The Riemann–Hilbert Problem in Higher Genus and Some Applications

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Abstract The role of (bi/multi/matrix) orthogonal polynomials in random matrices, integrable systems and combinatorics is well known. Our goal is to report on recent progress in the definition of suitable extensions of the notion of orthogonality where the polynomials are replaced by sections of appropriate line bundles on Riemann surfaces. We discuss their definition in the spirit of various generalizations of the Padé problem and the formulation of appropriate matrix Riemann Hilbert problems that allow to characterize them as well as control their asymptotic behaviour. Applications to Matrix Orthogonal Polynomials and the KP hierarchy will also be discussed.

Introduction

OPs are famously intertwined with Toda lattice equations; the Moser map linearizes the Toda flow in the space of (formal) measures.

$$\mu_j(\boldsymbol{t}) := \int_{\mathbb{R}} x^j \mathrm{e}^{\sum t_\ell x^\ell} \,\mathrm{d}\mu(x);$$

$$D_n(\boldsymbol{t}) := \det \left[\mu_{a+b-2}(\boldsymbol{t}) \right]_{a,b=1}^{n} \mapsto \text{Toda tau function } \tau(n, \boldsymbol{t}) \text{ and KP.}$$

The OPs

$$\int_{\mathbb{R}} P_n(x) P_m(x) \,\mathrm{d}\mu(x; \boldsymbol{t}) = h_n(\boldsymbol{t}) \delta_{nm}$$

There are "biorthogonal" extension; famous one is the one for the two–matrix model:

$$\int_{\mathbb{R}^2} P_n(x) Q_m(y) \mathrm{e}^{\sum t_\ell x^\ell + \sum s_\ell y^\ell - xy} \,\mathrm{d}\mu(x) \,\mathrm{d}\nu(y) = h_n(\boldsymbol{t}, \boldsymbol{s}) \delta_{nm}.$$

OTOH: Algebro–geometric solutions of KP (Krichever).

Can we merge these two worlds? (B)OPs on RSs?

Very little literature: recent work of Fasondini-Olver-Xu (2020) arXiv:2011.10884.

Padé and OPs: an old story

Given a measure the **Weyl-(Stiltjes)** function (or generating function of moments):

$$W(z) := \int \frac{\mathrm{e}^{w(x)}\,\mathrm{d}x}{z-x} = \sum_{j\geqslant 0} \frac{\mu_j}{z^{j+1}}$$

The Padé approximation is a rational approximation scheme:

$$W(z) = \frac{Q_{n-1}(z)}{P_n(z)} + \mathcal{O}(z^{-2n+1}), \quad |z| \to \infty.$$

Fact:

The denominators are the orthogonal polynomials for the measure.

Two generalization directions

Generalizations: either via meromorphic functions or meromorphic half–differentials.

$$\int P_n(z)P_m(z)\mathrm{e}^{w(x)}\,\mathrm{d}x = \int \underbrace{P_n(z)\sqrt{\mathrm{d}x}}_{\varphi_n}\ \underbrace{P_m(z)\sqrt{\mathrm{d}x}}_{\varphi_m}\ \mathrm{e}^{w(x)} \tag{1}$$

Meromorphic functions with pole at a given point.

I am going to describe only the second setting here. The first one is necessary for application to MOPs: also generalizes nicely multi–point Padé approximations.

Padé on Riemann surfaces

We need the following data:

- A smooth R.S. $\mathcal{C}$ of genus g;
- a (generic) divisor $\mathscr{D}$ of degree g;
- a fixed chosen point $\infty \in \mathcal{C}$;
- a local coordinate $z : \mathbb{D}_\infty \to \mathbb{C}$ such that $\frac{1}{z(\infty)} = 0$.
- a curve $\gamma \subset \mathcal{C}$;
- a density (measure) $\mathrm{d}\mu$ on γ.

The (scalar) Cauchy kernel

$\mathbf{C}_\infty(p, q)$ is a differential in p and function in q such that:

1. as a differential w.r.t. p it has poles at q, ∞ and residues $+1, -1$; zeros at $p \in \mathscr{D}$;
2. as a function w.r.t. q it has poles at $p, \mathscr{D}$ and zero at ∞.

Such object exists and is unique.

Example (genus 1)

$$\mathbf{C}_\infty(z, w) = \left(\zeta(z - w) + \zeta\left(w - a\right) - \zeta(z) + \zeta(a)\right)\, \mathrm{d}z.$$

∞ is $z = 0$ and $\mathscr{D} = a$.

Definition (Weyl differential)

We define it by

$$\mathcal{W}(p) = \int_{q \in \gamma} \mathbf{C}_\infty(p, q)\, \mathrm{d}\mu(q)$$

The space of polynomials of degree n is now replaced by the line bundle $\mathscr{L}(n\infty + \mathscr{D})$ (of dimension $n+1$ like the space of polynomials by Riemann–Roch).

Problem (Padé approximation problem)

Find $P_n \in \mathscr{L}(\mathscr{D} + n\infty)$ and $\mathfrak{Q}_{n-1} \in \mathcal{K}((n+1)\infty)$

$$\left(\frac{\mathfrak{Q}_{n-1}}{P_n} - W\right) \geqslant 2\mathscr{D} + (2n-1)\infty.$$

Theorem ("Orthogonality")

$$\int_\gamma P_n(p) P_m(p)\, \mathrm{d}\mu(p) = h_n \delta_{nm}.$$

What survives?

We now use a local coordinate z and define a reference basis of sections;

$$\zeta_j(q) := \operatorname*{res}_{p=\infty} z(p)^j \mathbf{C}(p,q) = z^j + \mathcal{O}(z^{-1}).$$

1. **Pseudo–moments** $\mu_{j,k}$ (not Hankel!):

$$\mu_{j,k} = \oint_\gamma \zeta_j(p)\zeta_k(p)\, \mathrm{d}\mu(p) = -\operatorname*{res}_{q=\infty} \oint_{p\in\gamma} \zeta_j(q)\mathbf{C}(q,p)\zeta_k(p)\, \mathrm{d}\mu(p)$$

$$D_n := \det\left[\mu_{j,k}\right]_{j,k=0}^{n-1}$$

2. **Heine formula**

$$P_n(p) := \frac{1}{D_n}\int_{\gamma^n} \det\left[\zeta_{a-1}(p_b)\right]_{a,b=1}^{n+1} \det\left[\zeta_{a-1}(p_b)\right]_{a,b=1}^{n} \prod_{j=1}^{n} \mathrm{d}\mu(p_j), \qquad p_{n+1} = p.$$

3. **Riemann–Hilbert problem** (see next).

The departed

1. Three term recurrence relation;

Problem

Let Y_n be a 2×2 matrix with functions in the first column and differentials in the second column, meromorphic in $\mathcal{C} \backslash \gamma$

$$Y_n(p_+) = Y_n(p_-) \left[\begin{array}{cc} 1 & \mathrm{d}\mu(p) \\ 0 & 1 \end{array} \right], \qquad p \in \gamma.$$

In addition we require that the matrix is such that it has poles at $\mathscr{D}$ in the first column and zeros in the second column, and also the following growth condition at ∞:

$$Y_n(p) = \left[\begin{array}{cc} \mathcal{O}(\mathscr{D} + n\infty) & \mathcal{K}(-\mathscr{D} - (n-1)\infty) \\ \mathcal{O}(\mathscr{D} + (n-1)\infty) & \mathcal{K}(-\mathscr{D} - (n-2)\infty) \end{array} \right]. \tag{2}$$

$$Y_n(p) = \left(\mathbf{1} + \mathcal{O}(z(p)^{-1})\right) \left[\begin{array}{cc} z^n(p) & 0 \\ 0 & \frac{\mathrm{d}z(p)}{z^n(p)} \end{array} \right], \qquad p \to \infty. \tag{3}$$

$\det Y_n \in \mathcal{K}(2\infty)$; it has $2g$ zeros! How to prove uniqueness? Existence? Tyurin divisor....

Theorem

The solution of the RHP exists and is unique if and only if $D_n \neq 0$.

Note that it is different from genus 0; the solution if it exists is unique. Now it may exist and be not unique (if $D_n = 0$).

$$Y_n(p) = \begin{bmatrix} P_n(p) & \mathfrak{R}_n(p) \\ \widetilde{P}_{n-1}(p) & \widetilde{\mathfrak{R}}_{n-1}(p) \end{bmatrix}$$

$$\mathfrak{R}_n(p) := \int_\gamma \mathbf{C}(p,q) P_n(q)\, \mathrm{d}\mu(q) \qquad \widetilde{\mathfrak{R}}_{n-1}(p) := \int_\gamma \mathbf{C}(p,q) \widetilde{P}_{n-1}(q)\, \mathrm{d}\mu(q).$$

$$P_n(p) = \frac{1}{D_n} \det \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n,0} \\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n,1} \\ \vdots & & & \vdots \\ \zeta_0(p) & \zeta_1(p) & \cdots & \zeta_n(p) \end{bmatrix} \in \mathscr{L}(\mathscr{D} + n\infty)$$

$$\widetilde{P}_{n-1}(p) = \frac{1}{D_n} \det \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} \\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1} \\ \vdots & & & \vdots \\ \zeta_0(p) & \zeta_1(p) & \cdots & \zeta_{n-1}(p) \end{bmatrix} \in \mathscr{L}(\mathscr{D} + (n-1)\infty).$$

Guaranteed existence: Harnack-curves I

If $\mathcal{C}$ has antiholomorphic involution fixing γ and $\mathrm{d}\mu$ is a positive measure, then $D_n > 0$ (easy to show).

Genus 1. Elliptic curve $E_\tau = \mathbb{C}/2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$, In Weierstraß form the elliptic curve is

$$Y^2 = 4X^3 - g_2 X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$$

with $e_1 + e_2 + e_3 = 0$ and $e_1 < e_2 < e_3$.

Antiholomorphic involution $z \to \frac{\omega_1}{\overline{\omega_1}}\overline{z} = \overline{z}$. We choose $\infty = \{0\}$ and $\mathscr{D} = \{a\}$, with $a \in (0, 2\omega_1)$.

$$\mathscr{L}(\mathscr{D} + n\infty) = \mathbb{C}\left\{1, \zeta(z) - \zeta(z - a) - \zeta(a), \wp(z), \wp'(z), \ldots, \wp^{(n-2)}(z)\right\}.$$

Real–analytic: $\overline{f(z)} = f(\overline{z})$.

Theorem

The orthogonal sections π_n exist and have $n + 1$ zeros. These lie all on γ for $(n + 1)$ even, while for $(n + 1)$ odd one zero belongs to α.

Question

Interlacing?

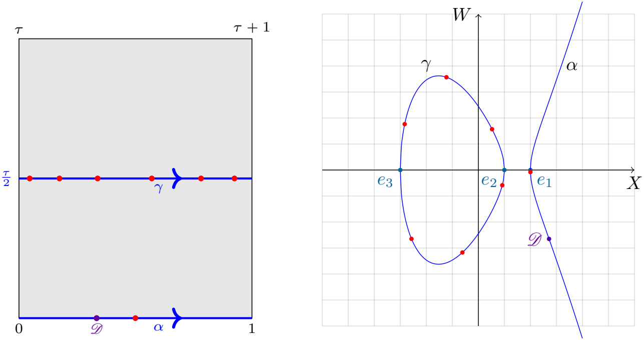


Figure: An example of real elliptic curve (specifically $W^2 = 4(X-1)(X-2)(X+3)$). On the left pane we have the "elliptic" parametrization as the quotient of $\mathbb{C}$ by the lattice Λ_τ. On the right the representation of the real section of $\mathcal{E}_\tau$ in the Weierstrass parametrization. The divisor $\mathscr{D}$ consists of a single point on the real oval of the α cycle (in this example $\mathscr{D} = 1/3$ in the elliptic parametrization), while the measure of orthogonality is defined on the cycle γ and it is given by an arbitrary smooth positive function $w(p)$ on γ times the holomorphic normalized differential $\mathrm{d}p = \frac{\mathrm{d}X}{2\omega_1 W}$. Also plotted are the zeros of the orthogonal section π_6 with respect to the "flat" measure with $w(p) \equiv 1$. Note that the zero on α is already (for $n=6$) extremely close to e_1: it is shown in Sec. **??** that this zero, for even n converges to e_1 exponentially fast.

Nonlinear Steepest descent analysis

In genus 1 no practical difference between functions/differentials.

Problem

Let $Y = Y_n(p)$ be the 2×2 matrix, meromorphic on $\mathcal{E}_\tau \setminus \gamma$ and with poles at $p = 0, \mathscr{D}$, such that

1. Near $p = 0 \equiv \Lambda_\tau$ we have the behaviour

$$Y(p) = (\mathbf{1} + \mathcal{O}(p)) \begin{bmatrix} p^{-n} & 0 \\ 0 & p^{n-2} \end{bmatrix}, \quad p \to 0 \mod \Lambda_\tau$$

2. Near $p = \mathscr{D} \mod \Lambda_\tau$ we have that

$$Y(p) = \begin{bmatrix} \mathcal{O}((p-\mathscr{D})^{-1}) & \mathcal{O}(p-\mathscr{D}) \\ \mathcal{O}((p-\mathscr{D})^{-1}) & \mathcal{O}(p-\mathscr{D}) \end{bmatrix}$$

3. The boundary values at $p \in \gamma$ are bounded and satisfy:

$$Y(p_+) = Y(p_-) \begin{bmatrix} 1 & \mathrm{e}^{w(p)} \\ 0 & 1 \end{bmatrix}$$

Note that $\det Y(p)$ has 2 **zeros**: usual argument for uniqueness fails. But the theorem earlier guarantees existence since (using Andreief) one sees $D_n > 0$.

A quick rundown of the DZ method and novelties

1. The g–function is found explicitly and along similar lines;
2. the steps of (i) normalization (using the g–function) of the singularity and (ii) opening lenses is also without major surprises.
3. The "model problem" (aka "outer parametrix") is found explicitly $M(p)$; alas, its determinant has also 2 zeros $\operatorname{div}\det M = (1/4) + (3/4)$. These zeros and the corresponding kernel spaces are the **Tyurin data**.
4. The issue is in the error analysis: to see consider the prototype

$$Y_+(z) = Y_-(z)J(z), \quad |z| = 1, \quad Y(\infty) = \mathbf{1}.$$

$$Y(z) = \mathbf{1} + \frac{1}{2i\pi} \oint_{|w|=1} Y_-(w)\big(J(w) - \mathbf{1}\big)\frac{\mathrm{d}w}{w-z}.$$

The latter expression needs a **matrix Cauchy kernel** that is defined given the Tyurin data: $\mathbf{C}_0(p,q)\,\mathrm{d}p$ is a matrix–valued differential with respect to the variable p and meromorphic function with respect to the variable q satisfying the following properties

1. It has a simple pole for $p = q$ and $p = 0$ and no other poles with respect to p;
2. The residue matrix for $p = q$ is $\mathbf{1}$ (and hence at $p = 0$ is $-\mathbf{1}$)
3. It has a simple pole for $q = p$ and at the Tyurin divisor $\mathscr{T} = (1/4) + (3/4)$ and all entries vanish for $q = 0$.
4. The expression $M^{-1}(p)\mathbf{C}_0(p,q)M(q)$ is locally analytic with respect to q and p at $\mathscr{T}$.

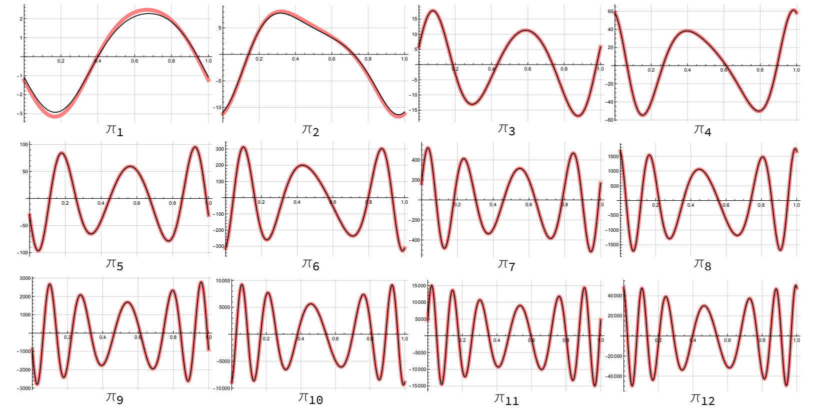


Figure: The first few monic orthogonal sections plotted as a function of $s \in [0, 1]$ via $p = \frac{\tau}{2} + s$; here $\pi_n(p) \in \mathscr{P}_n$ are the "monic" sections behaving like $\pi_n(p) = p^{-n}(1 + \mathcal{O}(p))$. The elliptic curve is $W^2 = 4X^3 - 19X + 15 = 4(X-1)(X-3/2)(X+5/2)$. Here $\tau \simeq 0.6563i$. We have set $\mathscr{D} = 1/3 \in \mathbb{R}$ and $\infty = 0$. The contour γ is the segment $[\tau/2, \tau/2 + 1]$ in $\mathcal{E}_\tau$; in the X–plane this is the segment $X \in [e_3, e_2]$ (on both sheets). The thick line is the plot of the orthogonal section obtained by computing explicitly the moments. The thin line is the approximation. Observe that the approximation is almost perfect starting from $n = 2$, confirming the exponential rate of convergence discussed in the text.

Asymptotic results I

1. For every compact subset of $\mathcal{E}_\tau \backslash \gamma$ we have

$$\pi_n(p) = \mathrm{e}^{-S_\infty} M_{11}(p) \mathrm{e}^{(n-1)g(p)+S(p)} \big(1 + \mathcal{O}(\mathrm{e}^{-nc_0})\big).$$

$$M_{11}(p) = \mathrm{e}^{-i\pi p} \frac{\theta_1(\mathscr{D}; 2\tau)\theta_1(p - \mathscr{D} - \tau; 2\tau)\theta_1'(0; 2\tau)\theta_{\{2,3\}}(p; 2\tau)}{\theta_1(\mathscr{D} + \tau; 2\tau)\theta_1(p - \mathscr{D}; 2\tau)\theta_1(p; 2\tau)\theta_{\{2,3\}}(0; 2\tau)}, \tag{4}$$

where the choice between θ_2, θ_3 is according to the parity of n. The function $S(p)$ is the "Szegö" function and:

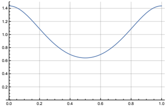
$$\mathrm{e}^{g(p)} = \mathrm{e}^{\ell} \begin{cases} \mathrm{e}^{i\pi(p - \frac{\tau}{2}) - \frac{i\pi}{2}} \dfrac{\theta_1(p; 2\tau)}{\theta_1(p - \tau; 2\tau)} & \Im \frac{\tau}{2} < \Im p < \Im \tau \\ \mathrm{e}^{-i\pi(p - \frac{\tau}{2}) + \frac{i\pi}{2}} \dfrac{\theta_1(p - \tau; 2\tau)}{\theta_1(p; 2\tau)} & 0 < \Im p < \frac{1}{2} \Im \tau. \end{cases} \qquad \mathrm{e}^{\ell} = -i \frac{\theta_1'(0; 2\tau)}{\theta_1(\tau; 2\tau)} \mathrm{e}^{-i\pi \frac{\tau}{2}} >$$

2. For $p \in \gamma$ we have the modulated oscillatory behaviour for $p = s + \frac{\tau}{2} + i0$:

$$\pi_n(p) = 2\mathrm{e}^{(n-1)\ell - S_\infty} \Re\left(M_{11}(p_+) \mathrm{e}^{S(p_+)} \left(\mathrm{e}^{i\pi s - \frac{i\pi}{2}} \frac{\theta_1\left(s + \frac{\tau}{2}; 2\tau\right)}{\theta_1\left(s - \frac{\tau}{2}; 2\tau\right)} \right)^{n-1} \right) \big(1 + \mathcal{O}(\mathrm{e}^{-nc_0})\big)$$

Asymptotic results II

3. For every continuous function ϕ defined on $\gamma \subset \mathcal{E}_\tau$



$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{2\lfloor \frac{n+1}{2} \rfloor} \phi(z_j^{(n)}) = \int_\gamma \phi(p)\sqrt{e_1 - \wp(p)}\,\frac{\mathrm{d}p}{2\pi}$$

4. The extra zero of π_n for n even tends at exponential rate to $p = \frac{1}{2}$ (i.e. $X = e_1$).
5. The square of the norms of the monic orthogonal sections have the asymptotics

$$\|\pi_n\|^2 = 2\pi \mathrm{e}^{2(n-1)\ell - 2S_\infty} \mathrm{e}^{-i\pi\tau} \frac{\mathrm{e}^{-2i\pi\mathscr{D}}\theta_1^2(\mathscr{D}; 2\tau)}{\theta_1^2(\mathscr{D}+\tau; 2\tau)} \frac{\theta_1'(0; 2\tau)}{\theta_4(0; 2\tau)} \left(\frac{\theta_3(0; 2\tau)}{\theta_2(0; 2\tau)}\right)^{\sharp n} (1 + \mathcal{O}(\mathrm{e}^{-nc_0}))$$

where $\sharp_n = 1$ for even n and -1 for odd n.

Matrix (Bi)Orthogonal Polynomials I

Matrix weight $W(z)$ on the real axis (or contour γ in $\mathbb{C}$) gives rise to matrix BOPs.

$$\int_\gamma P_n(z)W(z)P_m^\vee(z)\,\mathrm{d}z = \delta_{nm}\mathbf{H}_n,$$

Notable applications to the Aztec diamond (see Arno's talk).
Connection with scalar orthogonality on a Riemann surface already recognized by [Charlier '20] (implicitly in [Duits-Kuijlaars '17]).
It is sufficient that the eigenvectors of $W(z)$ live on an algebraic surface $\mathcal{C}$ (of genus g).

Example (arxiv:2107.12998)

$$Z : \mathcal{C} \to \mathbb{CP}^1, \quad \mathrm{div}\,(Z) \geqslant -r\infty$$

$$\int_\gamma \psi_n \psi_m^\vee \mathrm{e}^w, \quad \psi_n^{(\vee)} \in \sqrt{\mathcal{K}}((n+1)\infty) \otimes \mathcal{X}^{(\vee)}$$

$$\Psi_k^{(\vee)}(z) := \begin{bmatrix} \psi_{rk}^{(\vee)}(z^{(1)}) & \dots & \psi_{rk}^{(\vee)}(z^{(r)}) \\ \vdots & & \vdots \\ \psi_{rk+r-1}^{(\vee)}(z^{(1)}) & \dots & \psi_{rk+r-1}^{(\vee)}(z^{(r)}) \end{bmatrix}$$

Matrix (Bi)Orthogonal Polynomials II

Theorem

The matrices

$$P_n(z) := \Psi_k(z)\Psi_0^{-1}(z)\ , \qquad P_n^\vee(z) := (\Psi_0^\vee)^{-1}(z)\Psi_k^\vee(z)$$

are **polynomials** *and (bi)-orthogonal for the weight*

$$W(z) = W(z)\,\mathrm{d}z := \Psi_0(z)\Lambda(z)\Psi_0^\vee(z), \quad \Lambda(z) = \mathrm{diag}\Big(Y(z^{(1)}), \dots, Y(z^{(r)})\Big).$$

Example

It works also if $\mathcal{C}$ is the sphere! $Z(t) = (t-c)^2 : \mathbb{CP}^1 \to \mathbb{CP}^1$

$$W_L(z)\,\mathrm{d}z = \begin{bmatrix} 1 & \alpha+1-c-\sqrt{z} \\ \alpha+1-c-\sqrt{z} & (\alpha+1-c-\sqrt{z})^2 \end{bmatrix} (c+\sqrt{z})^\alpha \frac{\mathrm{e}^{-c-\sqrt{z}}}{2\sqrt{z}}\,\mathrm{d}z.$$

$$P_j(z) = \begin{bmatrix} \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j}^\alpha(s)(s+1+\alpha-2c)\,\mathrm{d}s}{z-Z(s)} & \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j}^\alpha(s)\,\mathrm{d}s}{Z(s)-z} \\ \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j+1}^\alpha(s)(s+1+\alpha-2c)\,\mathrm{d}s}{z-Z(s)} & \operatorname*{res}_{s=\infty} \frac{\mathrm{L}_{2j+1}^\alpha(s)\,\mathrm{d}s}{Z(s)-z} \end{bmatrix}.$$

$$\int_{c^2}^\infty P_j(z)W_L(z)P_k^t(z)\,\mathrm{d}z = \delta_{jk} \begin{bmatrix} \frac{\Gamma(2j+\alpha+1)}{(2j)!} & 0 \\ 0 & \frac{\Gamma(2j+\alpha+2)}{(2j+1)!} \end{bmatrix}.$$

Tensor $\mathscr{L}$ by a zero-degree bundle with transition function $\mathrm{e}^{\sum t_\ell z^\ell(p)}$ near ∞.
A section of $\mathscr{L}_{\boldsymbol{t}}(n\infty + \mathscr{D})$ satisfies:

$$(\psi_n) \geqslant -\mathscr{D}, \quad \psi_n(p) = z^n \mathrm{e}^{\sum t_\ell z^\ell(p)}(1 + \mathcal{O}(z^{-1})).$$

Note:

For $n = 0$ it is the Baker–Akhiezer function of Krichever.

Take

$$\psi \in \widehat{\mathscr{L}_{\boldsymbol{t}}} := \bigoplus_{n \geqslant 0} \mathscr{L}_{\boldsymbol{t}}(n\infty + \mathscr{D})$$

$$\phi \in \widehat{\mathscr{L}_{\boldsymbol{s}}} := \bigoplus_{n \geqslant 0} \mathscr{L}_{\boldsymbol{s}}(n\infty + \mathscr{D})$$

Pairing:

$$\langle \phi, \psi \rangle_{\boldsymbol{t},\boldsymbol{s}} = \int_\gamma \phi(p)\psi(p)\,\mathrm{d}\mu(p)$$

We can construct biorthogonal sections $\{\psi_n, \phi_n\}_{n \in \mathbb{N}}$ (if non-degenerate!)
A basis is

$$\zeta_j(p; \boldsymbol{t}) = z^j \mathrm{e}^{\sum t_\ell z^\ell}(1 + \mathcal{O}(z^{-1})) \text{ (similarly for } \boldsymbol{s})$$

Tau function

Definition (The Tau function)

The Tau function is defined by

$$\begin{aligned}\tau_n(\boldsymbol{t},\boldsymbol{s}) :=& \frac{1}{n!}\Theta(F(\boldsymbol{t}))\Theta(\mathbb{F}(\boldsymbol{s}))\mathrm{e}^{Q(\boldsymbol{t})+Q(\boldsymbol{s})+nA(\boldsymbol{t})+nA(\boldsymbol{s})}\times\\ &\times\int_{\gamma^n}\det\left[\zeta_{a-1}(r_b;\boldsymbol{t})\right]_{a,b=1}^n\det\left[\zeta_{a-1}(r_b;\boldsymbol{s})\right]_{a,b=1}^n\prod_{j=1}^n\mathrm{d}\mu(r_j) =\\ =&\tau_{Kr}(\boldsymbol{t})\tau_{Kr}(\boldsymbol{s})\mathrm{e}^{nA(\boldsymbol{t})+nA(\boldsymbol{s})}\det\left[\mu_{ab}(\boldsymbol{t},\boldsymbol{s})\right]_{a,b=0}^{n-1}\end{aligned}$$

The expression $Q(\boldsymbol{t})$ is a quadratic form and $A(\boldsymbol{t})$ is a linear form in the times.

Theorem

The tau function

1. *Is a KP tau function w.r.t. both sets of times (satisfies HBI):*

$$\operatorname*{res}_{x=\infty} \tau_n(\boldsymbol{t} - [x], \boldsymbol{s})\tau_n(\widetilde{\boldsymbol{t}} + [x], \boldsymbol{s})\mathrm{e}^{\xi(x;\boldsymbol{t})-\xi(x;\widetilde{\boldsymbol{t}})}\,\mathrm{d}z(x) \equiv 0$$

2. *It is a tau function for* $2-$*Toda Hierarchy (Adler-VanMoerbeke)*

$$\operatorname*{res}_{x=\infty} \tau_n(\boldsymbol{t} - [x]; \boldsymbol{s})\tau_{m+1}(\widetilde{\boldsymbol{t}} + [x]; \widetilde{\boldsymbol{s}})\frac{\mathrm{e}^{\xi(x;\boldsymbol{t})-\xi(x;\widetilde{\boldsymbol{t}})+A(\widetilde{\boldsymbol{t}}-\boldsymbol{t})}\,\mathrm{d}z(x)}{z(x)^{m-n+1}} =$$

$$= \operatorname*{res}_{x=\infty} \tau_{n+1}(\boldsymbol{t}; \boldsymbol{s} + [x])\tau_m(\widetilde{\boldsymbol{t}}; \widetilde{\boldsymbol{s}} - [x])\frac{\mathrm{e}^{\xi(x;\widetilde{\boldsymbol{s}})-\xi(x;\boldsymbol{s})+A(\boldsymbol{s}-\widetilde{\boldsymbol{s}})}\,\mathrm{d}z(x)}{z(x)^{n-m+1}}$$

3. *If* $P_n(p; \boldsymbol{t}, \boldsymbol{s}), Q_n(p; \boldsymbol{t}, \boldsymbol{s})$ *are the biorthogonal sections then the Baker and dual Baker functions are (up to prefactors)* $P_n(x; \boldsymbol{t}, \boldsymbol{s})$ *and*

$$\mathfrak{R}_n(x; \boldsymbol{t}, \boldsymbol{s}) := \int_{r\in\gamma} \mathbf{C}(x, r; \boldsymbol{t})Q_{n-1}(r; \boldsymbol{t}, \boldsymbol{s})\,\mathrm{d}\mu(r)$$

   respectively (note that dual BA is a differential).

4. $\tau_n(\boldsymbol{t}, \boldsymbol{s}) = 0$ *if and only if* $\tau_{Kr} = 0$ *or the pairing is degenerate on*

$$\mathscr{L}_{\boldsymbol{t}}(n\infty + \mathscr{D}) \otimes \mathscr{L}_{\boldsymbol{s}}(n\infty + \mathscr{D})$$

Outlook

1. Varying weights: this requires study of equilibrium problem on RS: we need appropriate Green functions.
2. One can study DRPF: the projection operator (in the Harnack case) gives a TP kernel defined on the curve.
3. New integrable systems? Connection with Hitchin systems (higher genus generalization of Calogero–Moser types).
4. Interface with algebraic geometry of vector bundles.