Lozenge tilings on a cylinder

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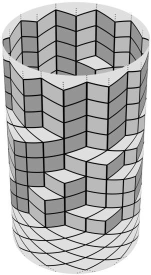
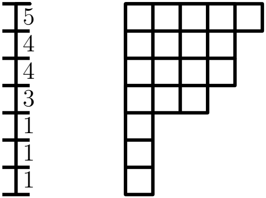


Based on joint work with A. Ahn and R. Van Peski.

Ordinary partition

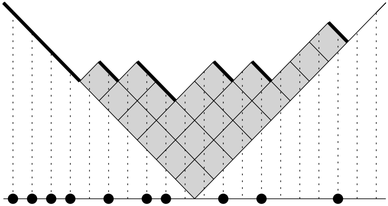
Young diagram



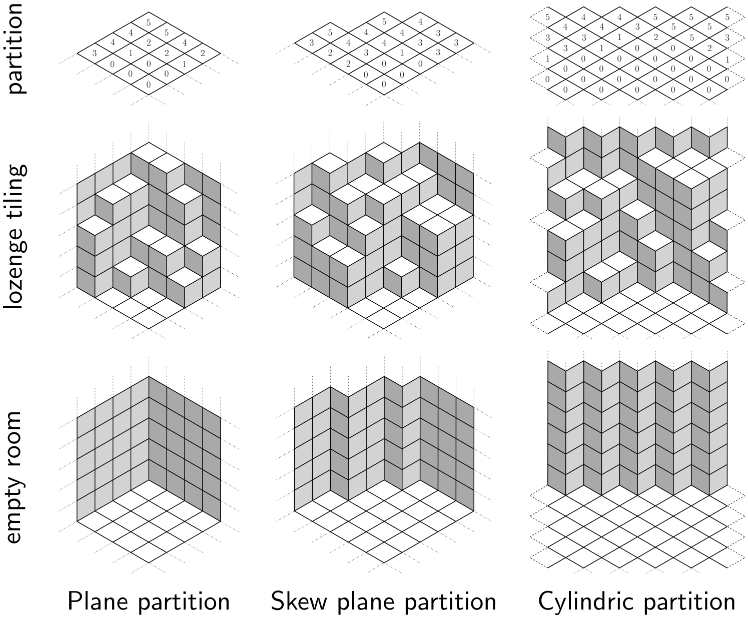
$$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq 0)$$

$$\lambda_i = 0 \quad \text{for} \quad i \gg 0$$

$$\lambda = (5, 4, 4, 3, 1, 1, 1)$$

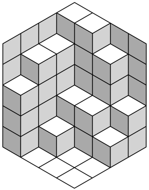


The partition $\lambda = (8, 5, 4, 2, 2, 1)$; $\lambda \quad \longleftrightarrow \quad \{\lambda_i - i + \frac{1}{2}\}$

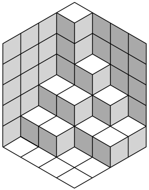

partition
lozenge tiling
empty room
Plane partition
Skew plane partition
Cylindric partition

Random tiling/partition

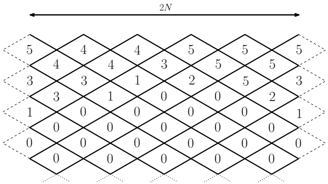
Uniform measure: uniformly random tilings.

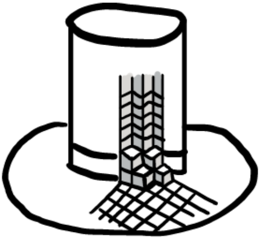


q^{vol} measure: $\mathbb{P}[\text{tiling}] \propto q^{\text{vol(tiling)}}$, $0 < q < 1$



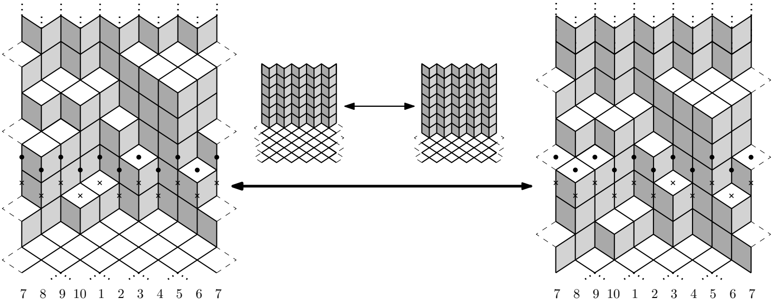
Cylindric partition





Let $q^N = t \in (0,1)$,
q^{vol} measure on cylindric partitions: $\mathbb{P}(\lambda) \propto q^{\mathrm{vol}(\lambda)}$.

Lozenge tilings on a cylinder

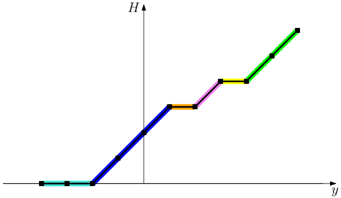


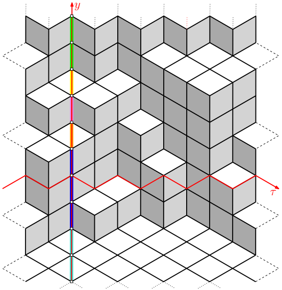
lozenge tilings of the cylinder = shifted cylindric partitions

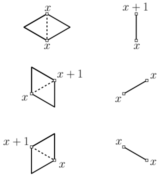
shift-mixed q^{vol} measure: $\mathbb{P}(\lambda, S) \propto (u^S q^{NS^2}) q^{\mathrm{vol}(\lambda)}$,

$u > 0$, S is a vertical shift of the wall-floor interface

Height function







The height function $H(\tau, y)$ vanishes for all sufficiently negative y and $H(\tau, y) = y - S$ for all sufficiently large positive y.

The key questions: the large-scale behavior of

(*a*) the limit shape of the height function,

(*b*) fluctuations of the height function.

Limit shape

Let $q^N = t \in (0, 1)$

Theorem (Ahn, R., Van Peski '21)

The height function h_N of a q^{vol}-distributed cylindric partition of width $2N$ converges in probability to the following limit shape uniformly:

$$\frac{1}{N} h_N(N\tau, Ny) \to \mathcal{H}(y) = \begin{cases} 0 & y \le \frac{\log 2}{\log t}, \\ \int_{\frac{\log 2}{\log t}}^{y} \frac{2\arctan(\sqrt{4t^{-2u} - 1})}{\pi}\, du & y \ge \frac{\log 2}{\log t}. \end{cases}$$

- [Borodin '07] showed result on local statistics which also computes the limit shape; our only real input here is showing concentration.
- The shift-mixed q^{vol} measure has the same limit shape above, as the distribution of the shift is independent of the tiling and is finite-order independent of N.

Fluctuations

Theorem (Ahn, R., Van Peski '21)

The fluctuations of the height function of a q^{vol}*-distributed cylindric partition converges on the liquid region to the Gaussian free field in the Kenyon-Okounkov complex structure.*

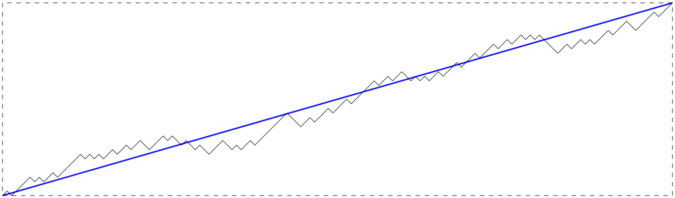
Theorem (Ahn, R., Van Peski '21)

The fluctuations of the height function of a shift-mixed q^{vol}*-distributed cylindric tiling are given by the same Gaussian free field with an additional discrete Gaussian shift component.*

Simple random walk

- Limit shape:

  As $X, T \to \infty$, $\frac{X}{T} = \text{const}$, $\frac{Z_{\lfloor sT \rfloor}}{X} \to s$ uniformly over $s \in [0,1]$.



- Fluctuations:

  $\frac{Z_{\lfloor sT \rfloor} - \mathbb{E}[Z_{\lfloor sT \rfloor}]}{C\sqrt{T}} \to B_s$, where B_s is a standard Brownian bridge.

$$G(s,s') := \text{Cov}(B_s, B_{s'}) = \min(s,s')(1 - max(s,s'))$$

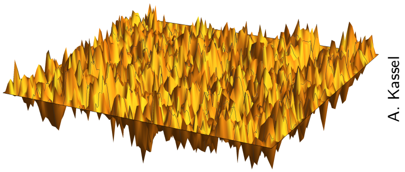
  is the Green's function for Laplacian $\Delta = \partial^2/\partial s^2$ on $[0,1]$ with zero Dirichlet boundary conditions.

Gaussian Free Field

Definition

The Gaussian free field Φ on $\mathcal{D}$ is the random distribution such that pairings with test functions $\int_{\mathcal{D}} f\Phi$ are jointly Gaussian with covariance

$$\mathrm{Cov}\left(\int_{\mathcal{D}} f_1\Phi, \int_{\mathcal{D}} f_2\Phi\right) = \int_{\mathcal{D}\times\mathcal{D}} f_1(z)G(z,w)f_2(w).$$



GFF with zero boundary conditions on a domain $\mathcal{D} \subset \mathbb{C}$ is a conformally invariant random generalized function:

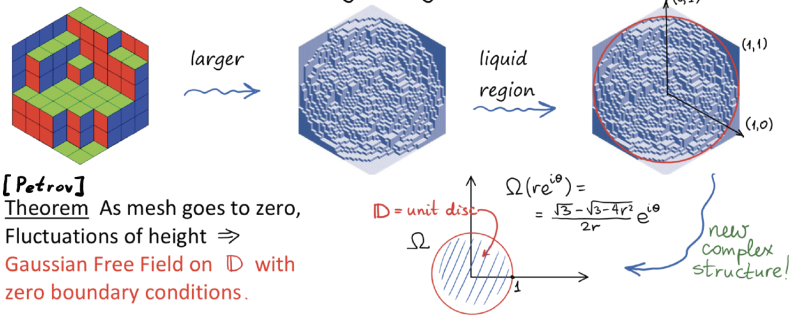
$$\Phi(z) = \sum_k \xi_k \frac{\phi_k(z)}{\sqrt{\lambda_k}},$$

[1d analog: Brownian Bridge]

where ϕ_k are eigenfunctions of $-\Delta$ on $\mathcal{D}$ with zero boundary conditions, λ_k is the corresp. eigenvalue, and ξ_k are i.i.d. standard Gaussians.
The GFF is not a random function, but a random distribution.

GFF is a Gaussian process on $\mathcal{D}$ with Green's function of the Laplacian as the covariance kernel.

Uniform lozenge tilings and GFF



Conjecture [Kenyon-Okounkov '05]
For lozenge tilings of simply connected planar regions, there exists a map ζ on liquid region $\mathcal{L}$ so that

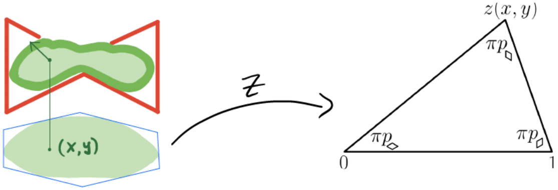
$$\sqrt{\pi}(H(x^\delta, y^\delta) - \mathbb{E}[H(x^\delta, y^\delta)]) \to \Phi \circ \zeta(x, y)$$

where Φ is the GFF and ζ is a local diffeomorphism onto its image.

Theorem (Kenyon-Okounkov '05)

In the liquid region (i.e. where $p_{\square}, p_{\square}, p_{\diamond} > 0$), there exists a function $z(x, y)$ taking values in the upper half plane such that

$$\nabla \mathcal{H} = \frac{1}{\pi}(\arg z, -\arg(1-z)) \quad \textit{and} \quad \frac{-z_x}{1-z} + \frac{z_y}{z} = 0.$$



Uniform measure: $\zeta = z$.

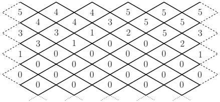
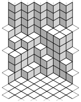
q^{vol} measure (volume-constrain): let $q = e^{-c\delta}$, then $\zeta = e^{cx} z$.

Known results

Limit shape	Fluctuations
[Cohn-Kenyon-Propp '00] proved a.s. convergence to certain entropy-maximizers for uniformly random domino tilings of simply connected domains in $\mathbb{R}^2$. **[Kenyon-Okounkov-Sheffield '03]** showed more generally (weighted doubly periodic bipartite dimer models on simply connected planar regions). **[Okounkov-Reshetikhin '01]** computed limit shape for q^{vol} ordinary plane partitions. **[Cerf-Kenyon '01]** Same limit shape for uniform measure on plane partitions of given volume.	Certain domains with no frozen regions (e.g. **[Kenyon '01]**, **[R. '18]**, **[R. '19]**; **[Kenyon '08]**, **[Berestycki-Laslier-Ray '20]**). **[Ahn '20]** q^{vol} plane partitions.
Certain polygonal domains (e.g. **[Borodin-Ferrari '08]**, **[Petrov '12]**, **[Bufetov-Knizel '18]**). **[Bufetov-Gorin '17]** Hexagon with a hole of fixed height (not simply connected).	
	[Chelkak-Laslier-R.'20] Certain general (not necessary doubly periodic) weighted bipartite planar graphs. **[Chelkak-Laslier-R.'21]** Appearance of Lorentz-minimal surfaces in the dimer model context.

Today: q^{vol}-distributed cylindric partitions and shift-mixed q^{vol}-distributed cylindric partitions.

Model

q^{vol}	shift-mixed q^{vol}
measure supported on: cylindric partitions 	shifted cylindric partitions = lozenge tilings of the cylinder 
$\mathbb{P}(\lambda) \propto q^{\text{vol}(\lambda)}$	$\mathbb{P}(\lambda, S) \propto u^S q^{\text{vol}(\lambda, S)} = (u^S q^{NS^2}) q^{\text{vol}(\lambda)}$, $u > 0$, S is a vertical shift of the wall-floor interface
periodic Schur process	shift-mixed periodic Schur process
	determinantal structure, comes from the dimer model

$h(\tau, y) := \sum_{x<y}$[there is no lozenge of type $\diamond$ at (τ, x)]

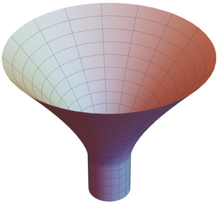
$h(\tau, y)$ vanishes for all sufficiently negative y and $h(\tau, y) = y - S$ for all sufficiently large positive y

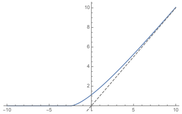
Limit shape

Define a function $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ by

$\mathcal{H}'(y) = \frac{2\arctan\left(\sqrt{4t^{-2y}-1}\right)}{\pi} \mathbb{1}(0 < t^y < 2)$

and $\quad \lim_{y\to-\infty} \mathcal{H}(y) = 0.$





Theorem (Ahn, R., Van Peski '21)

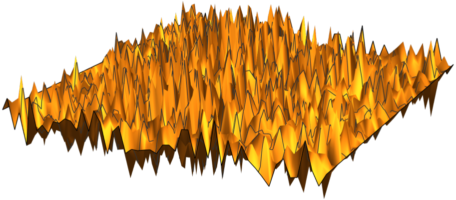
The height function $\frac{1}{N}h_N$ of a q^{vol} / shift-mixed q^{vol}-distributed cylindric partition of widh $2N$ converges in probability to the limit shape $\mathcal{H}$ uniformly.

$$p_{\lozenge} = p_{\lozenge} \quad \text{(symmetry)}$$

$$\mathcal{H}'(y) = 1 - p_{\diamond}$$

$$\mathcal{L} = \{(\tau, y) \in (0,1] \times \mathbb{R} : 0 < t^{2y} < 4\} = \{(\tau, y) \in (0,1] \times \mathbb{R} : y > \frac{\log 2}{\log t}\}.$$

q^{vol}



Theorem (Ahn, R., Van Peski '21)

Fix $t \in (0,1)$. Then the height function fluctuations of the unshifted q^{vol} measure converges as $N \to \infty$ to the η-pullback of the Gaussian free field on the cylinder $\mathcal{C} = (0, \frac{1}{2}) \times \mathbb{R}/\frac{|\log t|}{2\pi}$ with 0-Dirichlet boundary conditions, where $\eta : \mathcal{L} \to \mathcal{C}$ is given by

$$\eta(\tau, y) = \frac{1}{2\pi i} \log \left(t^{\tau} \frac{2 - t^{2y} + i\sqrt{4t^{2y} - t^{4y}}}{2} \right).$$

Remark: η defines the same conformal structure as the one conjectured by Kenyon-Okounkov.

shift-mixed q^{vol}

A discrete Gaussian $S \sim \mathcal{N}_{\text{discrete}}(C, m)$ is the $\mathbb{Z}$-valued random variable defined by

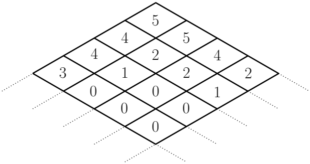
$$\Pr(S = x) \propto e^{-C(x-m)^2}.$$

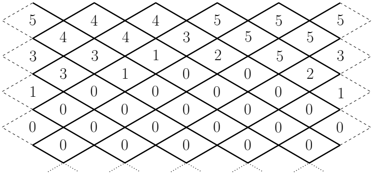
Theorem (Ahn, R., Van Peski '21)

Fix $u \in \mathbb{R}_{>0}$ and $t \in (0,1)$, set $q := q(N) := t^{1/N}$. Then the height function fluctuations of the shift-mixed q^{vol} measure converges to the η-pullback of the Gaussian free field with a discrete Gaussian shift $S \sim \mathcal{N}_{\text{discrete}}\left(\frac{|\log t|}{2}, \frac{\log u}{\log t}\right)$,

$$h(2N\tau, 2Ny) - \mathbb{E}[h(2N\tau, 2Ny)] \xrightarrow{N\to\infty} \Phi(\eta(\tau, y)) - S\mathcal{H}'(y).$$

Methods





- q^{vol} plane partitions are distributed as a certain Schur process **[Okounkov-Reshetikhin '01]**
- (shift-mixed) q^{vol} cylindric partitions are certain (shift-mixed) periodic Schur process **[Borodin '07]**

Methods

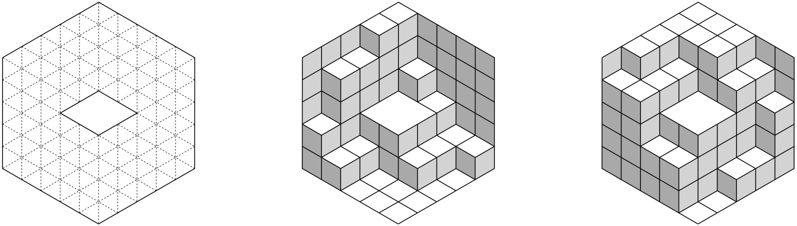
- new formulas for joint exponential moments of the height function of periodic Schur processes
- similar formulas for the joint moments, which obtained formulas for observables for periodic Macdonald processes **[Koshida '20]**
- similar methods for GFF convergence for random matrices and random tilings used in e.g. **[Borodin-Gorin '15]**, **[Ahn '20]**

shift-mixed q^{vol}**:** determinantal structure, Gaussian free field WITH an additional discrete Gaussian shift component

unshifted q^{vol}**:** NO determinantal structure, Gaussian free field

Holey hexagon

A domain topologically equivalent to the cylinder:



Height of hole depends on tiling. To choose random tiling either

- ★ allow hole height to vary
- ★ condition random tiling on fixed hole height

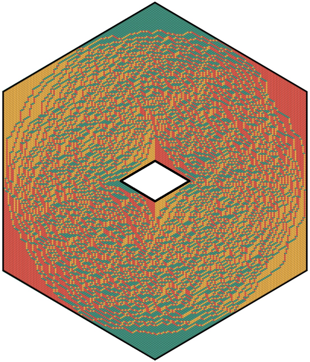
Analogy:

unrestricted tilings of cylinder	$\leftrightarrow$	tilings of holey hexagon
unshifted cylindric partitions	$\leftrightarrow$	tilings w/ fixed hole height.

Unshifted cylindric partitions ↔ tilings w/ fixed hole height

Theorem (Bufetov-Gorin '17)

The uniform measure on tilings of the holey hexagon conditioned on fixed hole height has Gaussian free field fluctuations in Kenyon-Okounkov complex structure.



(figure from V. Gorin, Lectures on random lozenge tilings, based on simulation by L. Petrov.)

Theorem (Ahn, R., Van Peski '21)

The fluctuations of the height function of a q^{vol}-distributed cylindric partition of width $2N$ converges on the liquid region to the Gaussian free field in the Kenyon-Okounkov complex structure.

Dirichlet energy

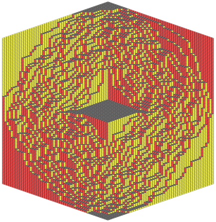
Conjecture

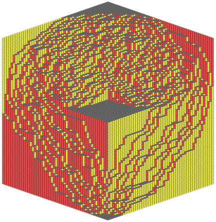
For a general planar domain with a hole, the limiting fluctuations of the hole height are discrete Gaussian $\mathcal{N}_{\text{discrete}}(C, m)$. Furthermore

$$C = \frac{\pi}{2} \int_{\zeta(\mathcal{L})} \|\nabla g\|^2 \, dx\, dy \qquad \textit{(Dirichlet energy)}$$

of unique harmonic function g which is 0 *on outer boundary,* 1 *on inner boundary.*

Rmk: To be proven for some domains in [Borot-Gorin-Guionnet, in prep.].





L. Petrov

Unrestricted tilings of cylinder $\leftrightarrow$ tilings of holey hexagon

For shift-mixed q^{vol} recall independent shift S has

$$\Pr(S = x) \propto u^x q^{Nx^2}.$$

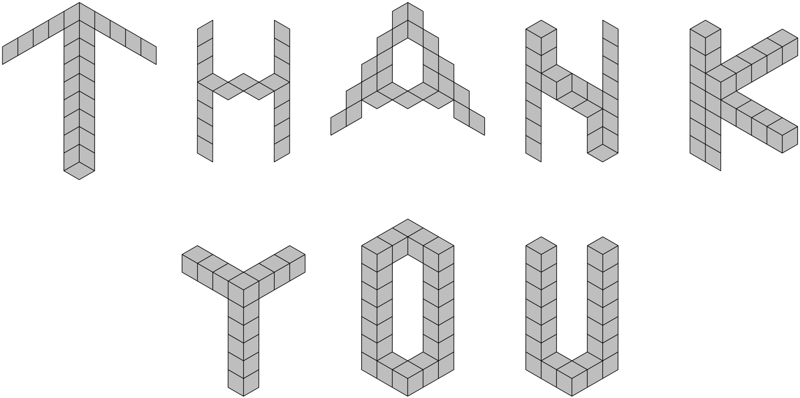
Equivalently (recall $t = q^N$)

$$S \sim \mathcal{N}_{\text{discrete}}\left(\frac{|\log t|}{2}, \frac{\log u}{\log t}\right)$$

and

$$C = \frac{|\log t|}{2}$$

is exactly the Dirichlet energy in previous conjecture for our case!


THANK
YOU