Local Universality of the Time-Time Covariance and of the Geodesic Tree for Last Passage Percolation

MSRI 18-22 October, 2021

Integrable structures in random matrix theory and beyond

based on joint works with

Alessandra Occelli (arXiv: 1905.08582)

and

Ofer Busani (arXiv: 2008.07844)

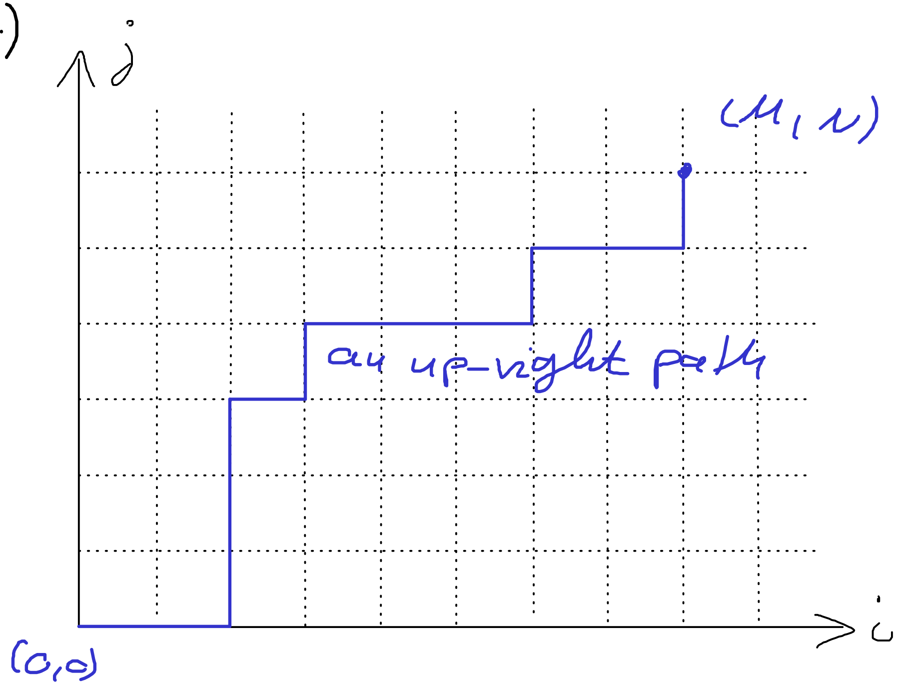
Last passage percolation: point-to-point

Let $\omega_{ij} \sim \exp(1), i,j \in \mathbb{Z}$ be independent r.v.

An *up-right path* $\pi = (\pi(0), \pi(1), \pi(2), \ldots)$

is a path on $\mathbb{Z}^2$ s.t.

$$\pi(k+1) - \pi(k) \in \{(0,1), (1,0)\}$$



The *last passage percolation*

from $(0,0)$ to (M,N) is given by

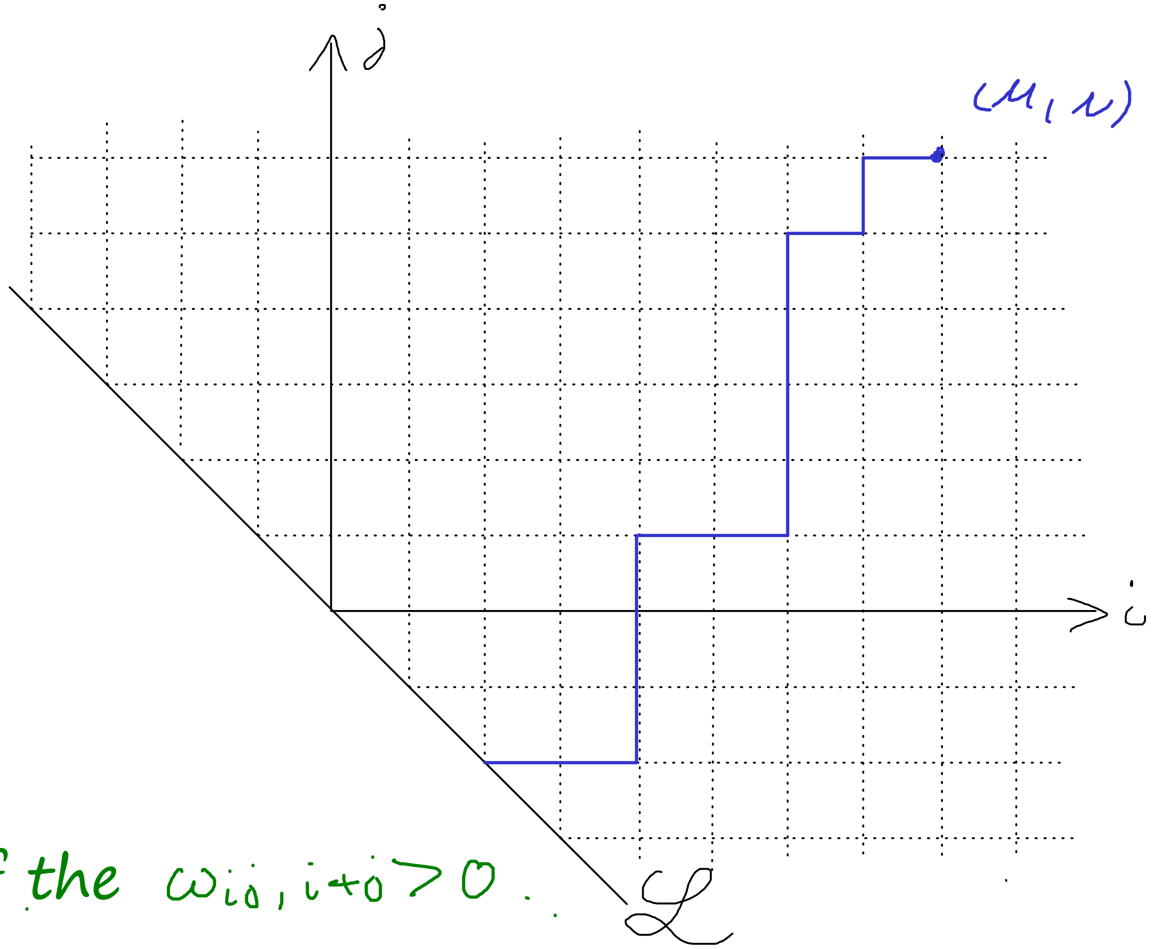
$$L^{PP}_{(M,N)} := \max_{\pi:(0,0)\to(M,N)} \sum_{(i,j)\in\pi} \omega_{ij}$$

Last passage percolation: point-to-line with weights

Let $\mathcal{L} = \{(i,j)\in\mathbb{Z}^2 \mid i+j=0\}$

Let the last passage percolation from $\mathcal{L}$ to (M,N) is

$$L^{\mathcal{L}}_{(M,N)} := \max_{\pi:\mathcal{L}\to(M,N)} \sum_{(i,j)\in\pi} \omega_{ij}$$



For the "weighted" generalization one takes $k \longmapsto \omega_{k,-k}$ to be a stochastic process independent of the $\omega_{ij},\ i+j>0$.

Remarks

The paths π satisfying $L_{(m,n)} = \sum_{(i,j)\in\pi} \omega_{ij}$

are also called geodesics

In our case, for each end-point the geodesic is a.s. unique

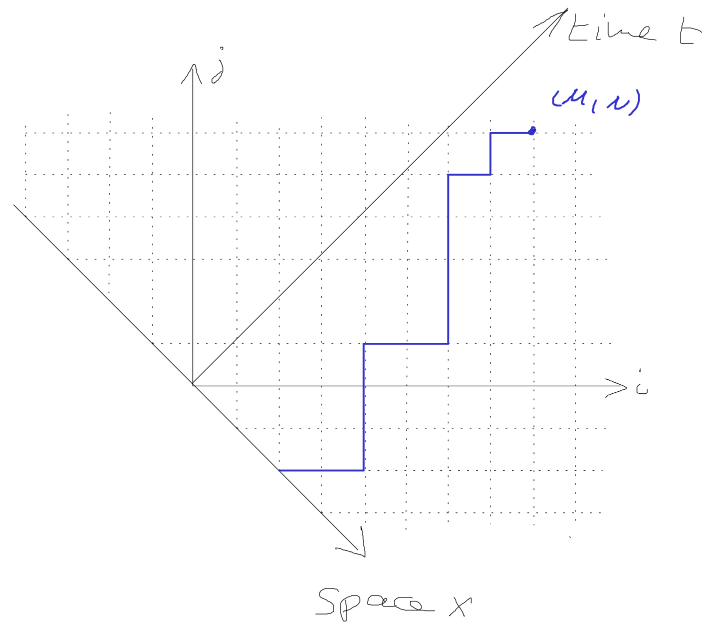
The stationary model with parameter $\rho \in (0,1)$ is given by

$$\omega_{k,-k} = \begin{cases} \sum_{j=1}^{k} (X_j - Y_j) & , k \geq 1 \\ 0 & , k = 0 \\ -\sum_{j=k+1}^{0} (X_j - Y_j) & , k \leq -1 \end{cases}$$

with $X_i \sim Exp(1-\rho)$, $Y_i \sim Exp(\rho)$

all independent r.v.

Relation with a growth model



For a given (m,n) we set

$$x = m-n \quad , \quad t = m+n$$

and $h(x,t) := L_{(m,n)}$ the height function

Due to $L_{(m,n)} = \max\{L_{(m,n-1)}, L_{(m-1,n)}\} + \omega_{m,n}$

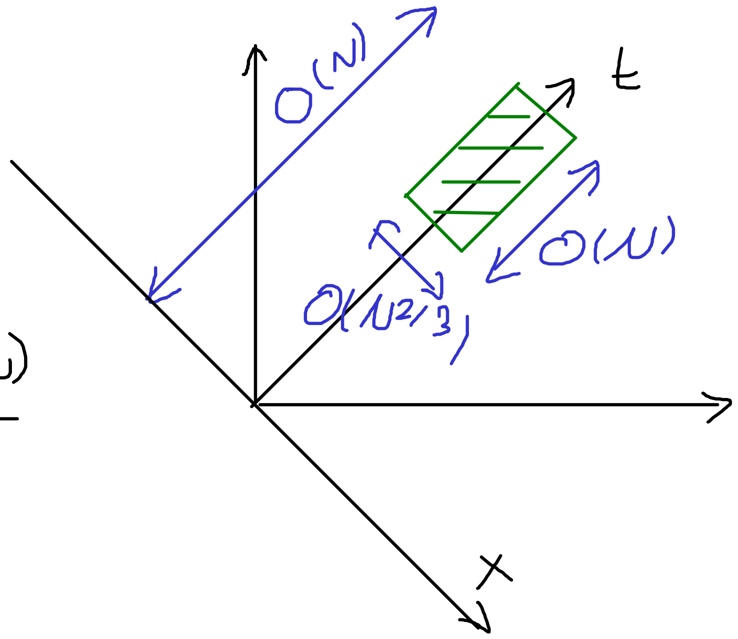
one has $h(x,t) = \max\{h(x-1,t-1), h(x+1,t-1)\} + \omega_{m(x,t),n(x,t)}$

$\Rightarrow h(x,0) = \omega_{x,-x}$ the process on $\mathcal{L}$.

The space-time scaling

Limit shape (deterministic):

$$L_{ma}(\xi,\tau) := \lim_{N\to\infty} \frac{L_{((\tau+\xi)N,(\tau-\xi)N)}}{N}$$



Scaling limit around $x=0$:

$$L_N^{resc}(u,\tau) := \frac{L_{(\tau N+uN^{2/3},\,\tau N-uN^{2/3})} - N\,L_{ma}(uN^{-1/3},\tau)}{N^{1/3}}$$

Special cases: fixed time

(Johansson) $\lim\limits_{N\to\infty} L_N^{pp,resc}(u,1) = \mathcal{A}_2(u) - u^2$

↑ Airy$_2$ process

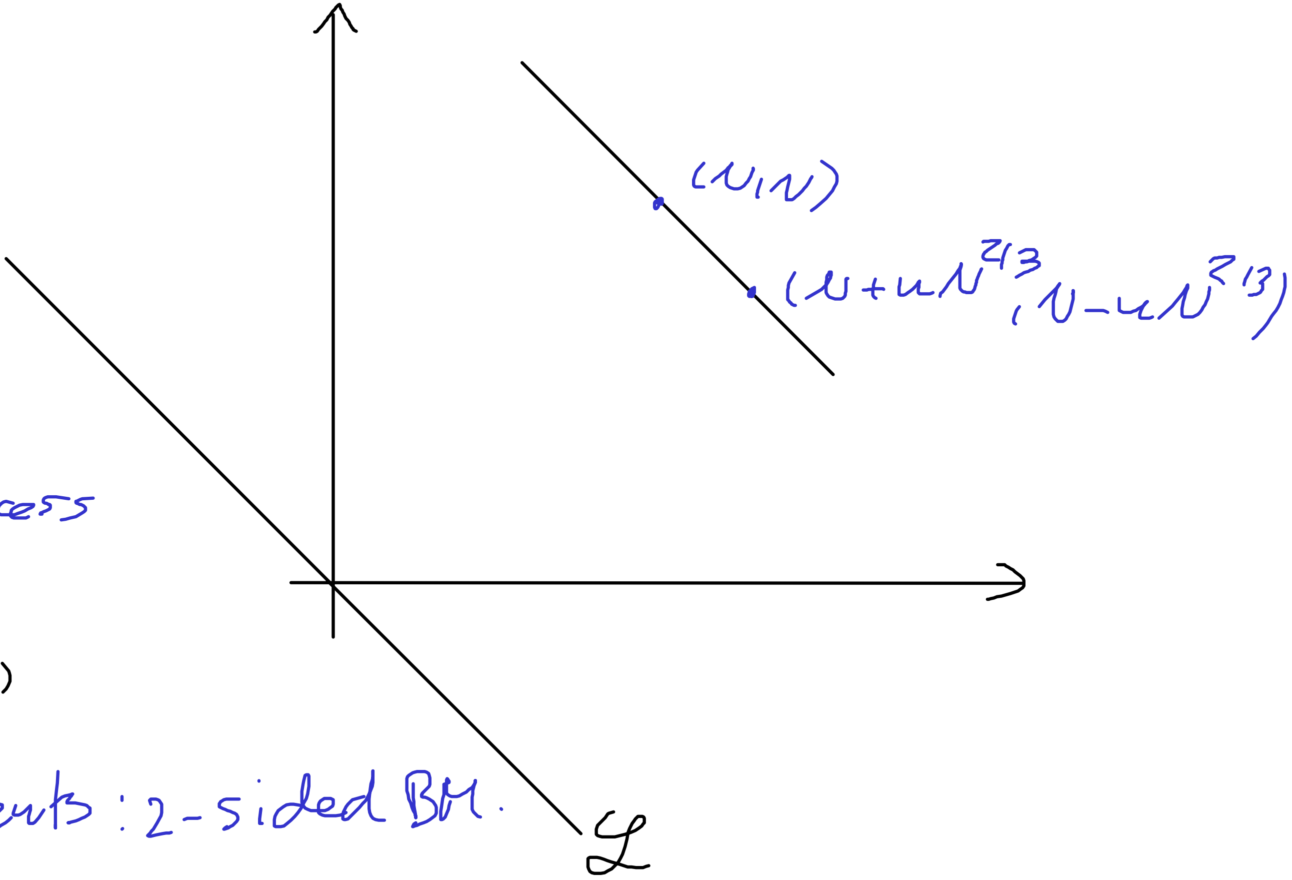
(Borodin, F., Sasamoto) $\lim\limits_{N\to\infty} L_N^{\mathcal{L},resc}(u,1) = \mathcal{A}_1(u)$

↑ Airy$_1$ process

(Baik, F., Péché) $\lim\limits_{N\to\infty} L_N^{stat,resc}(u,1) = \mathcal{A}_{stat}(u)$

↑ ($\rho=1/2$)

↑ Increments: 2-sided BM.



Local universality of the time-time covariance

with Alessandra Occelli

Let $\chi(\tau) := \lim_{N\to\infty} L_N^{\mathrm{resc}}(0,\tau)$; $\chi(u,\tau) := \lim_{N\to\infty} L_N^{\mathrm{resc}}(u,\tau)$

a) Stationary case:

$\forall \tau \in (0,1)$: $\mathrm{Cov}(\chi(\tau),\chi(1)) = \frac{1}{2}\left(1+\tau^{2/3} - (1-\tau)^{2/3}\right) \mathrm{Var}(\chi(1))$

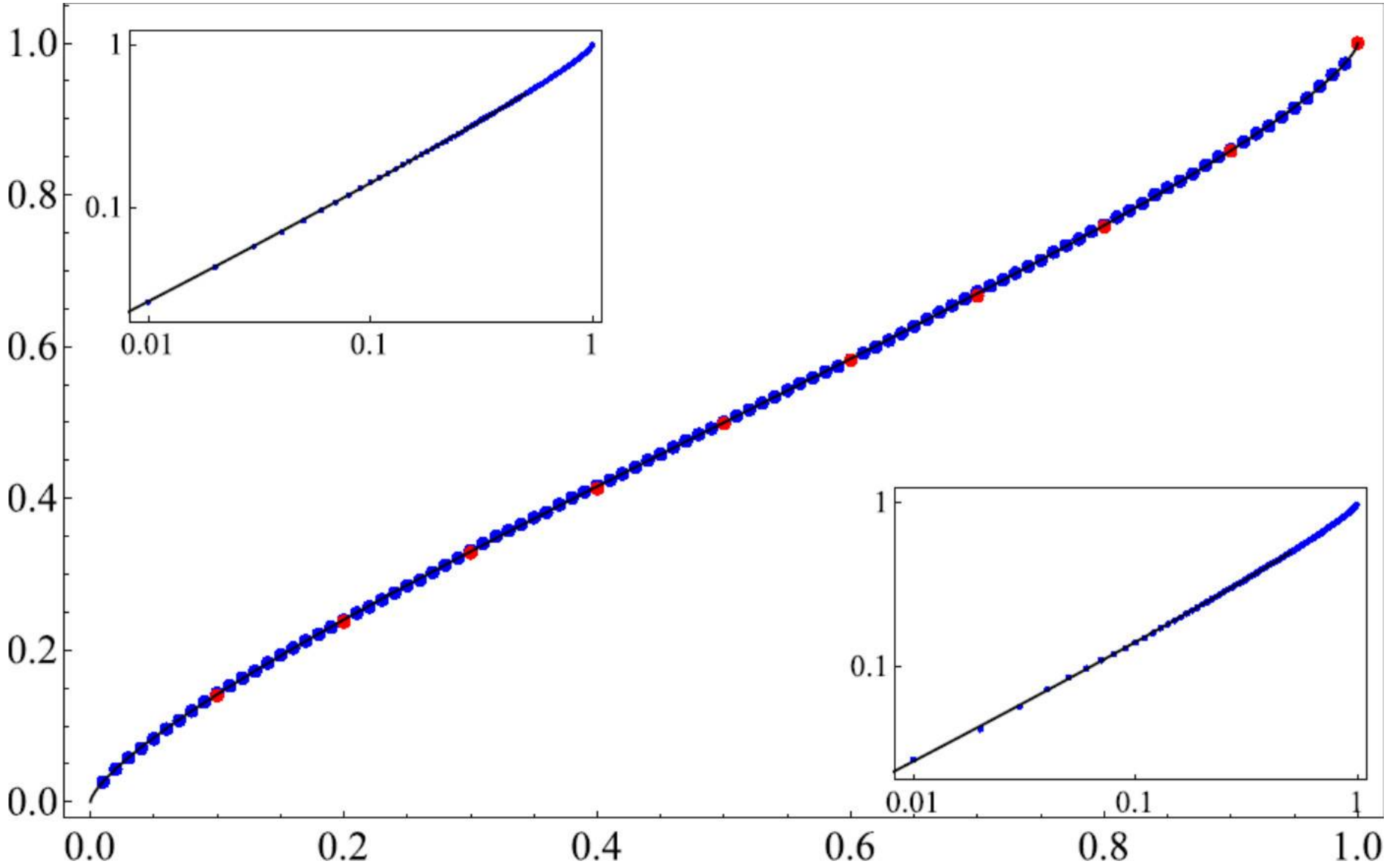
Baik-Rains distributed r.v.

b) Point-to-point & point-to-line cases:

$\mathrm{Cov}(\chi(\tau),\chi(1)) = \frac{1}{2}(1+\tau^{2/3})\mathrm{Var}(\chi(1)) - \frac{1}{2}(1-\tau)^{2/3}\mathrm{Var}(\xi_{BR}) + \mathcal{O}(1-\tau)$

GUE / GOE Tracy-Widom distributed r.v.

as $\tau \to 1$



$$\tau \mapsto \frac{\mathrm{Cov}(\chi(\tau), \chi(1))}{\sqrt{\mathrm{Var}(\chi(\tau))\,\mathrm{Var}(\chi(1))}}$$

for stationary TASEP, compared with simulations for t=1000 and t=10 000

(F., Spohn)

Origin of the first order universal behavior

$$\mathrm{Cov}(\chi(\tau),\chi(1)) = \tfrac{1}{2}\mathrm{Var}(\chi(\tau)) + \tfrac{1}{2}\mathrm{Var}(\chi(1)) - \tfrac{1}{2}\mathrm{Var}(\chi(1)-\chi(\tau))$$

and $$\chi(1) = \max_{u\in\mathbb{R}} \{\chi(u,\tau) + \chi^{pp}(u,1-\tau)\}$$

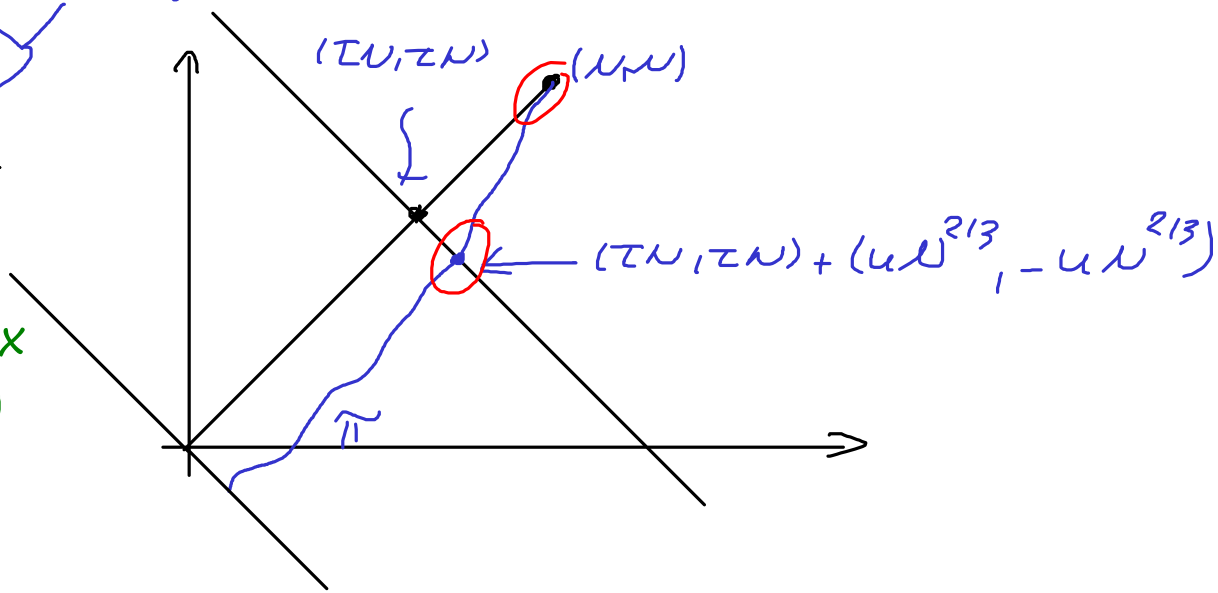
$$(1-\tau)^{1/3}\left[\mathcal{A}_2\left(\frac{u}{(1-\tau)^{2/3}}\right) - \frac{u^2}{(1-\tau)^{4/3}}\right]$$

indep.

$$\Rightarrow \chi(1)-\chi(\tau) = \max_u \{\chi(u,\tau)-\chi(0,\tau) + \chi^{pp}(u,1-\tau)\}$$

Locally like stationary case

It forces the arxmax to be $O((1-\tau)^{2/3})$



Strategy of the proof

1) Change of variable: $u = (1-\tau)^{2/3} v$

$$\chi(1) - \chi(\tau) = (1-\tau)^{1/3} \max_{v} \left\{ (1-\tau)^{-1/3} \left[\chi\left((1-\tau)^{2/3} v, \tau\right) - \chi(0,\tau) \right] + \mathcal{A}_2(v) - v^2 \right\}$$

2) Comparison with stationarity: (Cator–Pimentel comparison principle)

With high probability:

$$(1-\tau)^{-1/3} \left[\chi\left((1-\tau)^{2/3} v, \tau\right) - \chi(0,\tau) \right] \simeq \sqrt{2}\, \overset{(BM)}{B}(v) \quad \text{as } \tau \to 1.$$

This gives $\mathrm{Var}\left(\chi(1) - \chi(\tau)\right) = (1-\tau)^{2/3} \cdot \mathrm{Var}\Big(\underbrace{\max_{v} \left(\sqrt{2}\, B(v) + \mathcal{A}_2(v) - v^2 \right)}_{\overset{(d)}{=}\ \xi_{BR}} \Big) + O(1-\tau).$

(Quastel, Remenik)

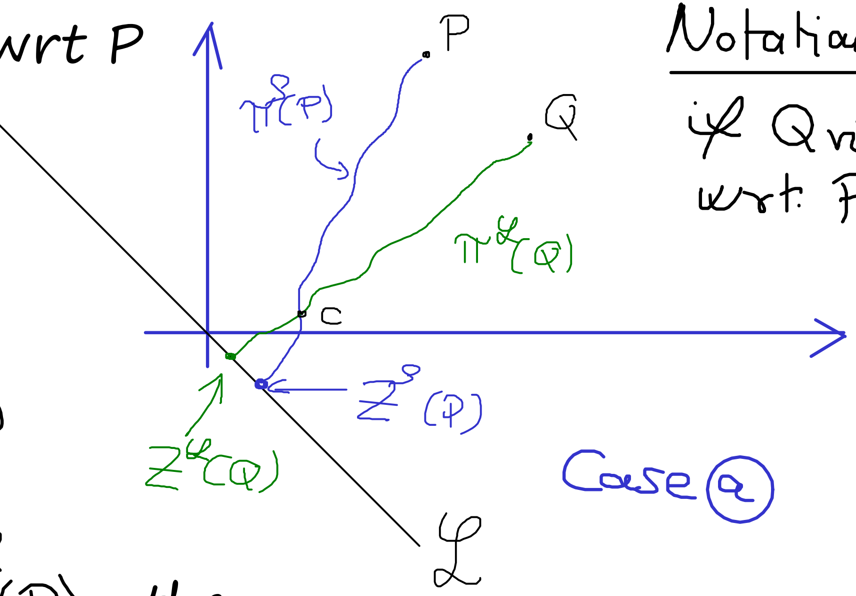
The comparison lemma: For Q right/down wrt P

Notation: $P \preceq Q$ if Q right/down wrt P.

(Stationary model)

(a) If the exit points satisfy: $Z^{\rho}(P)$ is to the right of $Z^{\alpha}(Q)$, then

$$L^{\alpha}(Q) - L^{\alpha}(P) \leq L^{\rho}(Q) - L^{\rho}(P)$$



(b) viceversa, if $Z^{\tilde{\rho}}(Q)$ is to the left of $Z^{\alpha}(P)$, then

$$L^{\alpha}(Q) - L^{\alpha}(P) \geq L^{\tilde{\rho}}(Q) - L^{\tilde{\rho}}(P)$$

Application: choose $\rho, \tilde{\rho}$ in a $N^{-1/3}$-neighborhood of 1/2, using the formulas from the integrable structure: show that the inequalities hold with high probability.

Other results concerning space-time processes are available:

- Johansson (2017-18): two-time distribution in Brownian and geometric LPP
- Johansson-Rahman (2019): multi-time distribution in discrete PNG
- Johansson (2019): The long and short time asymptotics of the two-time distribution in local random growth
- Johansson-Rahman (2020): On inhomogeneous polynuclear growth

- Baik-Liu (2017-18): multi-point distribution of periodic TASEP
- Liu (2019): Multi-time distribution of TASEP

- De Nardis, Le Doussal (2018): Two-time distribution for 1D KPZ growth

What about the geodesics?

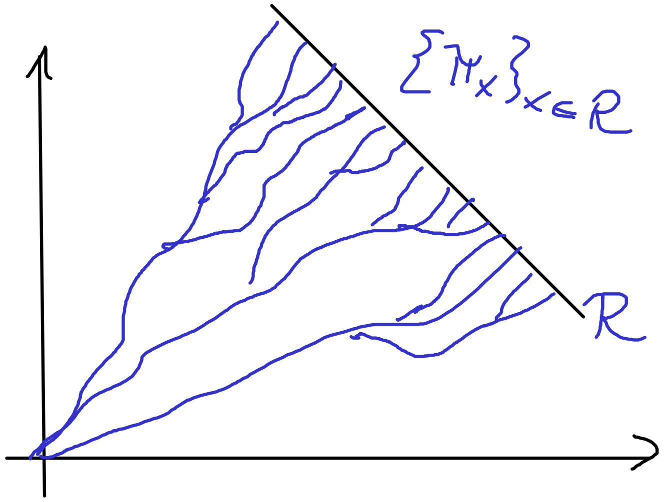
Is the universal behavior of the height-height correlation in a space-time window $o(\ell^{2/3}) \times o(\ell)$

reflected by other quantities, like the geodesics?

The geodesic tree

For each end-point x

we have a geodesic π_x^{PP}.



Notation: For point-to-point LPP: π_x^{PP}

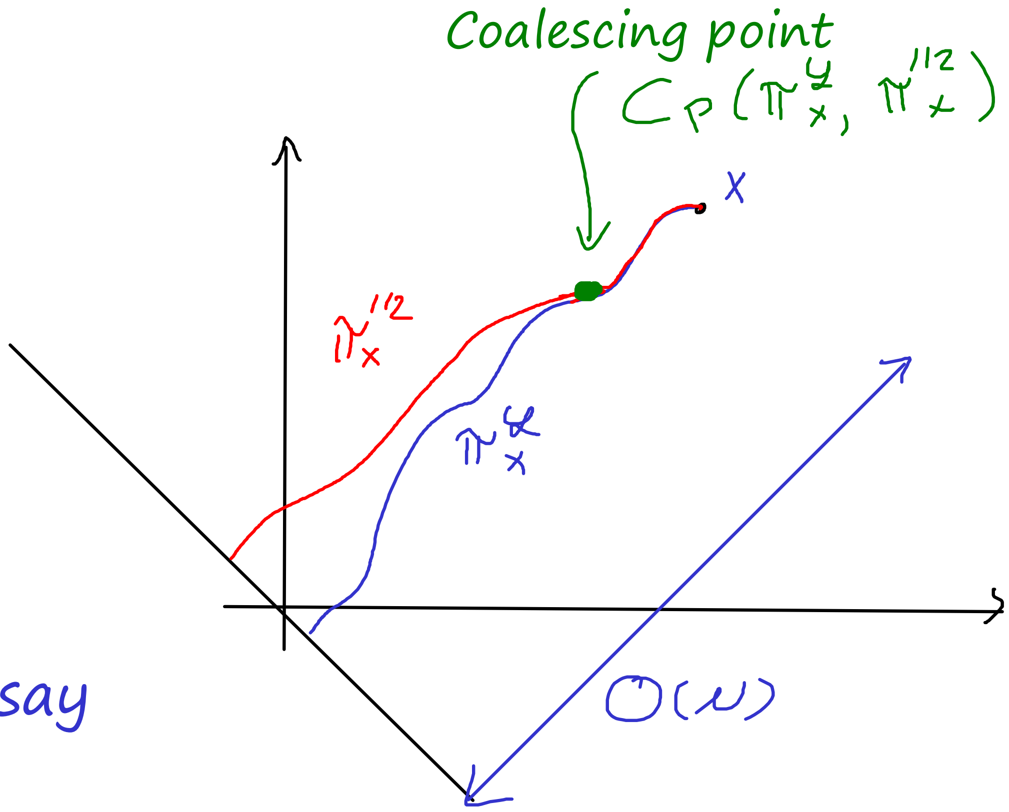
For point-to-line LPP: $\pi_x^{\mathcal{L}}$

For stationary LPP with parameter ϱ: π_x^{ϱ}

The geodesic tree is then given by $T^{\cdot} = \{\pi_x^{\cdot}\}_{x \geq \mathcal{L}}$

Our question

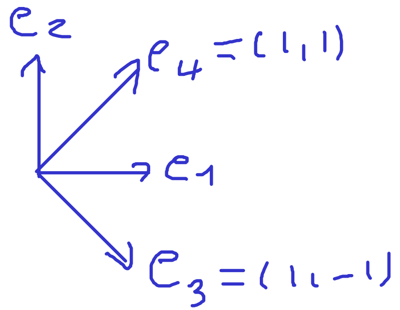
How "close" is the geodesic tree for the point-to-point LPP (or point-to-line LPP) to the geodesic tree of stationary LPP?

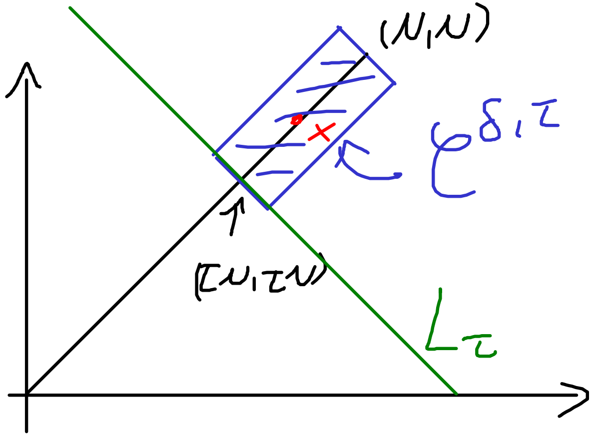


Define for any x and two LPP, say π_x^{α} and π_x^{β} consider the coalescing point $C_p(\pi_x^{\alpha}, \pi_x^{\beta})$

Point-to-point result

with Ofer Busani





Define the cylinder of width $\delta N^{2/3}$

and length $(1-\tau)N$: $\mathcal{C}^{\delta,\tau} = \{ i e_4 + j e_3 \mid \tau N \le i \le N \text{ and } -\frac{\delta}{2} N^{2/3} \le j \le \frac{\delta}{2} N^{2/3} \}$

Define $L_\tau = \{ \tau N e_4 + i e_3,\ i \in \mathbb{Z} \}$.

Theorem: $\exists\, C, \delta_0 > 0$ s.t. $\forall\, \delta \in (0, \delta_0)$ and $1-\tau \le \dfrac{\delta^{3/2}}{(\ln(\delta^{-1}))^3}$,

$$\mathbb{P}\left(C_p(\pi^{1/2}_x, \pi^{pp}_x) \le L_\tau,\ \forall x \in \mathcal{C}^{\delta,\tau} \right) \ge 1 - C\delta^{1/2} \ln(\delta^{-1})$$

uniformly in N.

Theorem: $\exists\, C, \delta_0 > 0$ s.t. $\forall\, \delta \in (0, \delta_0)$ and $1-\tau \leq \frac{\delta^{3/2}}{(\ln(\delta^{-1}))^3}$,

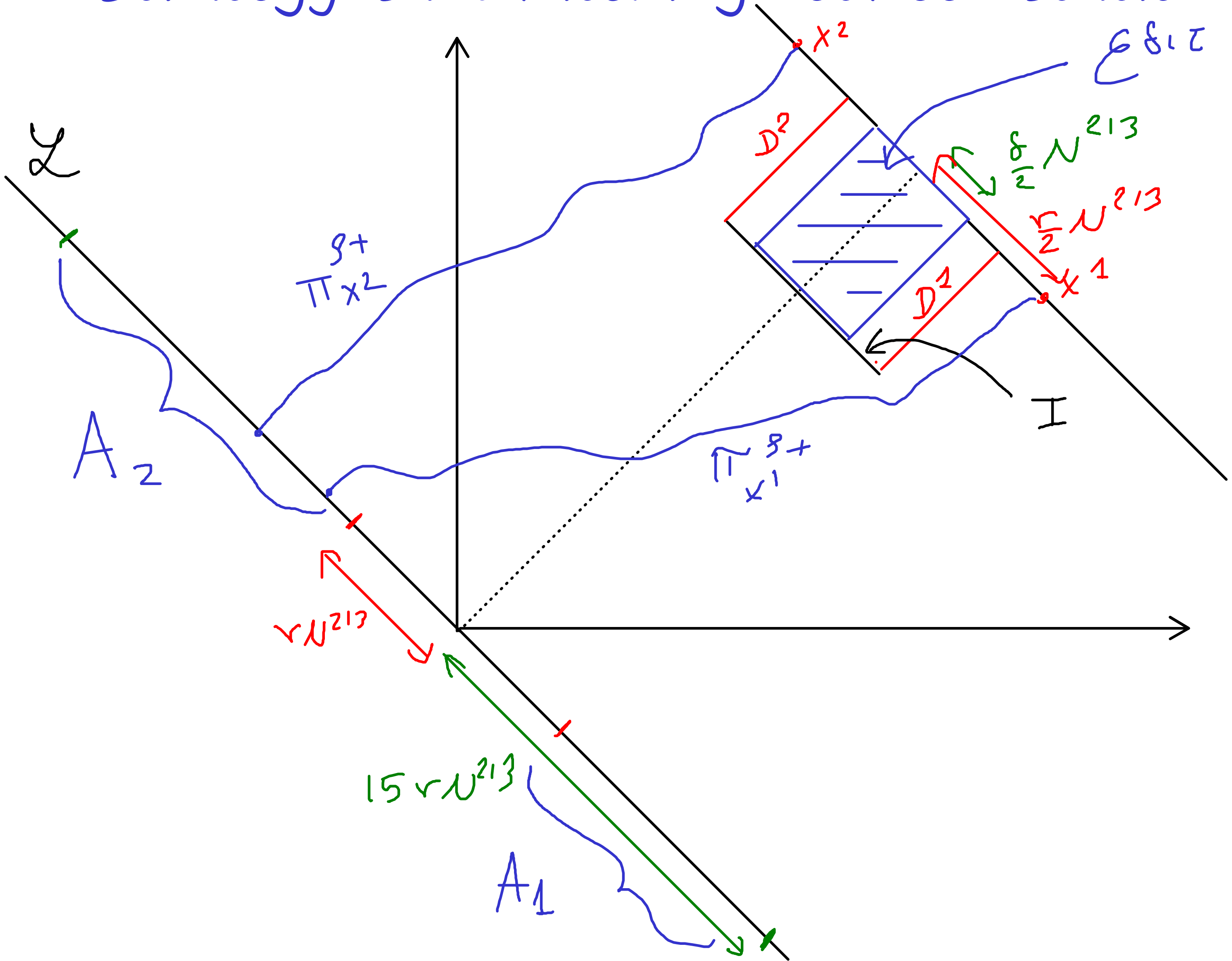
$$\mathbb{P}\left(C_P(\pi_x^{1/2}, \pi_x^{pp}) \leq L_\tau, \; \forall x \in \mathcal{E}^{\delta,\tau} \right) \geq 1 - C\delta^{1/2} \ln(\delta^{-1})$$

uniformly in N.

The probability that the geodesic tree generated by all end-points in $\mathcal{E}^{\delta,\tau}$ of the point-to-point LPP agrees with the stationary geodesic tree

We also generalize this result to the case of LPP with weights on $\mathcal{L}$

Strategy: sandwitching between stationary geodesics



$\rho_\pm = \frac{1}{2} \pm r N^{-1/3}, \quad r >> 1$

1 Whp: $\pi^{\rho_+}_{x^2} \cap D^2 = \emptyset$

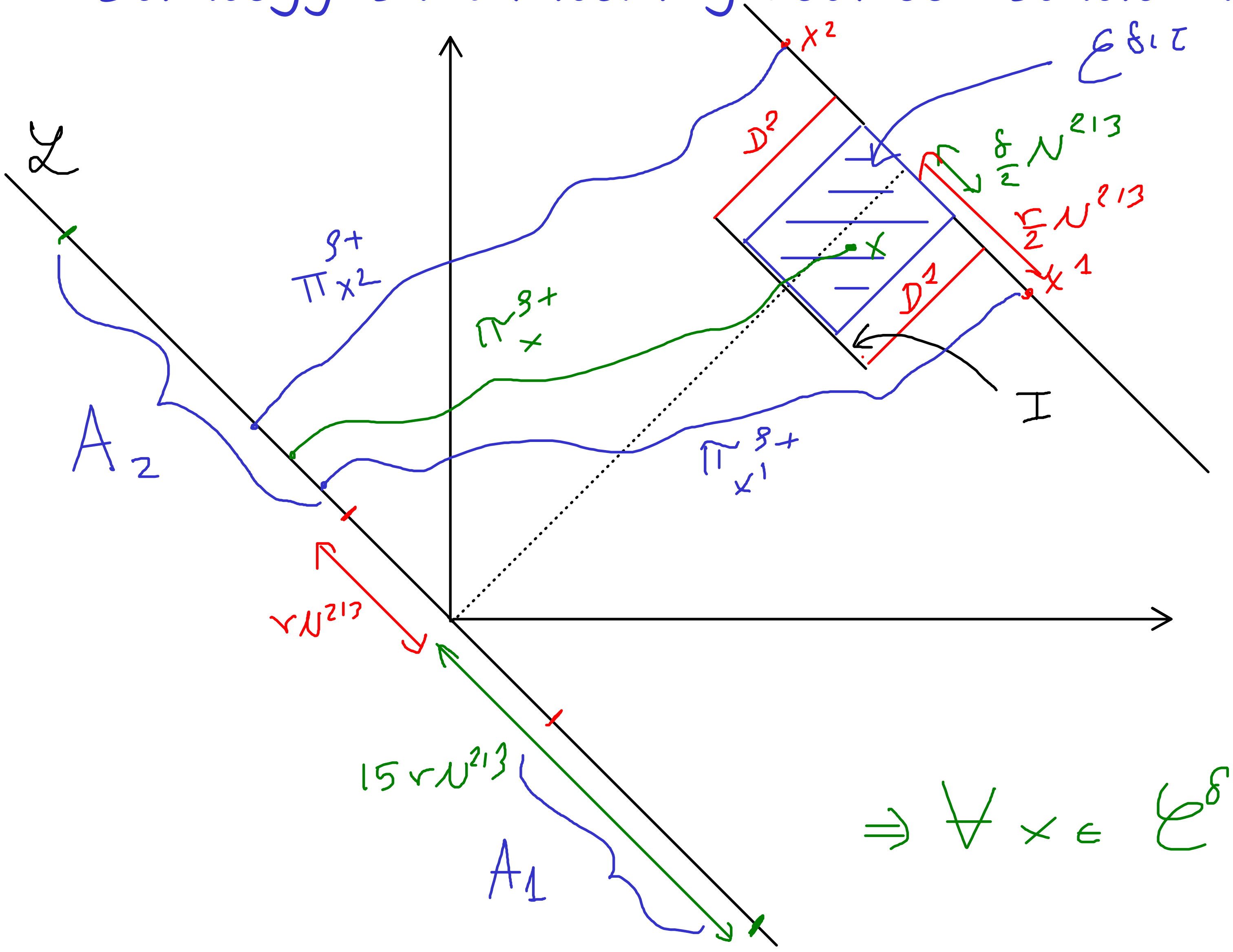
(As $\delta \to 0$ & $r \to \infty$)

$\cdot\ \pi^{\rho_+}_{x^1} \cap D^1 = \emptyset$

$\cdot\ \pi^{\rho_+}_{x^2} \cap \mathcal{L} \in A_2$

$\cdot\ \pi^{\rho_+}_{x^1} \cap \mathcal{L} \in A_2$

Strategy: sandwitching between stationary geodesics



$\rho_\pm = \frac{1}{2} \pm rN^{-1/3}$, $r \gg 1$

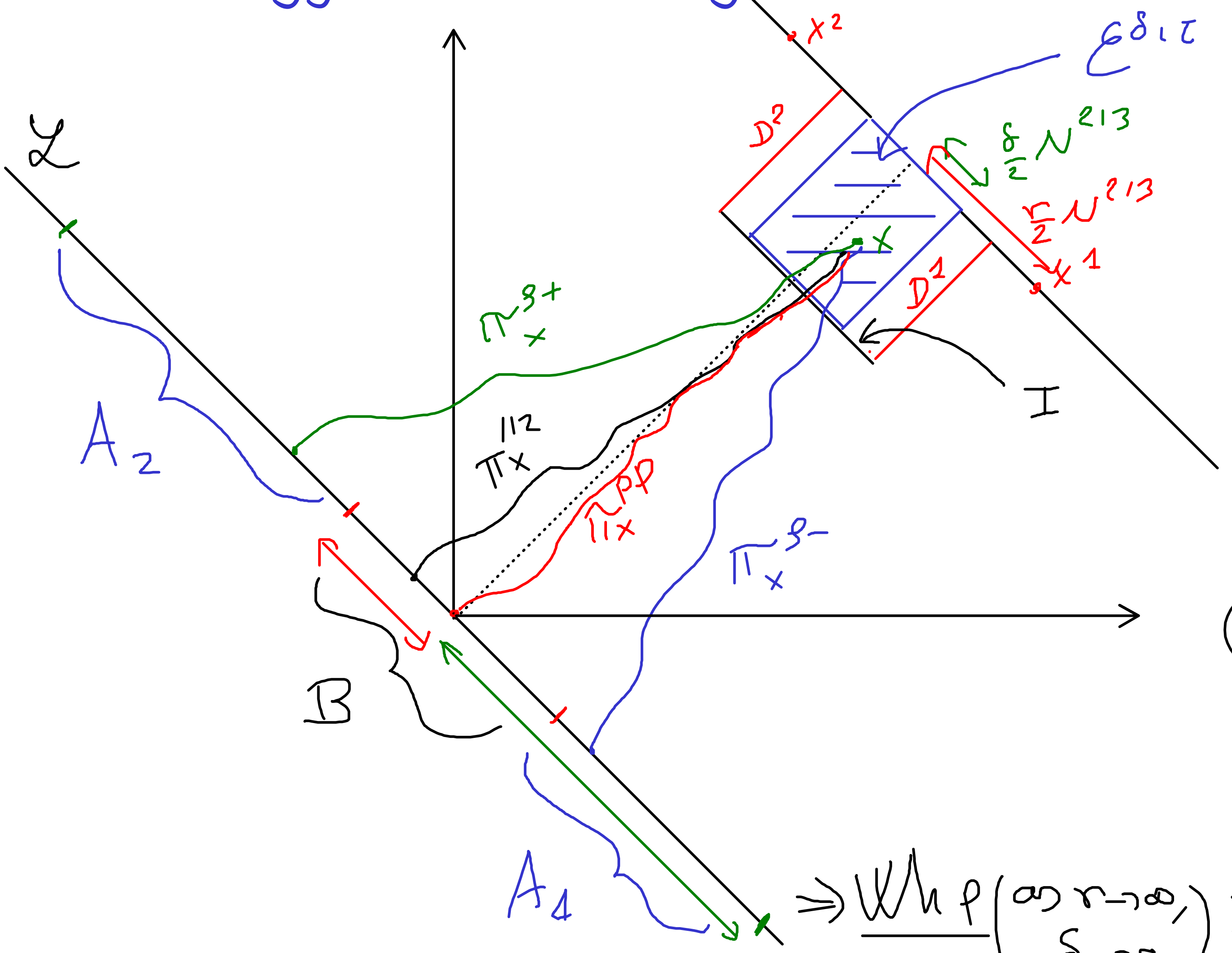
(1) Whp:
- $\pi^{\rho+}_{x^2} \cap D^2 = \emptyset$
- $\pi^{\rho+}_{x^1} \cap D^1 = \emptyset$
- $\pi^{\rho+}_{x^2} \cap \mathcal{L} \in A_2$
- $\pi^{\rho+}_{x^1} \cap \mathcal{L} \in A_2$

(As $\delta \to 0$ & $r \to \infty$)

(2) Whp: $\pi^{\rho\pm}_{x} \cap I \neq \emptyset$, $\forall x \in \mathcal{C}^{\delta,\tau}$

$\Rightarrow \forall x \in \mathcal{C}^{\delta,\tau}$, $\pi^{\rho+}_{x^2} \leq \pi^{\rho+}_{x} \leq \pi^{\rho+}_{x^1}$

Strategy: sandwitching between stationary geodesics



$\rho_\pm = \frac{1}{2} \pm r N^{-1/3}$

$\Rightarrow \forall\, x \in \mathcal{E}^{\delta,\tau},$

$\pi^{\rho+}_{x^2} \le \pi^{\rho+}_x \le \pi^{\rho+}_{x^1}$

$\&\ \pi^{\rho-}_{x^2} \le \pi^{\rho-}_x \le \pi^{\rho-}_{x^1}$

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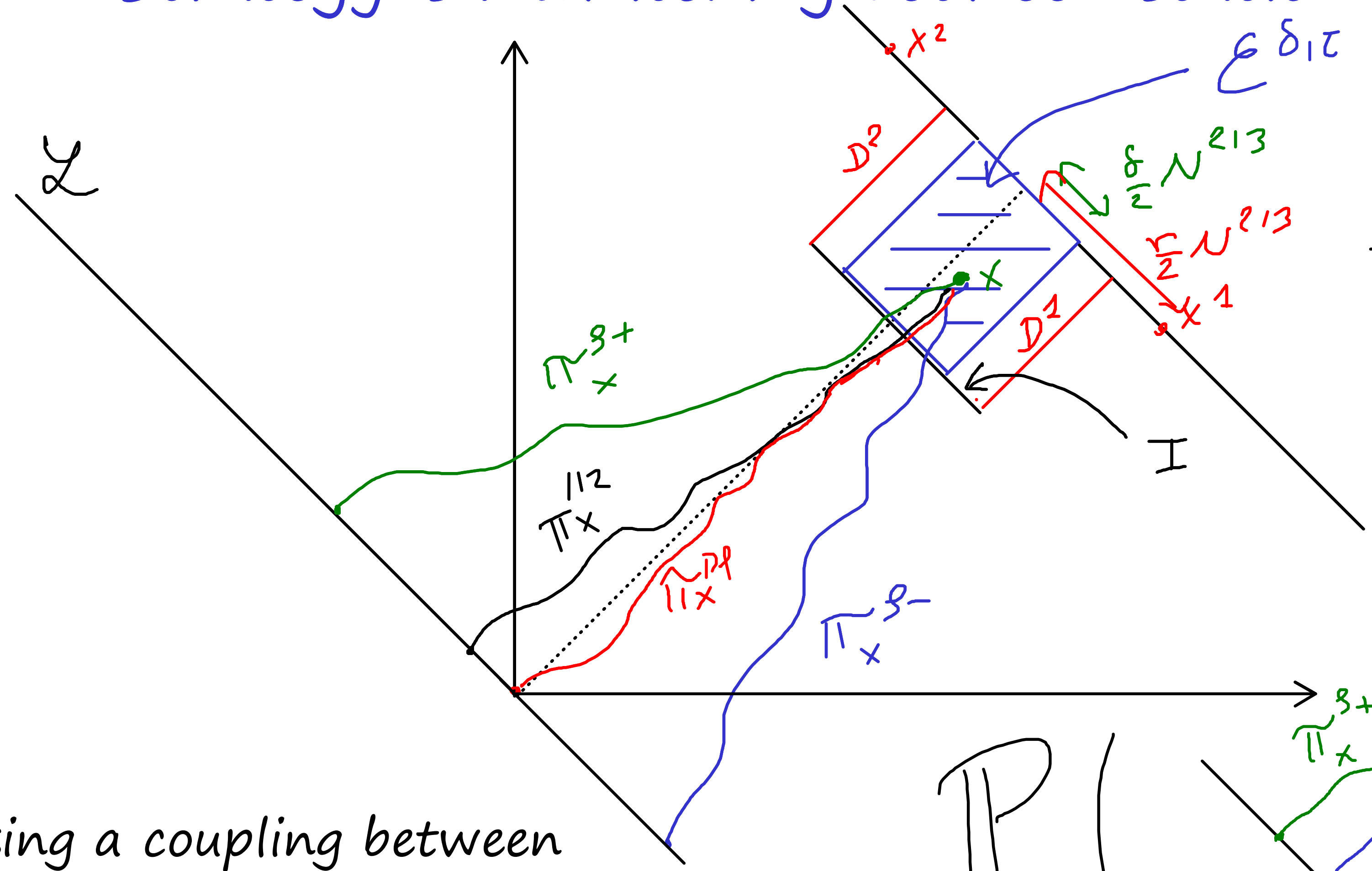
③ Whp: $\pi^{1/2}_x \cap \mathcal{L} \in \mathcal{B}$

$\tilde{\pi}^{pp}_x \cap \mathcal{L} =$ [illegible]

$\Rightarrow$ <u>Whp</u> $\left(\begin{matrix}\text{as } r \to \infty, \\ \delta \to 0\end{matrix}\right)$: $\underline{\pi^{\rho+}_x \le \pi^{1/2}_x, \tilde{\pi}^{pp}_x \le \pi^{\rho-}_x.}$

Strategy: sandwitching between stationary geodesics

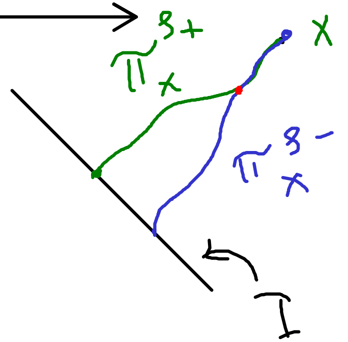


$\rho_{\pm} = \frac{1}{2} \pm r N^{-1/3}$

$\Rightarrow$ whp, $\forall x \in \mathcal{E}^{\delta,\tau}$:

$\pi_x^{\rho+} \leq \pi_x^{1/2}, \tilde{\pi}_x^{pp} \leq \pi_x^{\rho-}$.

Using a coupling between stationary models by Seppäläinen-Shen, one obtains the bound:

$\mathbb{P}\Bigg($  $\Bigg) \leq \delta^{1/2} r \xrightarrow[\substack{\delta\to 0\\ r\to\infty}]{} 0$

if $r \ll \frac{1}{\sqrt{\delta}}$.

Other results on aspects of geodesics in LPP can be found in:

- Pimentel (2016): Duality between coalescence times and exit points in LPP models
- Basu, Sarkar, Sly (2019): Coalescence of geodesics in exactly solvable models of LPP
- Zhang (2020): Optimal exponent for coalescence of finite geodesics in exponential LPP
- Hammond (2020): Exponents governing the rarity of disjoint polymers in Brownian LPP
- Basu, Ganguly, Hammond (2021): Fractal geometry of Airy2 processes coupled via the Airy sheet
- Basu, Ganguly, Hammond, Hedge (2020): Interlacing and scaling exponents for the geodesic watermelon in LPP