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Two periodic Aztec diamond and matrix valued orthogonality

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Integrable Structures in Random Matrix Theory and Beyond, MSRI, October 20, 2021

0 References

Based on

- M. Duits and A.B.J. Kuijlaars
  The two periodic Aztec diamond and matrix valued orthogonal polynomials,
  J. Eur. Math. Soc. 23 (2021), 1075–1131.

- C. Charlier, M. Duits, A.B.J. Kuijlaars, and J. Lenells
  A periodic hexagon tiling model and non-Hermitian orthogonal polynomials,
  Comm. Math. Phys. 378 (2020), 401–466.

1 Outline

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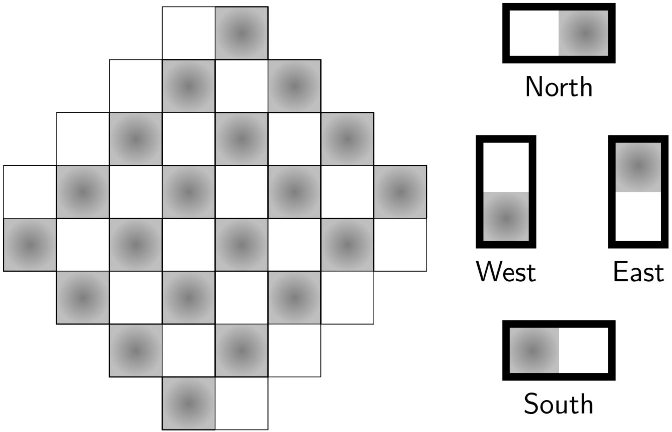
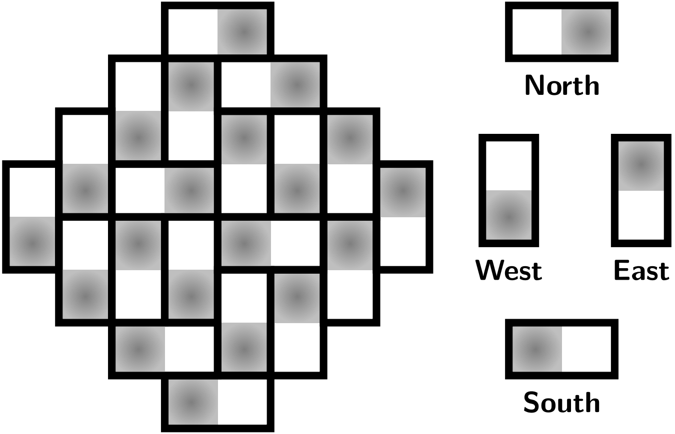
1 Aztec diamond

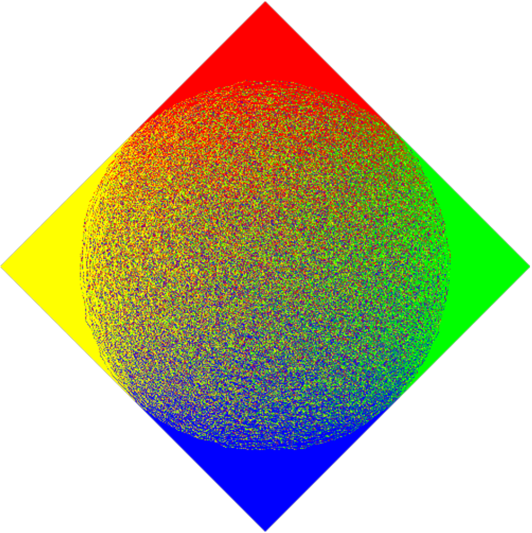


1 Tiling of an Aztec diamond



- **Tiling with 2×1 and 1×2 rectangles (dominos)**
- **Four types of dominos**

1 Aztec diamond: Large random tiling



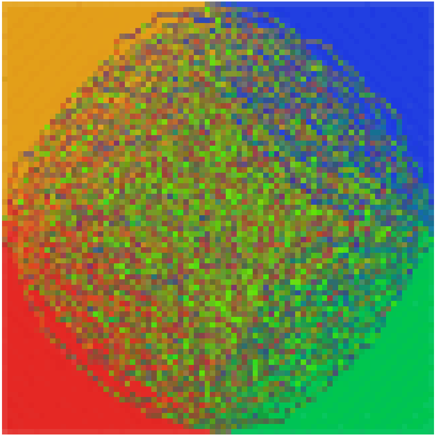
Deterministic pattern near corners
Solid region
or Frozen region

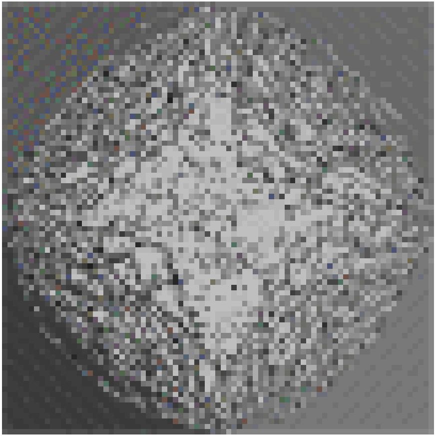
Disorder in the middle Liquid region
or Rough region

Boundary curve
Arctic circle

Jockush, Propp, Shor (1995)
Johansson (2002)

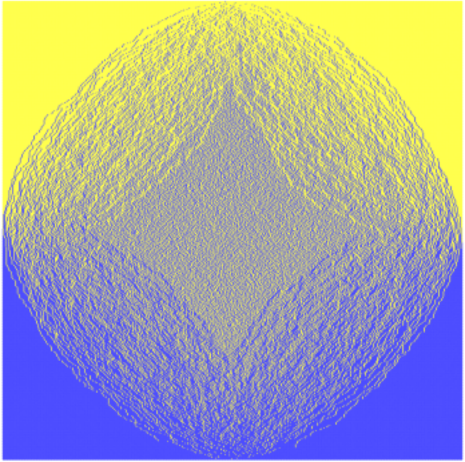
1 Aztec diamond: two periodic weighting





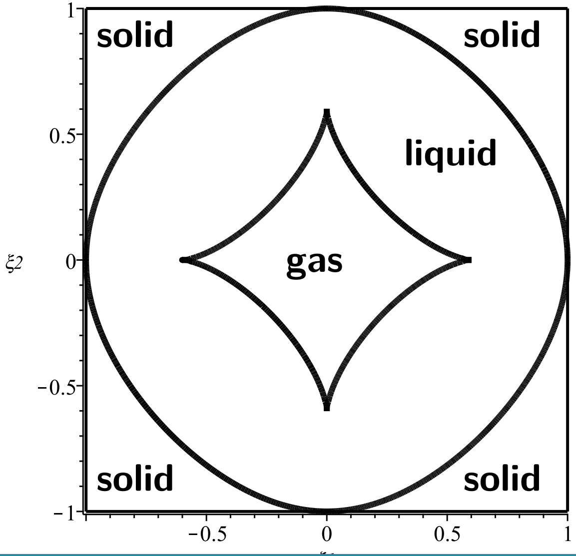
Chhita, Johansson (2016)
Beffara, Chhita, Johansson (2018)

1 Aztec diamond; two-periodic weighting



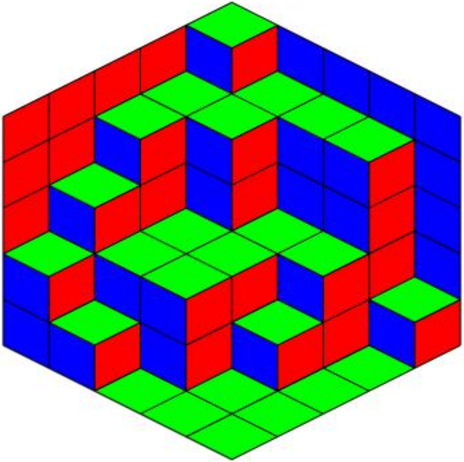
New phase within liquid region: gas region (or smooth region)

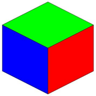
1 Two periodic Aztec diamond: Phase diagram



2 Outline

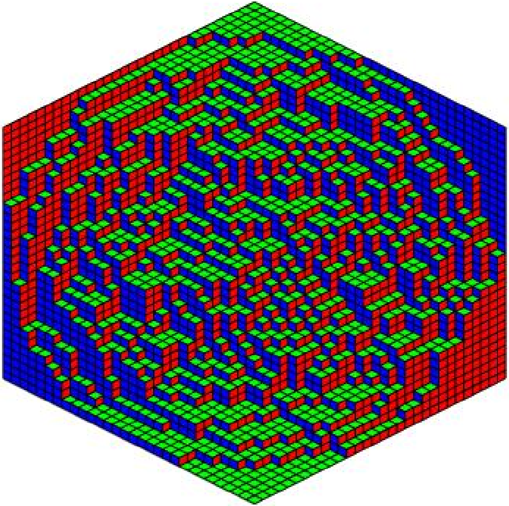
2 Lozenge tiling of a hexagon



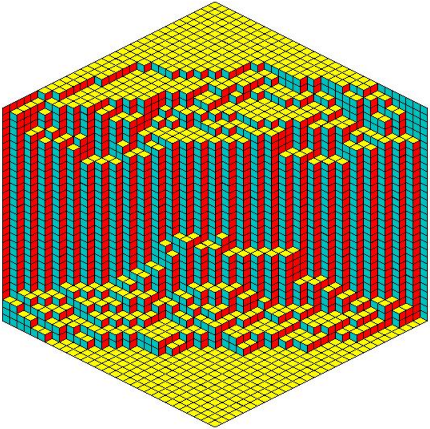


three types of lozenges

2 Arctic circle phenomenon



2 Two periodic hexagon



Hexagon tiling with **two periodic weighting** depends on parameter

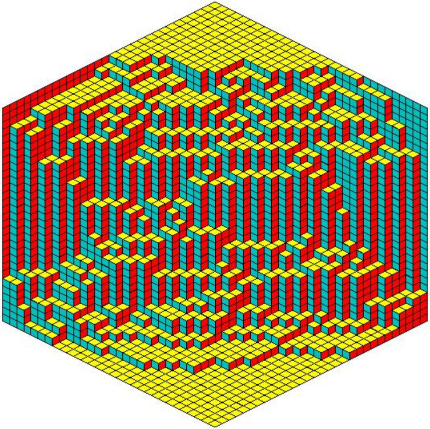
$$0 < \alpha < 1$$

There is **no gas region**

Instead there is a **phase transition**

Liquid region consists of two **disjoint ellipses** if $\alpha < \alpha_{crit}$

2 Two periodic hexagon



Ellipses come together at α_{crit}

For $\alpha > \alpha_{crit}$ the liquid region is connected

The boundary is more complicated algebraic curve

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3 Outline

3 Matrix valued orthogonal polynomials

$$\int_X P_k(x)P_j(x)w(x)dx = h_j\delta_{j,k} \qquad h_j \neq 0$$

3 Matrix valued orthogonal polynomials

$$\int_X P_k(x)P_j(x)w(x)dx = h_j\delta_{j,k} \qquad h_j \neq 0$$

Matrix valued extension (size $p \times p$)

- $W(x)$ **is** $p \times p$ **matrix for every** x
- P_k **is matrix valued polynomial**

$$P_k(x) = C_0x^k + C_1x^{k-1} + \cdots, \qquad C_i \text{ is } p \times p \text{ matrix.}$$

Orthogonality

$$\boxed{\int_X P_k(x)W(x)P_j^T(x)dx = H_j\delta_{j,k},} \qquad \det H_j \neq 0$$

3 Matrix valued orthogonal polynomials

$$\int_X P_k(x)W(x)P_j^T(x)dx = H_j\delta_{j,k} \qquad \det H_j \neq 0$$

Questions

- **Existence and uniqueness, examples**
- **Algebraic properties: recurrence relations, generating functions, differential equations**
- **Asymptotic properties**
- **Applications: do MVOP appear in "real life" problems?**

3 Our setting

MVOP that we encounter in random tiling problems have the form

$$\frac{1}{2\pi i}\oint_\gamma P_k(z)W_N(z)P_j^T(z)dz = H_j\delta_{j,k}$$

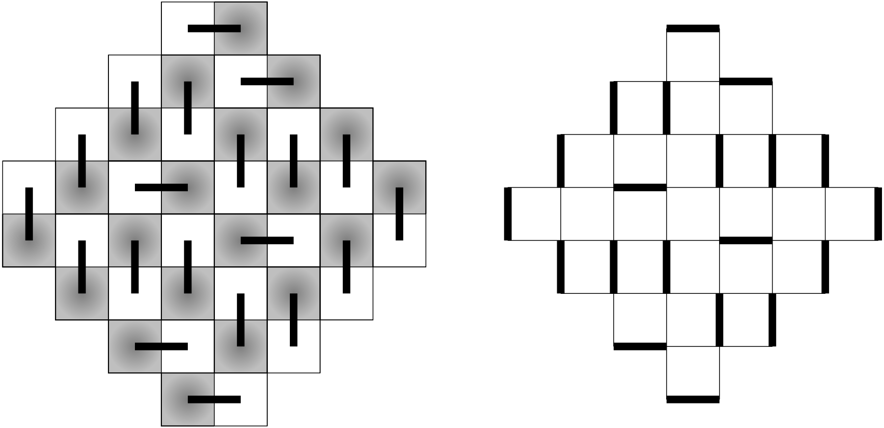
- γ is closed contour in the complex plane
- W_N is rational and varies with N
- orthogonality is non-Hermitian

Of interest is the reproducing kernel

$$R_N(w,z) = \sum_{j=0}^{N-1} P_j^T(w)H_j^{-1}P_j(z)$$

4 Outline

4 Aztec diamond as a dimer model



A domino tiling of the Aztec diamond is a dimer configuration on part of the square lattice (a.k.a. perfect matching)

survey on dimer model Kenyon (2006)

4 Aztec diamond with periodic weights

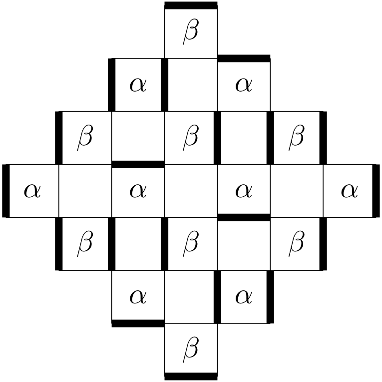
Put weights on the faces

Weight of a dimer (domino) is the product of the weights of adjacent faces

Weight of a domino tiling is the product of the weights of dominos

Probability of a domino tiling is proportional to its weight

$$\mathrm{Prob}(\mathcal{T}) = \frac{w(\mathcal{T})}{\sum_{\text{tilings } \mathcal{T}'} w(\mathcal{T}')}$$



Empty faces have weight 1

We may assume $\alpha\beta = 1$

and $\alpha > 1$

$\alpha = 1$ is the uniform model

4 Overview, step 1

Tiling of an Aztec diamond (or hexagon) is equivalent to a multi-level particle system that is determinantal

- **For periodic weightings, the correlation kernel has a double contour integral representation containing the reproducing kernel of certain MVOP.**
- **Double contour integral simplifies for periodic Aztec diamond.**

4 Overview, step 2

Orthogonality weight $W^N(z)$ of MVOP is rational.

- **Eigenvalues of W live on spectral curve**

$$\boxed{y^2 = z(z+\alpha^2)(z+\beta^2)}$$

that has genus 1

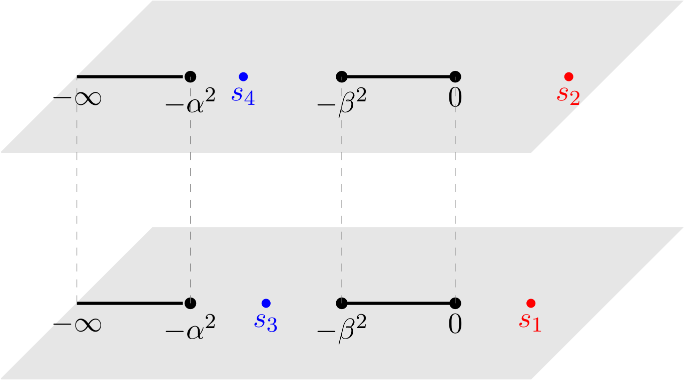
- **For a point with asymptotic coordinates (ξ_1, ξ_2) in the Aztec diamond there is action function**

$$\Phi(z; \xi_1, \xi_2)$$

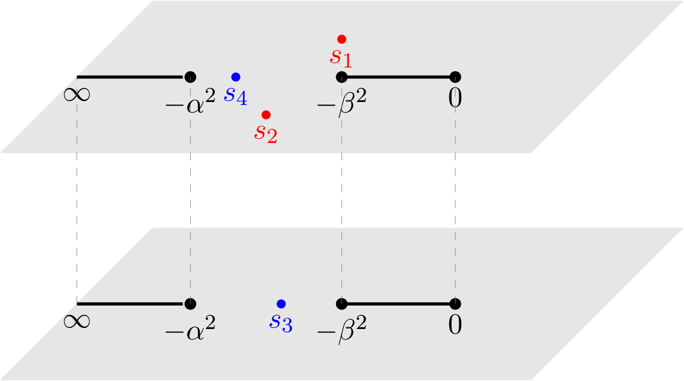
with four saddles on the spectral curve.

- **Two saddles are in the gap $[-\alpha^2, -\beta^2]$.**
- **Location of other two saddles determines the phase.**

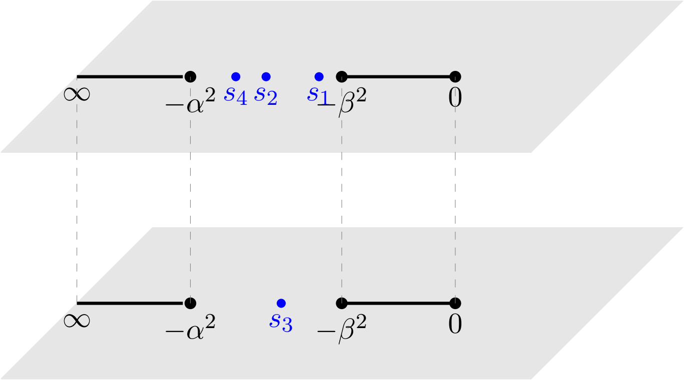
4 Solid phase: saddles s_1 and s_2 are in $[0, \infty)$



4 Liquid phase: saddles s_1 and s_2 are not on the real part



4 Gas phase: all saddles are in $[-\alpha^2, -\beta^2]$



4 Overview, step 3

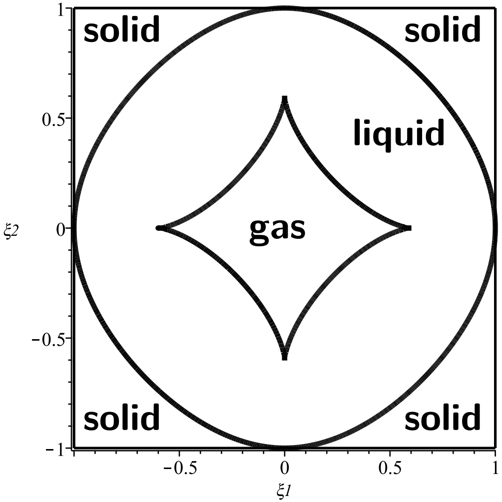
Steepest descent analysis.

- Deform contours in the double contour integral representation to let them pass through the saddles s_1 and s_2.
- Leading contributions to the correlation kernel come from residues at poles that we cross while deforming contours.

Phase diagram

- Phase transition occurs when two (or more) saddles coalesce
- Degree 8 algebraic curve in ξ_1-ξ_2 variables.

4 Phase diagram: degree 8 algebraic curve



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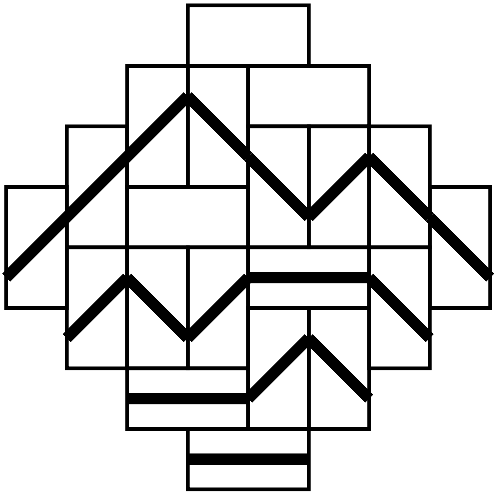
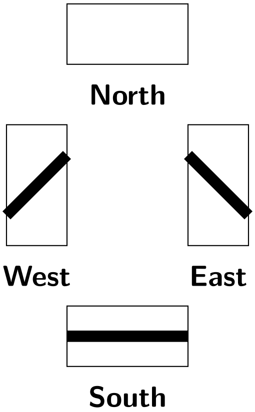
5 Outline

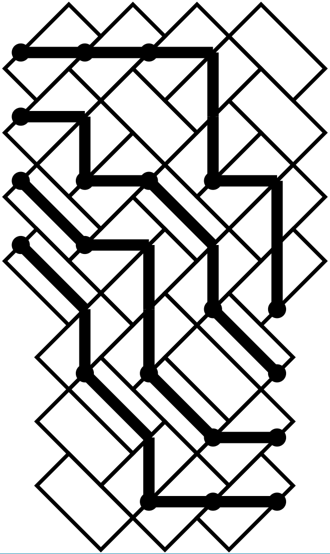
5 Paths in the Aztec diamond



Line segments on West, East and South dominos



5 Transformations and extension; particle system



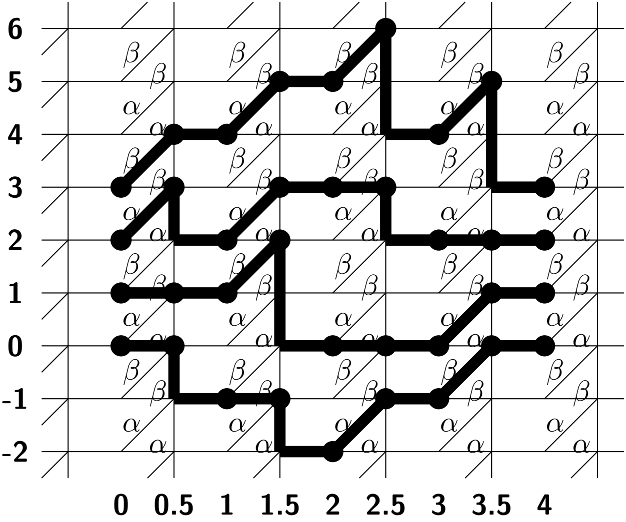
- Rotate the Aztec diamond
- Extend the tiling to a **double Aztec diamond**
- Put particles on the paths
- Particles are a **determinantal point process**

5 Non-intersecting paths on a weighted graph

- **Apply affine transformation**

Two types of steps

- **Bernoulli step up: weight α/β**
- **Steps down followed by horizontal step: weight 1**
- **Weight of a path system: product of weights of edges**



5 Lindström Gessel Viennot lemma

At each level $m = 0, 1, \ldots, L$ there are N particles $x_j^{(m)}$

Proposition [consequence of LGV lemma]

$$\mathrm{Prob}\left[\left(x_j^{(m)}\right)_{j=0,m=1}^{N-1,L-1}\right] = \frac{1}{Z_n} \prod_{m=0}^{L-1} \det\left[T_m\left(x_j^{(m)}, x_k^{(m+1)}\right)\right]_{j,k=0}^{N-1}$$

with transition matrices

$T_m(x, y) =$ **weight on edge from** (m, x) **to** $(m+1, y)$

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with transition matrices

$$T_m(x,y) = \text{ weight on edge from } (m,x) \text{ to } (m+1,y)$$

Corollary The point process is determinantal:

$$\mathrm{Prob}\left[\exists \text{ particle at each } (m,x) \in \mathcal{A}\right] =$$

$$\det\left[K((m,x),(m',x'))\right]_{(m,x),(m',x')\in\mathcal{A}}$$

for finite $\mathcal{A} \subset \{0, 1, \dots, L\} \times \mathbb{Z}$ **Eynard-Mehta**

5 Periodic transition matrices

Suppose each T_m is 2-periodic: $T_m(x+2, y+2) = T_m(x, y)$

Block Toeplitz matrix

with block symbol

$$A_m(z) = \sum_{j=-\infty}^{\infty} B_j z^j$$

$$T_m = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & B_0 & B_1 & \ddots & \\ \ddots & B_{-1} & B_0 & B_1 & \ddots \\ & \ddots & B_{-1} & B_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

- **Notation** $A_{[m',m]}(z) = \prod_{j=m'}^{m-1} A_j(z)$ **for** $m' < m$

5 Double contour integral formula

Theorem (Duits + K for 2-periodic case)

Suppose $2N$ non-intersecting paths of length L, with consecutive starting and ending positions, shifted by M. Then

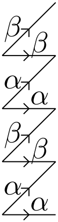
$$\begin{pmatrix} K(2m,2x;2m',2y) & K(2m,2x+1;2m',2y) \\ K(2m,2x;2m',2y+1) & K(2m,2x+1,2m',2y+1) \end{pmatrix}$$
$$= -\frac{\chi_{m>m'}}{2\pi i} \oint_\gamma A_{[m',m]}(z) z^{y-x} \frac{dz}{z}$$
$$+ \frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma A_{[m',L]}(w) R_N(w,z) A_{[0,m]}(z) \frac{w^y}{z^{x+1} w^{M+N}} dz dw$$

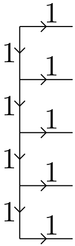
where $R_N(w,z)$ is the reproducing kernel for MVOP with matrix weight

$$W(z) = \frac{A_{[0,L]}(z)}{z^{M+N}}$$

6 Outline

6 Two periodic Aztec diamond





Bernoulli step with symbol

$$\begin{pmatrix} \alpha & \alpha \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} z = \begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix}$$

Geometric step down with symbol

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} z^{-1} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} z^{-2} + \cdots = \frac{1}{z-1} \begin{pmatrix} z & 1 \\ z & z \end{pmatrix}$$

6 Two periodic Aztec diamond

In two periodic Aztec diamond of size $2N$, we find $L = 2N$, $M = 0$, and weight matrix

$$\frac{A_{[0,L]}(z)}{z^{M+N}} = W^N(z)$$

with

$$W(z) = \frac{1}{z(z-1)^2} \begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix} \begin{pmatrix} z & 1 \\ z & z \end{pmatrix} \begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix} \begin{pmatrix} z & 1 \\ z & z \end{pmatrix}$$

$$= \frac{1}{(z-1)^2} \begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

6 Matrix valued orthogonal polynomials

$$W(z) = \frac{1}{(z-1)^2} \begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

- **MVOP of degree N with respect to W^N has explicit formula (if N is even)**

$$\boxed{P_N(z) = (z-1)^N W(\infty)^{N/2} W^{-N/2}(z)}$$

- **The double contour integral for the correlation kernel simplifies considerably**
- **Different approach is due to Berggren-Duits (2019)**

6 Geometry of the problem

Eigenvalues of

$$(z-1)^2 W(z) = \begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

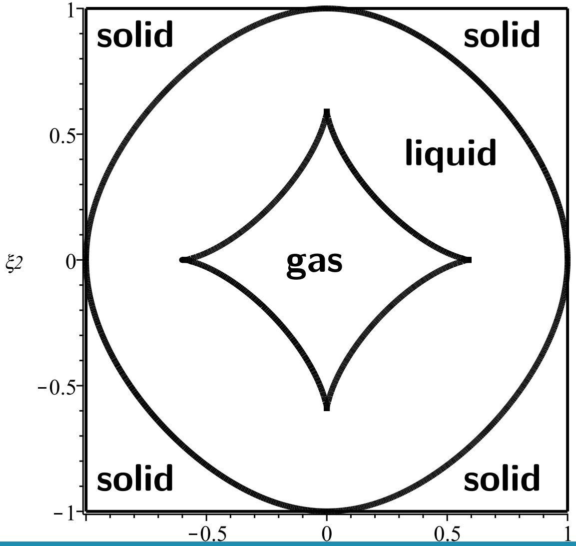
are

$$(\alpha+\beta)z \pm \sqrt{z(z+\alpha^2)(z+\beta^2)}$$

Eigenvalues "live" on spectral curve $\boxed{y^2 = z(z+\alpha^2)(z+\beta^2)}$

- **The genus is one (unless $\alpha = \beta$)**
- **Similar calculations for a two periodic hexagon tiling lead to genus zero spectral curve $\implies$ no gas region.**

Thank you for your attention



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