FOURTH ZiF SUMMER SCHOOL

RANDOMNESS IN PHYSICS AND MATHEMATICS

FROM INTEGRABLE PROBABILITY TO DISORDERED SYSTEMS

1 – 13 AUGUST 2022

CONVENORS
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Properties of the chGUE at the hard edge: Spacing distributions and universality with external field

Gernot Akemann

(MSRI & Bielefeld University)

joint work with V. Gorski, M. Kieburg & T.R. Würfel

[arXiv:2110.03617, arXiv:2111.xxxxx]

Outline

Disussion of local properties of the chGUE (= complex Wishart = Laguerre Unitary Ensemble) at the **hard edge**

- Part I: **spacing distribution** between kth and $(k+1)$st: **determinantal expression** $\sim \int^{k} \mathrm{det}_{k\times k}$ for finite-N & $\to \infty$ VERY close to GUE bulk spacing for $k=1$

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  determinantal expression $\sim \int^k \det_{k\times k}$ for finite-N & $\to \infty$

  VERY close to GUE bulk spacing for $k = 1$

- Part II: chGUE + **external field** A = polynomial ensemble:

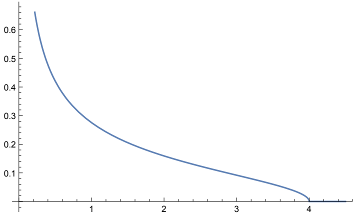
  correlation functions R_k **universal**, agree with $A = 0$

  $\exists$ determinantal expressions for R_k of **different sizes**, all equivalent

Part I - Spacing Distributions

Global density = Marchenko-Pastur law

- Gaußian random matrix W of size $N \times (N + \nu)$, density of eigenvalues $x_1, \ldots, x_N$ of WW^*



with $\nu = O(1)$

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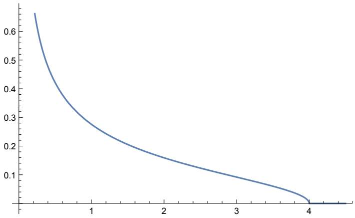

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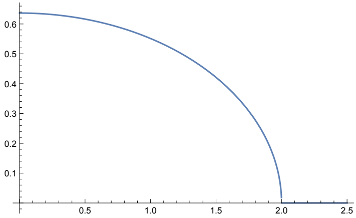
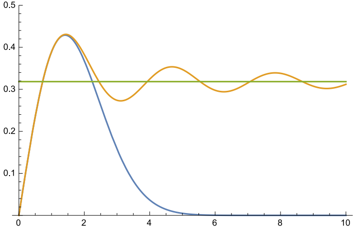


$x \to x^2$ quartercircle "unfolding"

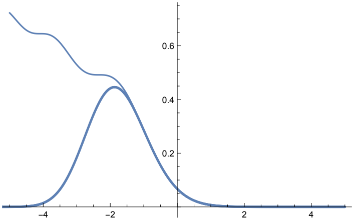
- here: zoom into hard edge = origin with scale $1/N$

Local correlations = Bessel, Sine, Airy

- distinguish hard edge, bulk and soft edge scaling limits:



hard edge = Bessel



soft edge = Airy

- **Bessel density** ($\nu = 0$):

$$R_1(x) = \frac{x}{2}(J_0^2(x) + J_1^2(x)) \sim \frac{1}{\pi}\left(1 - \frac{\cos(2x)}{2x} + \mathcal{O}(1/x^2)\right)$$

$P_1(x) = \frac{x}{2} e^{-x^2/4}$ smallest eigenvalue distribution

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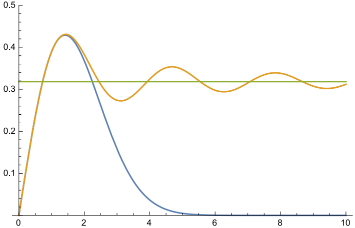

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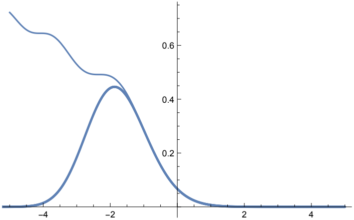

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- **What is the *k*th spacing distribution?** application QCD

- similarly Airy density and Tracy-Widom (+ unfold)

Joint density of WW^* eigenvalues^2 = singular values

$$\mathcal{P}_N^{N_f,\nu}(\{x\}) = \prod_{k<l}^N (x_l^2 - x_k^2)^2 \prod_{j=1}^N x_j^{2\nu+1} e^{-x_j^2}$$

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- motivated by **application to QCD** add N_f characteristic polynomials $D_N(-m^2) = \prod_{i=1}^{N}(x_i^2 + m^2)$

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- normalising **partition function**
  $Z_N^{N_f,\nu} = \int_0^\infty dx_1 \cdots dx_N \mathcal{P}_N^{N_f,\nu}(\{x\}) \sim \mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)\right]$
- explicitly known at finite and large-N:

$$Z_N^{N_f,\nu}(\{m\}) = C \Delta_{N_f}(\{m^2\})^{-1} \det\left[L_{N+j-1}^{(\nu)}(-m_i^2)\right]_{i,j=1}^{N}$$

$\exists$ formulae with kernels [Baik, Deift, Strahov '03; GA, Vernizzi '03]

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  $\exists$ formulae with kernels [Baik, Deift, Strahov ’03; GA, Vernizzi ’03]
- Why not spacing from Fredholm det?
  $\rightarrow$ orthogonal polynomials wrt N_f-dependent weight
  here use $Z_N^{N_f,\nu}$

Spacing $p_{k,N}(s)$ between kth and $(k+1)$st

- **kth gap probability:** $b > a$

$$E_k([a,b]) \sim \int_{(0,a)^k} dx_1 \ldots dx_k \int_{(b,\infty)^{N-k}} dx_{k+1} \ldots dx_N \mathcal{P}_N^{N_f,\nu}(\{x\})$$

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- idea:
  shift $\underbrace{\int_{(x_k+s,\infty)}}$ $\to \int_0^\infty$ $\Rightarrow$ integrate: $\overbrace{Z_{N-k-1}^{N_f,\nu}} \times$ rest
  and $Z_{N-k-1}^{N_f,\nu} = \det_{N_f+\nu+k}$ of fixed size known

Closed form results

k*th spacing distribution at finite-*N (GA, Gorski, Kieburg '21):
For $N_f, \nu = 0$ it holds

$$p_{k,n}^{(0,0)}(s) = \frac{2N}{(k-1)!}\int_0^\infty dx_k \int_0^{x_k} dx_1 \ldots dx_{k-1} \prod_{j=1}^{k}((x_k+s)^2 - x_j^2)^2 x_j e^{-x_j^2}$$

$$\times (x_k+s)e^{-(N-k)(x_k+s)^2} \det\left[K_{N-1}^{\mathrm{chGUE}}(x_i^2-(x_k+s)^2, x_j^2-(x_k+s)^2)\right]_{i,j=1}^{k}$$

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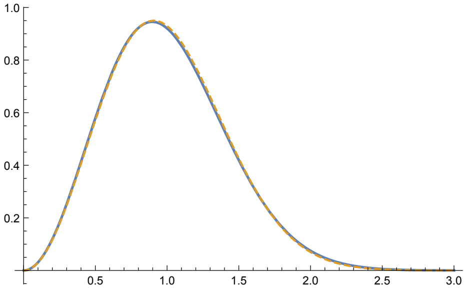
- corresponding results hold for $N_f, \nu \neq 0$ with larger det's
- hard edge limit straight forward
  $\lim_{N\to\infty} N^{-\nu} L_N^{(\nu)}\left(-\frac{z^2}{N}\right) = z^{-\nu} I_\nu(2z)$
- **example limiting spacing** $k = 1$ ($\sigma^2 = Ns^2$)

$$p_2^{(0,0)}(\sigma) = 4 \int_0^\infty dx x(x+\sigma) e^{-(x+\sigma)^2} (I_2(y)^2 - I_3(y) I_1(y))\Big|_{y=2\sqrt{\sigma(2x+\sigma)}}$$

Example spacing distribution $k = 1$

- Left: $k = 1$ spacing for $N = 2$ (blue) vs. $N \to \infty$ (yellow dashed)



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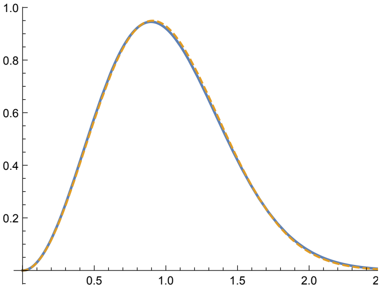
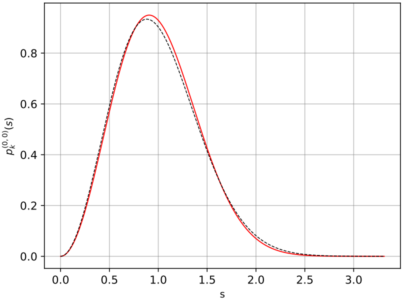
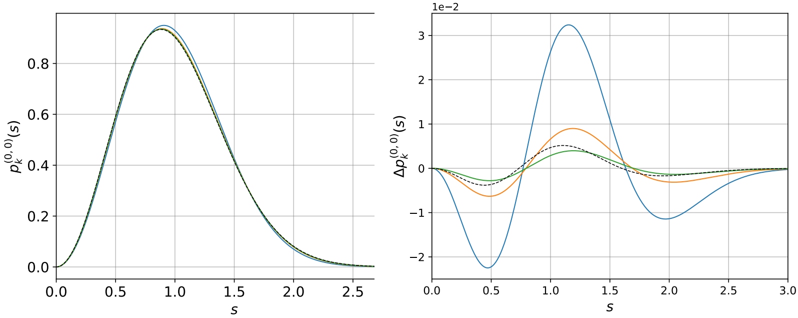




- Right: limiting $k = 1$ spacing (red) vs. limiting **GUE bulk spacing** (black dashed)

Convergence $k \nearrow$ to bulk spacing

- Left: spacings $k = 1$ (blue), $k = 2$ (orange), $k = 3$ (green) vs. GUE bulk (black dashed)



- Right: **difference** to GUE bulk spacing $k = 1$ (blue), $k = 2$ (orange), $k = 3$ (green)

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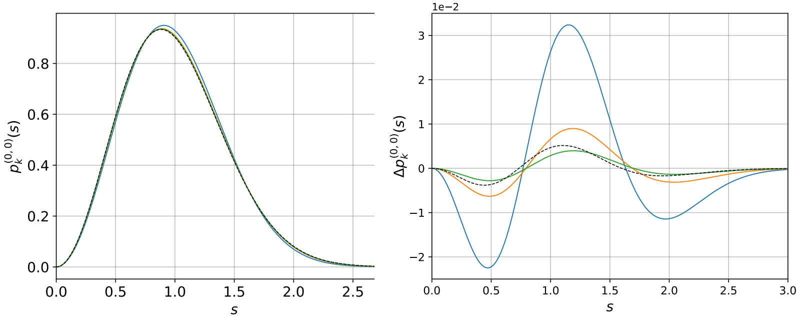
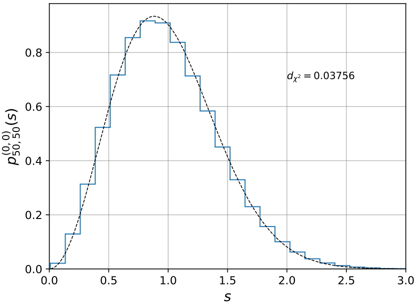


- Right: **difference** to GUE bulk spacing $k = 1$ (blue), $k = 2$ (orange), $k = 3$ (green) vs.
  GUE **Wigner surmise** $p_{\mathrm{GUE}}(s) \sim s^2 e^{-b_2 s^2}$ (black dashed)

Soft edge spacing

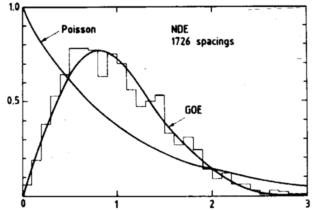
- numerically generated spacing for $N = 50$ with $k = 49$, 10^6 realisations

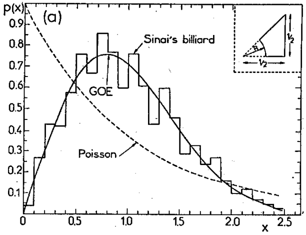


- **same phenomenon:** spacing between largest (Tracy-Widom) and second largest VERY close to bulk spacing

Comparison with real data - can we see this?

- GOE spacing distribution $p(s)$ [Bohigas et al. 83]: neutron scattering (left) and billiards (right)





- distinction between Wigner surmise $p_{\mathrm{GOE}}(s) \sim s e^{-b_1 s^2}$ and exact GOE bulk spacing impossible

- **hard (and soft) edge spacing as close as surmise**
  $\Rightarrow$ spacing is not a good measure to distinguish edges

- we have a closed form result at the hard edge

Part II - chGUE with external field

RMT for QCD with temperature $T \neq 0$

- chGUE $W \in \mathbb{C}^{N\times(N+\nu)}$ + **external field** $A \approx \pi T$

  [Jackson, Verbaarschot '96; Wettig, Schäfer, Weidenmüller '96]

  $$Z_N^{N_f,\nu} = \int d[W] \exp[-\mathrm{Tr}\, WW^\dagger] \prod_{f=1}^{N_f} \det[\mathcal{D}(A) + m_f \mathbb{1}_{2N+\nu}]$$

- "QCD" **Dirac operator** $\mathcal{D}(A) = \begin{pmatrix} 0 & W+A \\ W^\dagger + A^\dagger & 0 \end{pmatrix}$

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- use **group integral** [Jackson, Sener, Verbaarschot '96; Guhr, Wettig '96]

  $$\int d[U] \int d[V] e^{\mathrm{Tr}\,(UXV^\dagger A^\dagger + VX^\dagger U^\dagger A)} \sim \frac{\det[I_\nu(2\sqrt{a_i x_j})]_{i,j=1}^N}{\prod_{j=1}^N (a_j x_j)^{\frac{\nu}{2}} \Delta_N(\{a\})\Delta_N(\{x\})}$$

  singular values $(a_1, \ldots, a_N)$ of A; $(x_1, \ldots, x_N)$ of $W = UXV^\dagger$

chGUE + A = polynomial ensemble

- $\mathcal{P}_N \sim \Delta_N(\{x\}) \det[\varphi(a_i, x_j)]_{i,j=1}^N$ [Kuijlaars '16]
- subset of bi-orthogonal ensembles [Borodin '98]

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- **determinantal point process**

  $R_{k,A}^{(N_f)}(x_1, \ldots, x_k) = \det\left[K_{N,A}^{(N_f)}(x_i, x_j)\right]_{i,j=1}^k$

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- or **ratio of characteristic polynomials** [Desrosiers, Forrester '08]

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- has application in **overlaps of eigenvectors** [Fyodorov, Grela, Strahov '18]

Kernel for invertible polynomial ensembles

- Definition **invertible polynomial ensemble:**

  $\boxed{\mathcal{P}_N \sim \Delta_N(\{x\}) \det[\varphi(a_i, x_j)]_{i,j=1}^N}$ and $F(s,z)$ such that

  $\pi_k(a) = \int_I dx x^k \varphi(a,x)$ monic polynomials

  $z^k = \int_{I'} ds F(s,z) \pi_k(s)$

- in our case: $\pi_k = L_k^\nu$ Laguerre, $F(s,z) = \varphi(-s,-z)$ and $I = -I' = \mathbb{R}_+$ (for $N_f = 0$)

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- in our case: $\pi_k = L_k^\nu$ Laguerre, $F(s,z) = \varphi(-s,-z)$ and $I = -I' = \mathbb{R}_+$ (for $N_f = 0$)

- **Proposition** (GA, Strahov, Würfel '20)

  $$\boxed{K_N(x,y) = \frac{1}{2\pi i} \int_{I'} ds F(s,x) \prod_{n=1}^N (s - a_n) \oint_C du \frac{\varphi(u,y)}{(s-u)\prod_{n=1}^N (u-a_n)}}$$

  where C encircles $a_1, \ldots, a_N$ counter clockwise

Hard edge limit: Saddle point analysis

- for simplicity $N_f = 0$, hard edge scaling of

$$\boxed{\frac{K^{(0)}_{N,A}(\frac{\rho}{N},\frac{\eta}{N})}{N} = \int_0^\infty dt t^{\nu/2} J_\nu\left(2\sqrt{\rho t}\right) e^{-N\mathcal{L}_2(t)} \oint_C \frac{du}{2\pi i} \frac{u^{-\nu/2} I_\nu(2\sqrt{\eta u})}{-(s+u)} e^{N\mathcal{L}_2(-u)}}$$

- **saddle point** $e^{-Nt}\prod_{n=1}^N (t+a_n) = e^{-N\mathcal{L}_2(t)}$

$$\boxed{\mathcal{L}_2'(t) = 1 - \frac{1}{N}\sum_{n=1}^N \frac{1}{a_n+t} \stackrel{!}{=} 0}$$

- for critical $t_c \equiv \frac{1}{N}\sum_{n=1}^N \frac{1}{a_n} \in (0,1)$ (else no hard edge!):
  $\exists!$ saddle point $t = \Xi(A)$ on integration domain

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- final answer = **universal Bessel kernel** $\mathcal{K}^{\mathrm{chGUE}}(\zeta,\eta)$

$$\lim_{N\to\infty} \frac{K^{(0)}_{N,A}\left(\frac{\zeta^2}{4N\Xi(A)},\frac{\eta^2}{4N\Xi(A)}\right)}{2N\Xi(A)} = \frac{\zeta J_{\nu+1}(\zeta)J_\nu(\eta)-\eta J_{\nu+1}(\eta)J_\nu(\zeta)}{(\zeta^2-\eta^2)}$$

using SUSY [Guhr, Wettig '97, Jackson, Sener, Verbaarschot '97]

- generalisation to $N_f, \nu \neq 0$ [GA, Würfel '21]

$N_f \neq 0$ flavours - previous results for $A = 0$

- **Christoffel Theorem:** orthogonal polynomial $P_k^{(0)}(x)$ for $w(x) \Rightarrow P_k^{(1)}(x)$ for $w^{(1)}(x) = (x + m^2)w(x)$

$$\boxed{P_k^{(1)}(x; m^2) = \frac{P_k^{(0)}(x)P_{k+1}^{(0)}(-m^2) - P_k^{(0)}(x)P_{k+1}^{(0)}(-m^2)}{x+m^2}} \qquad 2 \times 2 \text{ det}$$

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$$K_N^{(1)}(x, y) = \frac{P_{N-1}^{(1)}(x)P_N^{(1)}(y) - P_{N-1}^{(1)}(x)P_N^{(1)}(y)}{x-y} \sim P_{N-1}^{(2)}(x; -y, m^2)$$

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- iterate $\Rightarrow$ kernel $\boxed{\text{size } N_f+2}$, **hard edge limit:**

$$\mathcal{K}_{A=0}^{(N_f)}(\zeta,\eta) = \frac{\begin{vmatrix} J_\nu(\zeta) & \zeta J_{\nu+1}(\zeta) & \dots & \zeta^{N_f+1}J_{\nu+N_f+1}(\zeta) \\ J_\nu(\eta) & \eta J_{\nu+1}(\eta) & \dots & \eta^{N_f+1}J_{\nu+N_f+1}(\eta) \\ I_\nu(\mu_1) & -\mu_1 I_{\nu+1}(\mu_1) & \dots & (-\mu_1)^{N_f+1}I_{\nu+N_f+1}(\mu_1) \\ \vdots & \vdots & \dots & \vdots \\ I_\nu(\mu_{N_f}) & -\mu_{N_f} I_{\nu+1}(\mu_{N_f}) & \dots & (-\mu_{N_f})^{N_f+1}I_{\nu+N_f+1}(\mu_{N_f}) \end{vmatrix}}{\sqrt{|\zeta\eta|}^{-1}(\eta^2-\zeta^2)\prod_{f=1}^{N_f}\sqrt{(\zeta^2+\mu_f^2)(\eta^2+\mu_f^2)}\det_{1\le f,g\le N_f}\left[(-\mu_f)^{g-1}I_{\nu+g-1}(\mu_f)\right]}$$

[Damgaard, Nishigaki '98; Wilke, Guhr, Wettig '98] at $\nu = 0$

$N_f \neq 0$ flavours - results for $A \neq 0$

- **polynomial ensemble**: kernel A-independent in units $\Xi(A)$

$$\mathcal{K}_A^{(N_f)}(\zeta,\eta) = \lim_{N\to\infty} \frac{\sqrt{|\zeta\eta|}}{2N\Xi(A)} K_N^{(N_f)}\left(\frac{\zeta^2}{4N\Xi(A)}, \frac{\eta^2}{4N\Xi(A)}\right)$$

$$= \frac{\begin{vmatrix} \mathcal{K}^{\mathrm{chGUE}}(\zeta,\eta) & \hat{\mathcal{K}}^{\mathrm{chGUE}}(\mu_1,\eta) & \dots & \hat{\mathcal{K}}^{\mathrm{chGUE}}(\mu_{N_f},\eta) \\ J_\nu(\zeta) & I_\nu(\mu_1) & \dots & I_\nu(\mu_{N_f}) \\ \zeta J_{\nu+1}(\zeta) & -\mu_1 I_{\nu+1}(\mu_1) & \dots & -\mu_{N_f} I_{\nu+1}(\mu_{N_f}) \\ \vdots & \vdots & \dots & \vdots \\ \zeta^{N_f-1} J_{\nu+N_f-1}(\zeta) & (-\mu_1)^{N_f-1} I_{\nu+N_f-1}(\mu_1) & \dots & \end{vmatrix}}{\sqrt{|\zeta\eta|}^{-1} \det\left[(-\mu_f)^{j-1} I_{\nu+j-1}(\mu_f)\right]_{j,f=1}^{N_f}}$$

kernel $\boxed{\text{size } N_f + 1}$ [GA, Würfel '21]

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- from SUSY $R_k(\zeta_1,\ldots,\zeta_k) = \det_{N_f+k} \neq \det[\text{kernel}]$ [Seif, Guhr, Wettig ’99], numerical agreement to $A = 0$
- **Do these 3 forms all agree?**

Identity relating det's

- **Product of characteristic polynomials** (GA,Vernizzi '03)
  For ensemble with OP p_k, norms h_k, kernel K_N with $K \geq L$:

$$\mathbb{E}\left[\prod_{k=1}^{K} D_N(v_k) \prod_{l=1}^{L} D_N(u_l)\right] = \frac{\begin{vmatrix} K_{N+L}(v_1, u_1) & \dots & K_{N+L}(v_K, u_1) \\ \vdots & \dots & \vdots \\ K_{N+L}(v_1, u_L) & \dots & K_{N+L}(v_K, u_L) \\ p_{N+L}(v_1) & \dots & p_{N+L}(v_K) \\ \vdots & \dots & \vdots \\ p_{N+K-1}(v_1) & \dots & p_{N+K-1}(v_K) \end{vmatrix}}{\prod_{j=N}^{N+L-1} h_j^{-1} \Delta_K(\{v\}) \Delta_L(\{u\})}$$

- RHS of size $K \times K$, many possibilities to split $K + L$

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- RHS of size $K \times K$, many possibilities to split $K + L$

- choice $\boxed{K = N_f + 2,\ L = 0}$ vs. $\boxed{K = N_f + 1,\ L = 1}$ proves
  equivalence of kernels at $A = 0$ and $A \neq 0$ when $N \to \infty$
  $K(x,y) \to \frac{g(x)}{g(y)} K(x,y)$

Identity relating det's of det's

- **Consistency conditions** (GA, Damgaard '98) Among limiting partition functions it holds [Braden, Mironov, Morozov '01]

$$\Delta_k(\{\xi\})\Delta_k(\{\eta\})\frac{\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\},\xi_1,\ldots,\xi_k,\eta_1,\ldots,\eta_k)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})} = \det\left[\frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\},\xi_a,\eta_b)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\right]_{a,b=1}^{k}$$

- choice of $\xi_a = \eta_a \; \forall a = 1, \ldots, k$ and express

  SUSY $R_k(\xi_1, \ldots, \xi_k)$ in terms of $\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\}, \{i\xi\})$

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  SUSY $R_k(\xi_1, \ldots, \xi_k)$ in terms of $\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\}, \{i\xi\})$

- identify

$$K_N^{(N_f)}(x,y) \sim \frac{\mathbb{E}\left[D_{N-1}(x)D_{N-1}(y)\prod_{f=1}^{N_f} D_{N-1}(-m_f^2)\right]}{\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)\right]} \sim \frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\},ix,iy)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}$$

$\Rightarrow$ **equivalence** for SUSY to $R_k = \det[\text{kernel}]$ at $A = 0, \neq 0$

Conclusions

- Novel features of the chGUE at the hard edge:
- Part I:
  - spacing distribution at hard edge in closed form $N \leq \infty$
  $\rightarrow$ almost identical to bulk spacing

  - ditto at soft edge (numerics)

  - spacing NOT useful to detect vicinity of an edge
- Part II:
  - chGUE $+N_f$ universal for $A \neq 0$

  - point process enjoys different representations for kernel and correlation functions

  - next step: tune to criticality via A: hard meets soft edge
  $\rightarrow$ expect different spacing