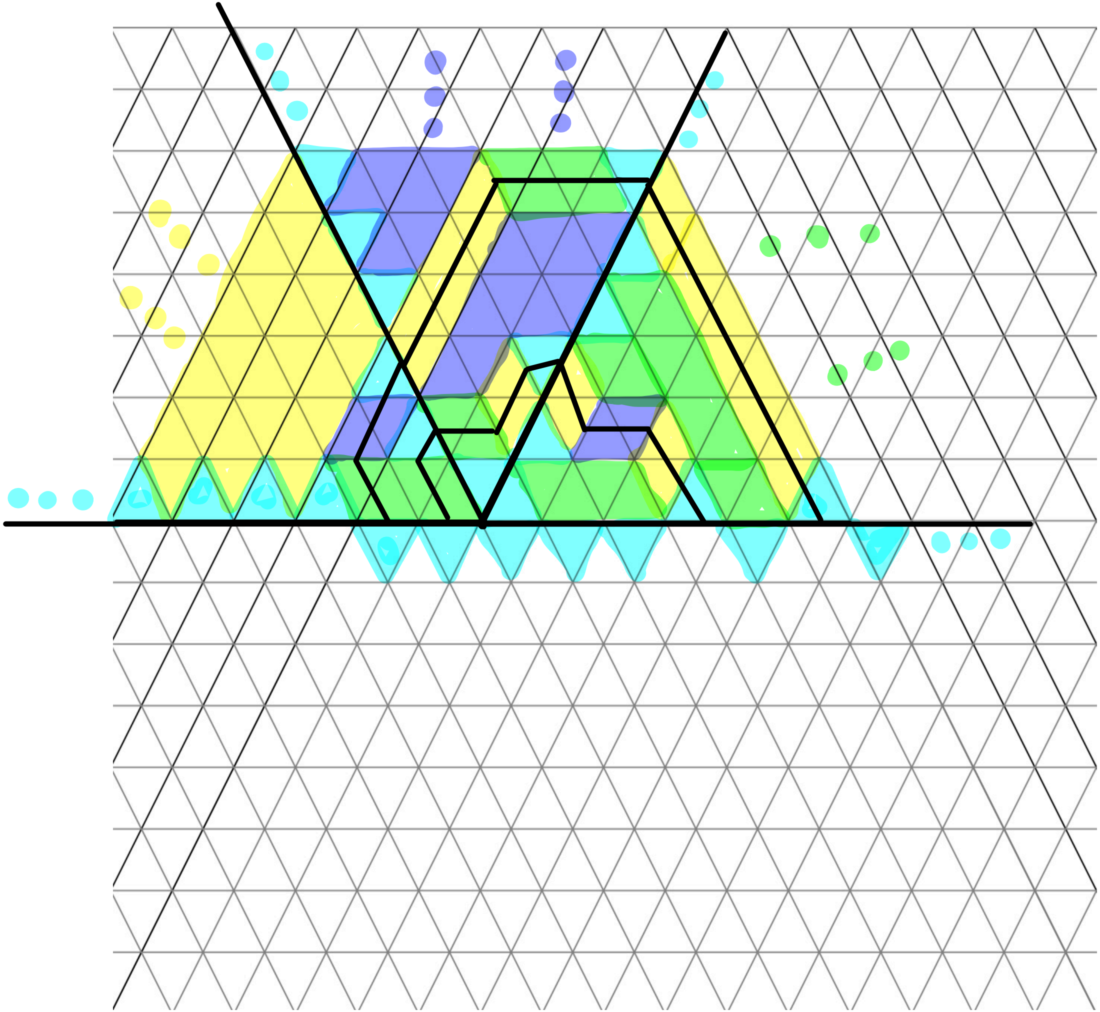


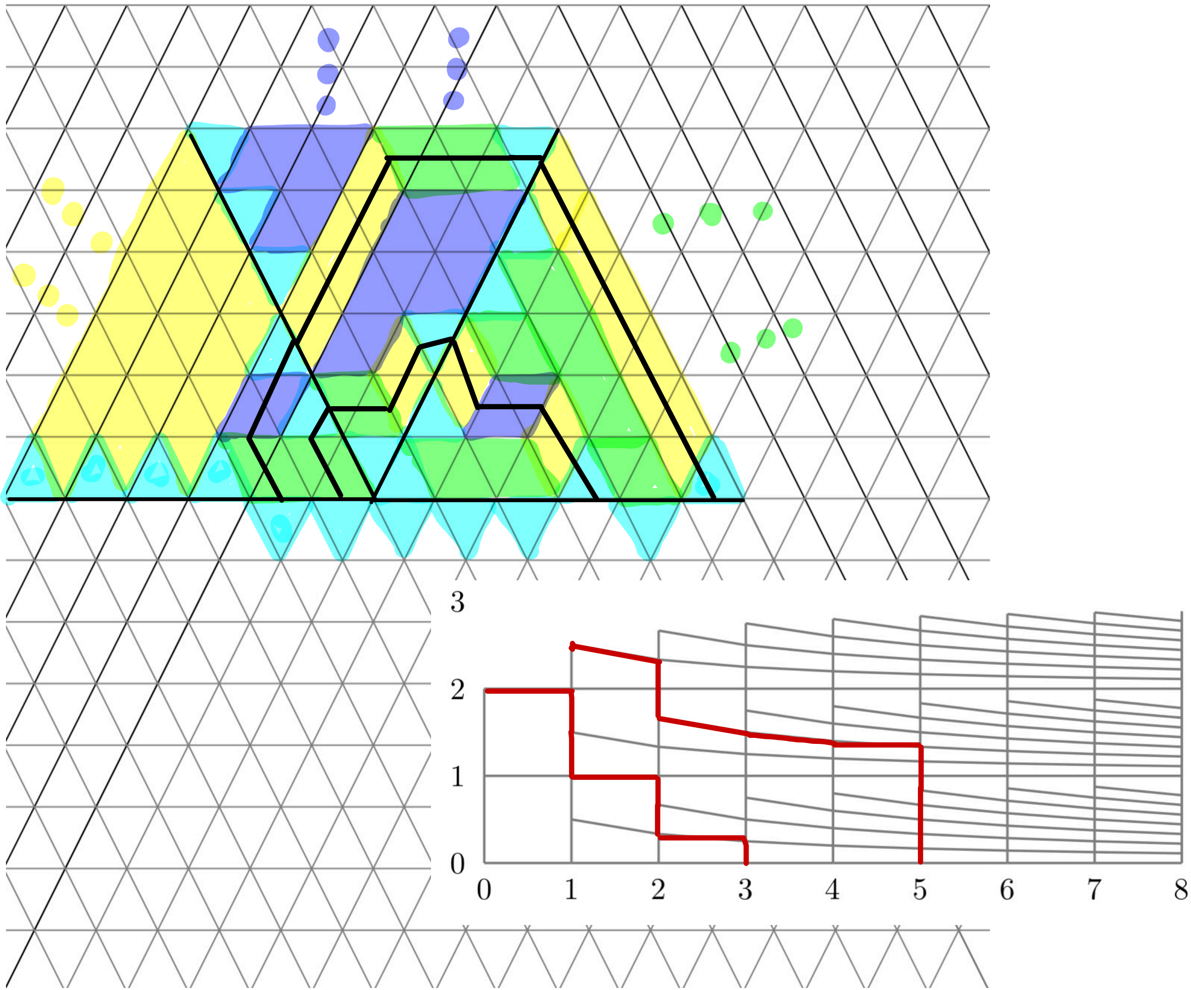
Tilings (Kealing)



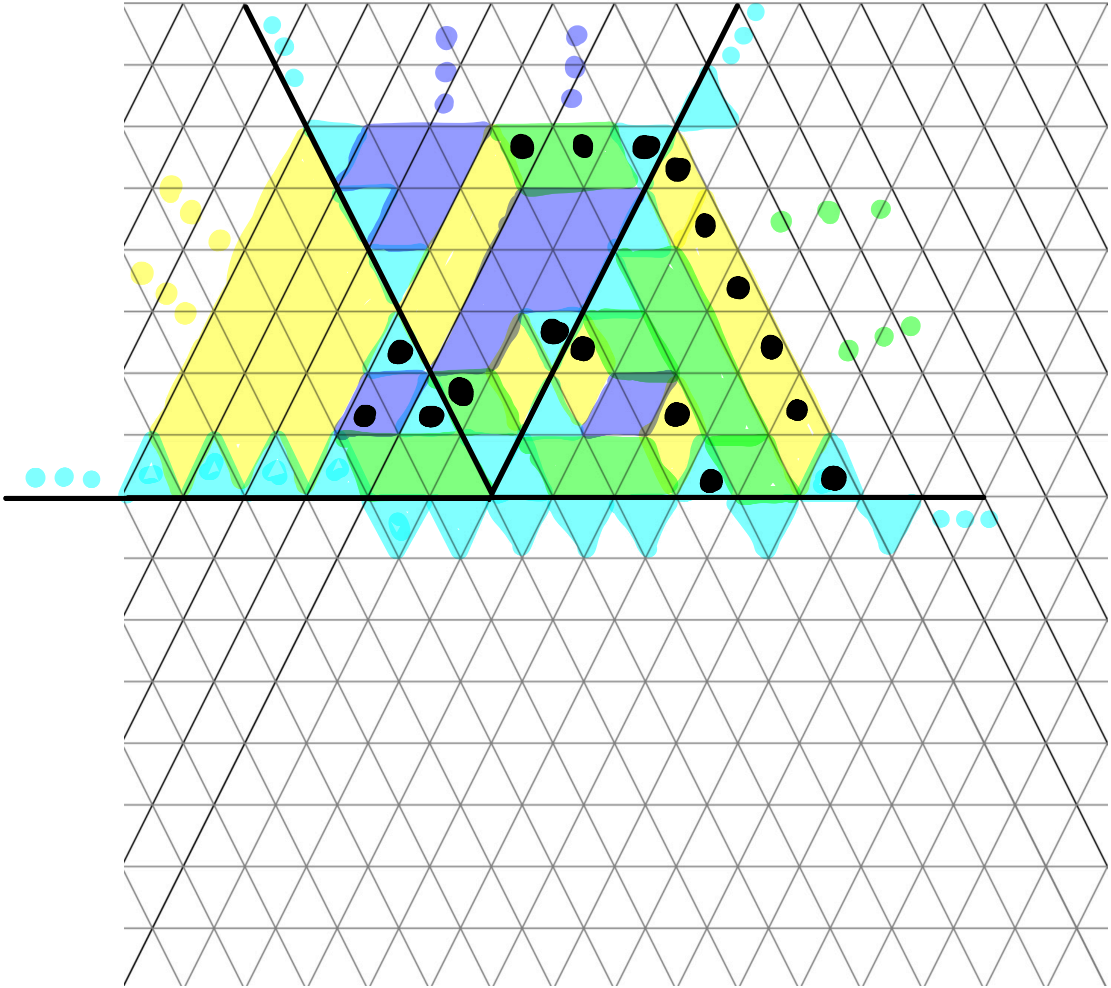


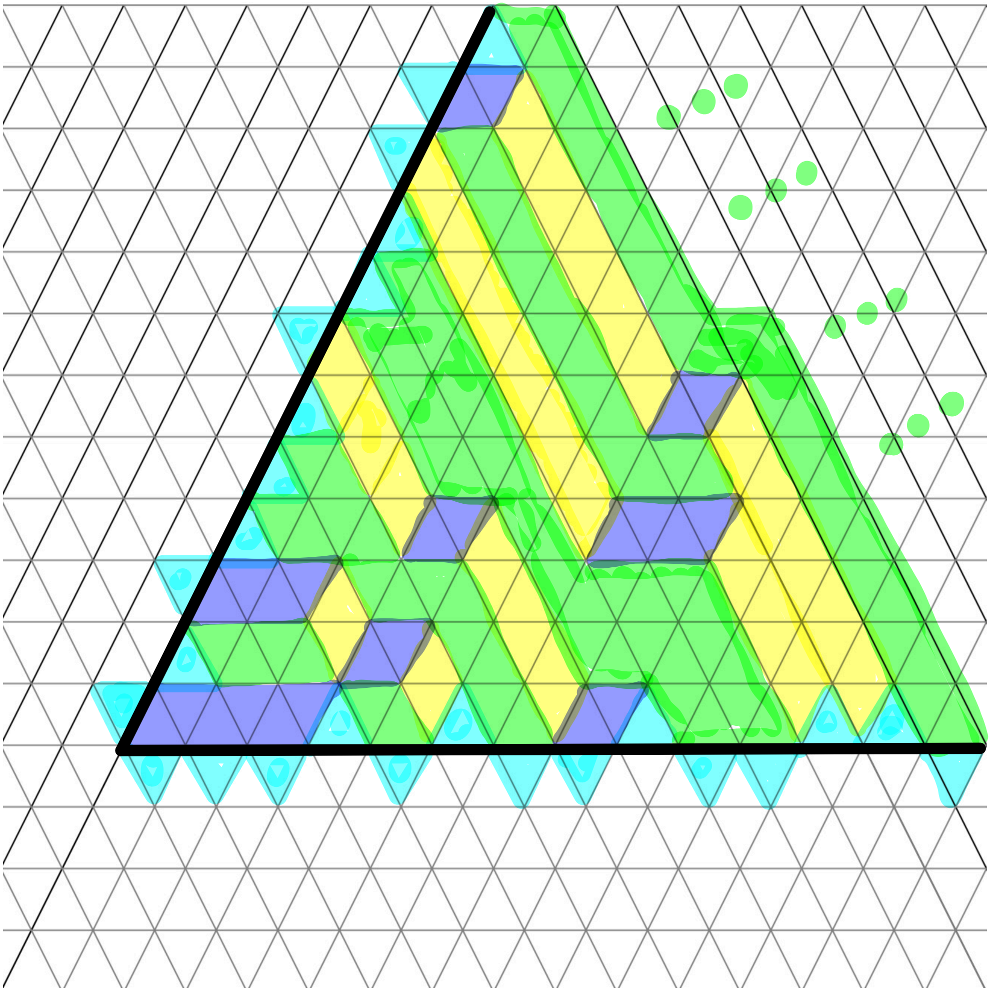


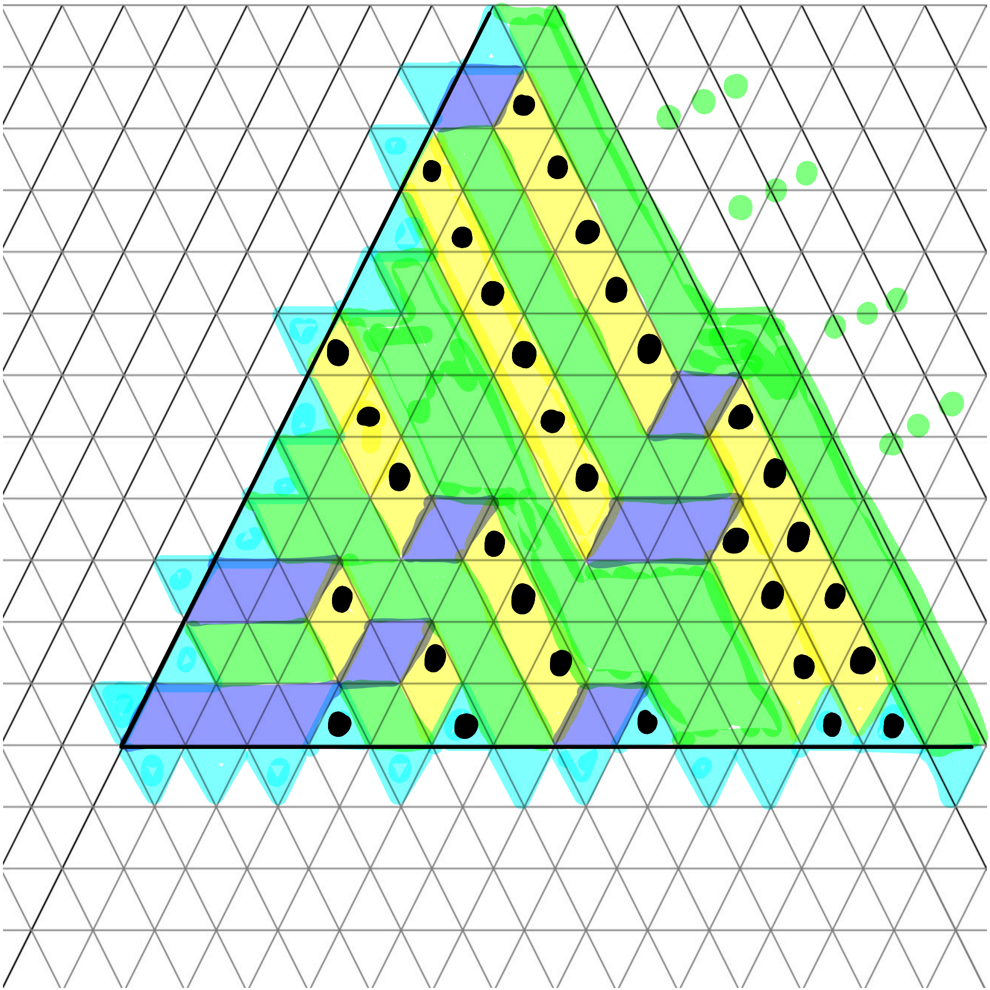




3
2
1
0
0
1
2
3
4
5
6
7
8



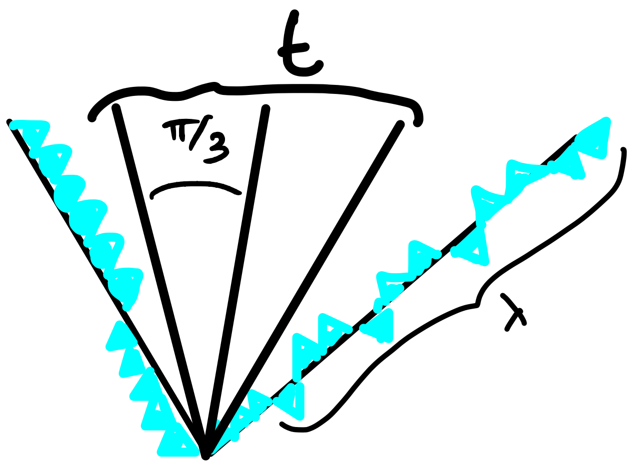




Q: ① Can we understand the distribution of the vertical tiles?

② Asymptotic behavior of the tilings?

(Li ⇒ asymptotic behavior on each slice)



General question

1. other type of slices?
2. other lattices? tilings?

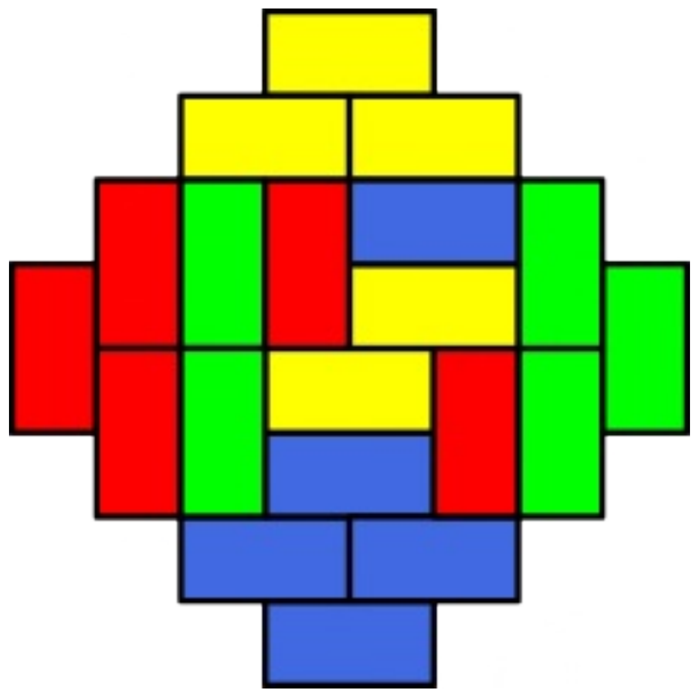
General question

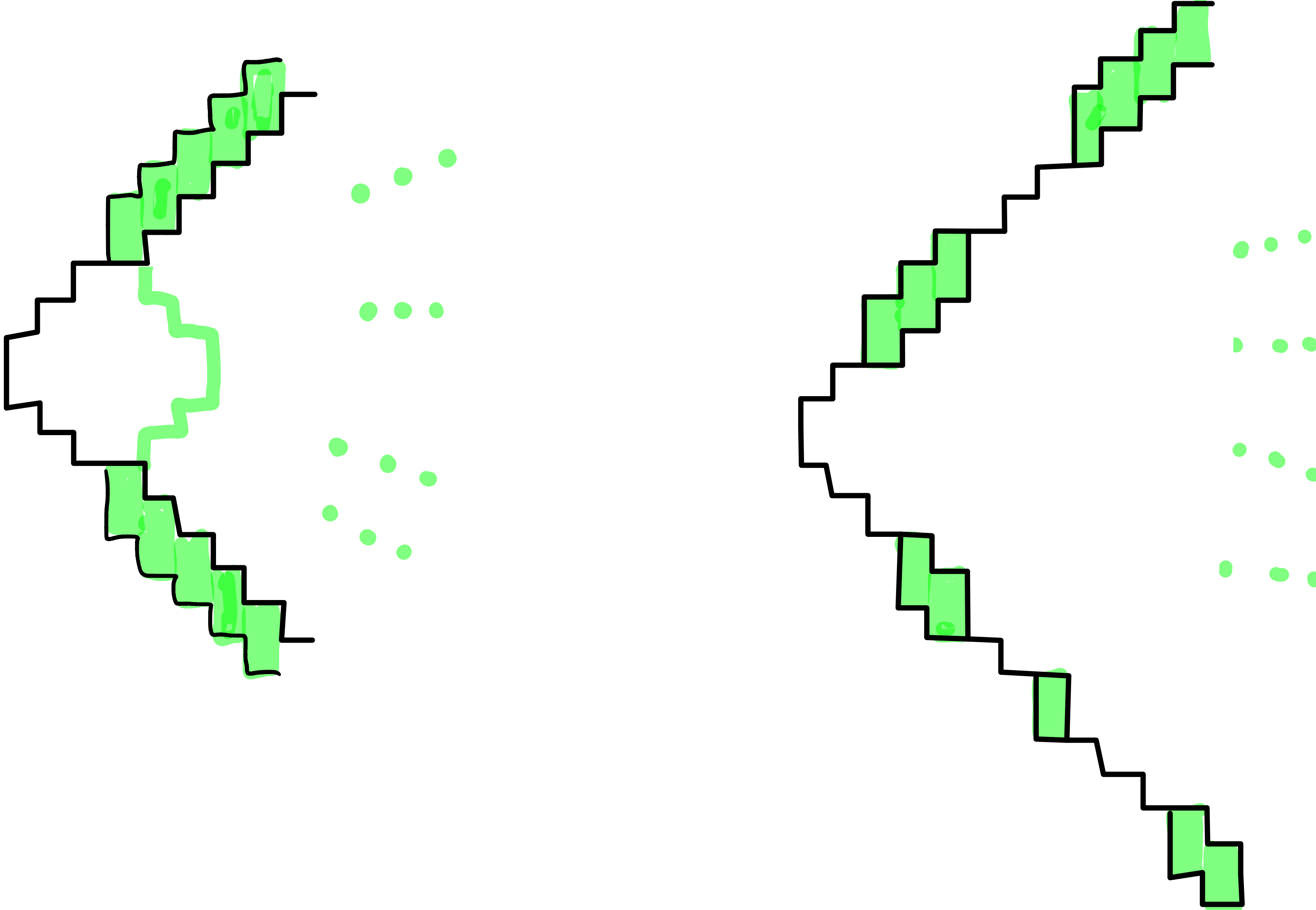
① other type of slices?

② other lattices? tilings?

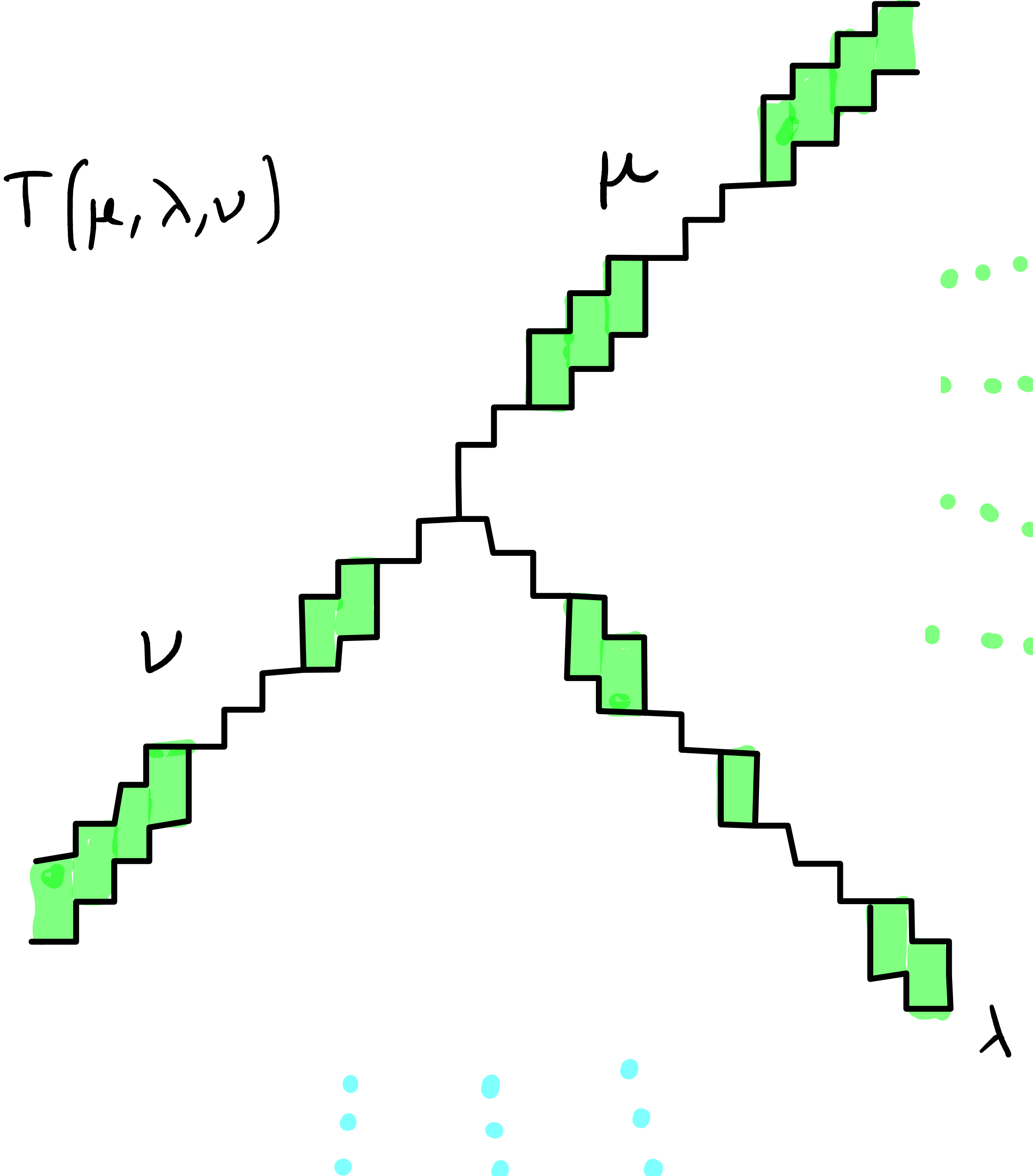
Domino tilings

Aztec diamond





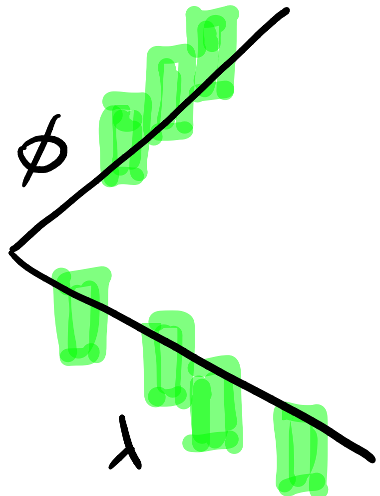
$$Z_{\mu,\nu} = \sum_{\lambda} x^{|\lambda|} \, T(\mu,\lambda,\nu)$$



Theorem (Elkies, Kuperberg, Larsen & Propp '92)

One slice

$\lambda = (\lambda_1, \dots, \lambda_n)$



The number of tilings is

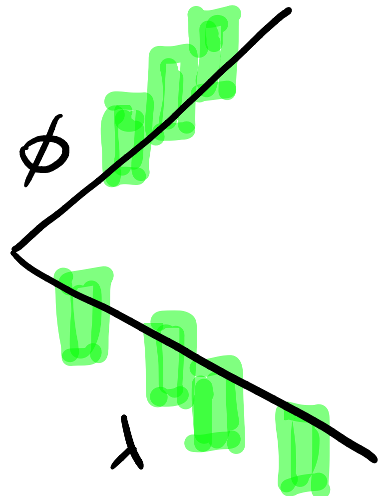
$$2^{\binom{n+1}{2}} \underbrace{s_\lambda(1, \dots, 1)}_{\text{Schur pols}}$$

Two slices? Infinite number of tilings

Theorem (Elkies, Kuperberg, Larsen & Propp '92)

One slice

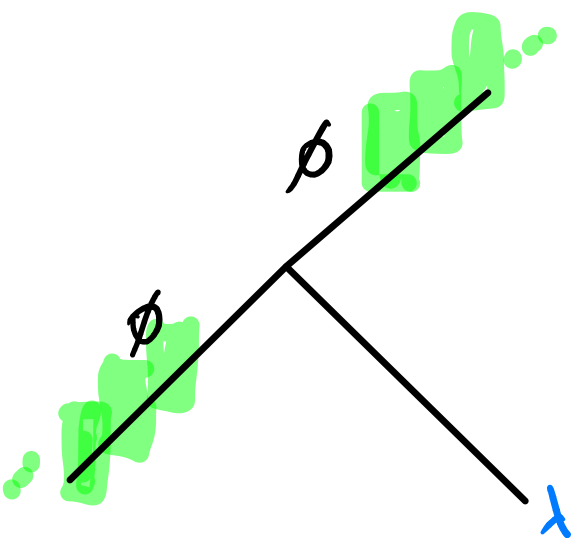
$\lambda = (\lambda_1, \ldots, \lambda_n)$



The number of tilings is

$$2^{\binom{n+1}{2}} \underbrace{S_\lambda(1, \ldots, 1)}_{\text{Schur pols}}$$

Two slices

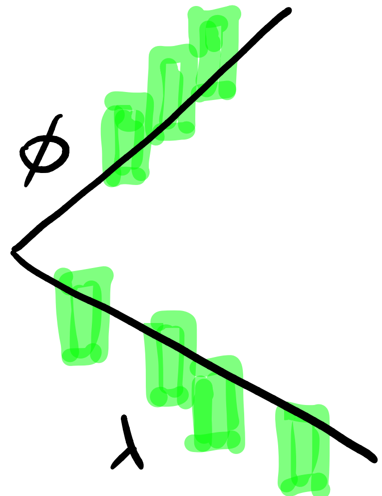


$$\sum_{\substack{\lambda \\ \ell(\lambda)=n}} \left(2^{\binom{n+1}{2}} S_\lambda(1, \ldots, 1)\right)^2 x^{|\lambda|} = \frac{2^{n(n-1)}}{(1-x)^{n^2}}$$

Theorem (Elkies, Kuperberg, Larsen & Propp '92)

One slice

$\lambda = (\lambda_1, \dots, \lambda_n)$



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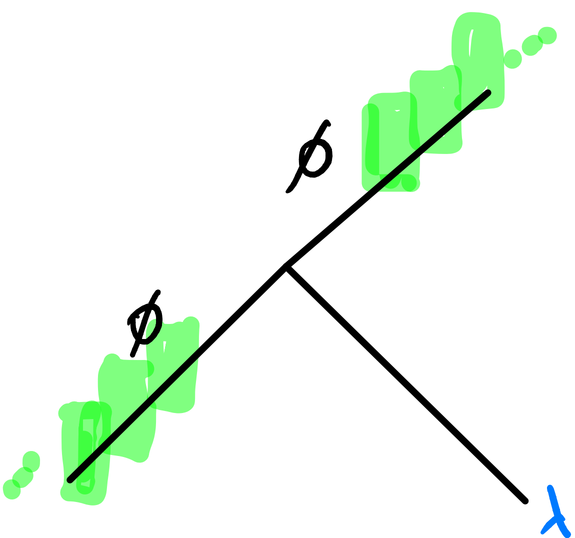
Two slices



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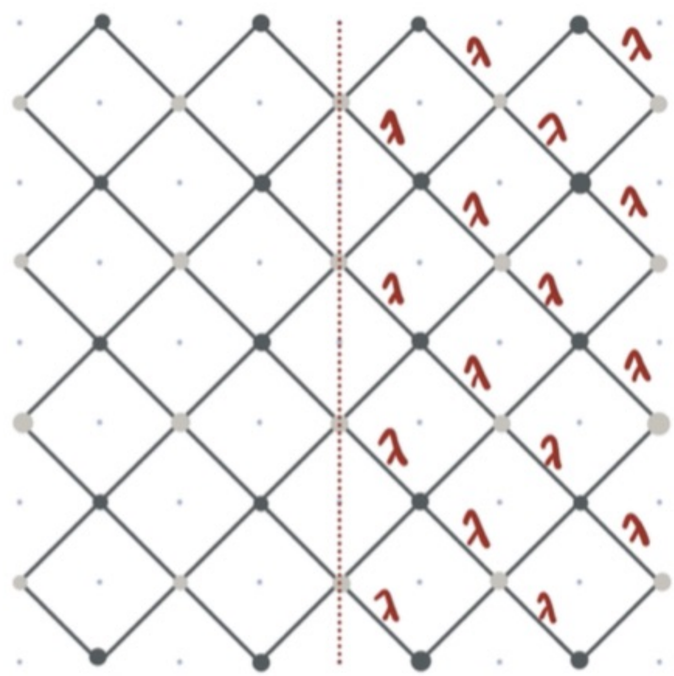
$$= \frac{2^{n(n-1)}}{(1-x)^{n^2}}$$

$\boxed{\text{More slices? General } \mu\text{?}}$

Can we say something about those tilings with slices?

Other work: M. Shea (Berkeley)



Thank you