Stationary half-space last passage percolation

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joint work with D. Betea and P. Ferrari

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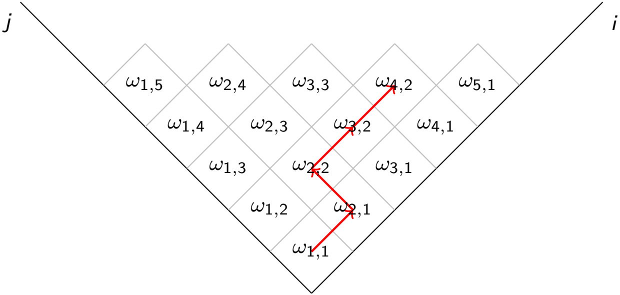
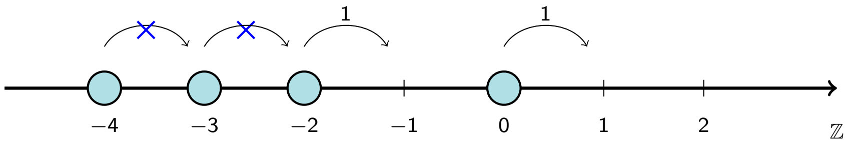
Last passage percolation (LPP)

- O, E point in $\mathbb{Z}^2$
- $\omega_{i,j} \sim$Exp(1), i.i.d. r.v.'s, $i, j \in \mathbb{Z}$
- Directed path π composed of $\nwarrow$ and $\nearrow$ s.t. $\pi(0) = O$ and $\pi(n) = E$
- Last passage time: $L_{O \to E} = \max_{\pi: O \to E} \sum_{1 \leq k \leq n} \omega_{\pi(k)}$

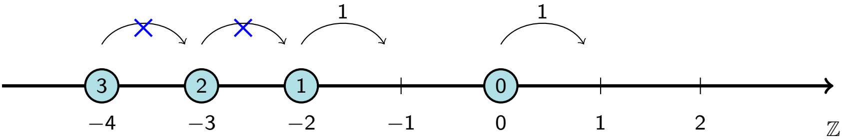


LPP and TASEP



- There is at most one particle per site
- Particles jump independently on the right with rate 1, provided the site is empty
- The dynamics preserves the order of particles

LPP and TASEP



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- Particles jump independently on the right with rate 1, provided the site is empty
- The dynamics preserves the order of particles

Right-to-left ordering of particles

$$\dots < x_3(0) < x_2(0) < x_1(0) < 0 \leq x_0(0) < x_{-1}(0) < \dots$$

$$x_k(t) = \text{position of particle } k \text{ at time } t$$

$$\mathbb{P}(L_{m,n} \leq t) = \mathbb{P}(x_n(t) \geq m - n)$$

Stationary TASEP

- At time $t = 0$ occupation variables $\eta_x^{stat}(0)$ are i.i.d. Bernoulli(ρ) random variables

- Equivalently,

$$x_0^{stat}(0) \sim Geom(1-\rho), \quad x_k^{stat}(0) - x_{k+1}^{stat}(0) - 1 \sim Geom(1-\rho)$$

- Fluctuations of the particle in the origin are asimptotically described by the Baik–Rains distribution (here $\rho = 1/2$)

$$\lim_{t\to\infty} \mathbb{P}(x^{stat}_{t/4-w(t/2)^{2/3}}(t) \geq 2w(t/2)^{2/3} - s(t/2)^{1/3}) = F_{BR,w}(s)$$

where $F_{BR,w}(s) = \frac{d}{ds}[F_{GUE}(s+w^2)g(s,w)]$

One-point: Baik–Rains '00 (PNG with external sources)
Ferrari-Spohn '06 (TASEP with Bernoulli IC)

Multi-point: Baik–Ferrari–Péché '10 (Airy$_{stat}$ process)

Half-space last passage percolation

- LPP in the half-quadrant of $\mathbb{Z}^2$

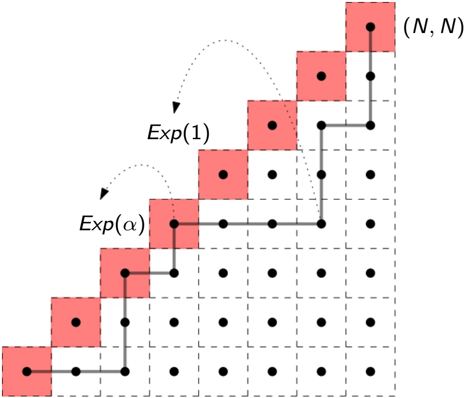
$$\omega_{i,j} \sim \begin{cases} Exp(1), & i \geq j+1 \\ Exp(\alpha), & i = j \end{cases}$$

- Equivalent to LPP on the full quadrant with weights symmetric w.r.t. the diagonal $\omega_{i,j} = \omega_{j,i}$

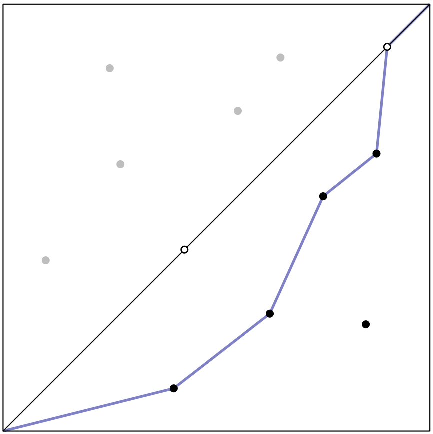
Hammersley LPP in half-space
Baik–Rains '01
Sasamoto–Imamura '04

Symmetrized LPP with geometric weights
Baik–Rains '01

and exponential weights
Baik–Barraquand–Corwin–Suidan '18



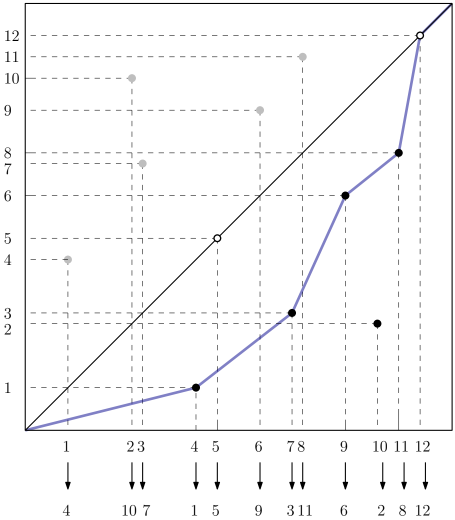
Ulam's problem for random involutions



PPP(θ^2) in $[0,1]^2$ with points reflected by the diagonal, PPP($\alpha\theta$) on the diagonal

$L =$ longest up-right path from $(0,0)$ to $(1,1)$

Ulam's problem for random involutions



L is the length of the longest increasing subsequence in the corresponding random involution

TASEP on the half-line

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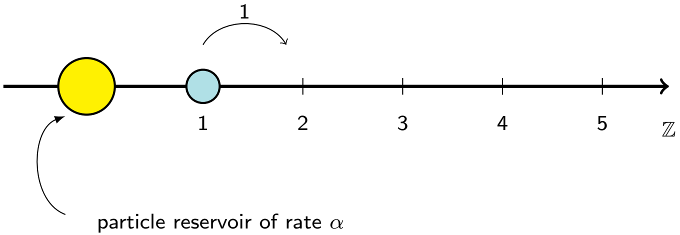
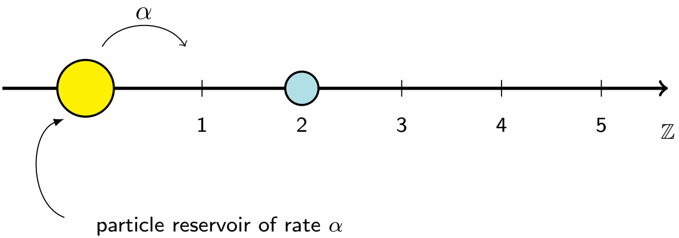


TASEP on the half-line



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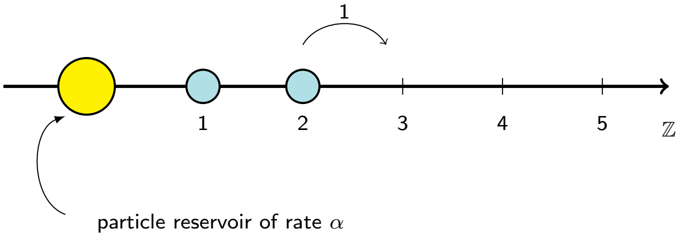
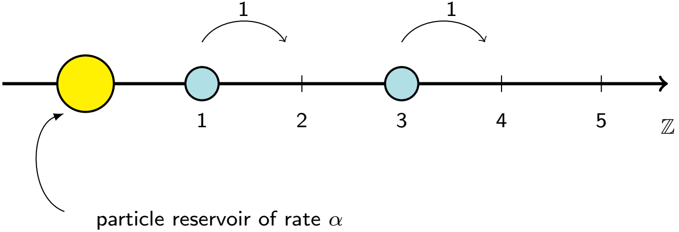
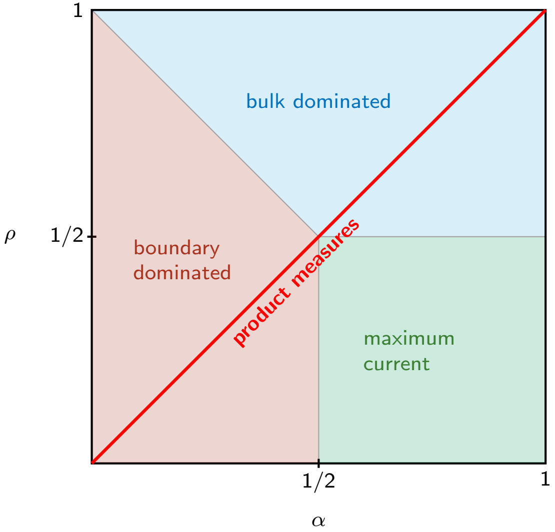


TASEP on the half-line



Stationary measures



Liggett '75
Grosskinksy '04

Stationary half-space LPP

We consider the half-space LPP from the origin to $(N, N-n)$ with the following weights

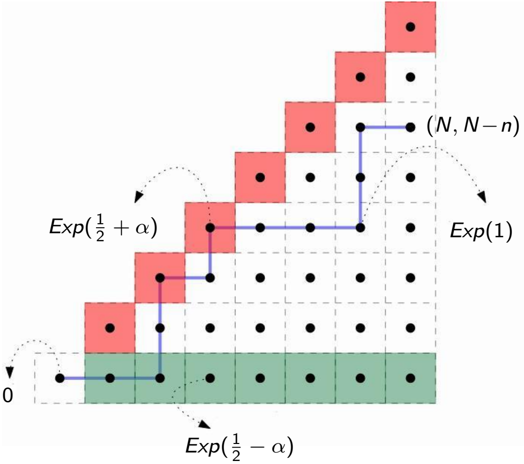
$$\omega_{i,j} \sim \begin{cases} Exp(\frac{1}{2}+\alpha) & i=j>1 \\ Exp(\frac{1}{2}-\alpha) & j=1, i>1 \\ 0 & i=j=1 \\ Exp(1) & \text{otherwise} \end{cases}$$

$\alpha \in (-1/2, 1/2)$

$L_{N,N-n}$ is stationary in the sense of Balász–Cator–Seppäläinen '16:

$$L_{m,n} - L_{m,n-1} \sim Exp(\tfrac{1}{2}+\alpha)$$
$$L_{m,n} - L_{m-1,n} \sim Exp(\tfrac{1}{2}-\alpha)$$

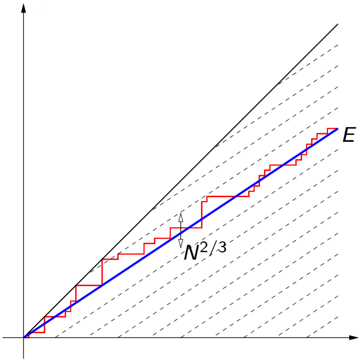
Stationary full-space LPP: Baik–Rains '00



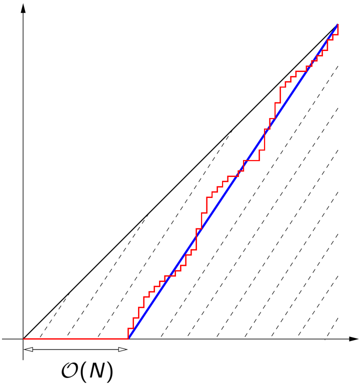
Scaling

Characteristics are lines with slope $((\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha))^2$

Case $\alpha < 0$



Case $\alpha > 0$



- $E = (N, N) \Rightarrow$ Gaussian fluctuations
- $E = N(1, ((\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha))^2) \Rightarrow$ path visits diagonal in a region $\mathcal{O}(N^{2/3})$ around the origin

- $E = (N, N) \Rightarrow$ path visits horizontal line in a region $\mathcal{O}(N) \Rightarrow$ Gaussian fluctuations

pictures by P. Ferrari

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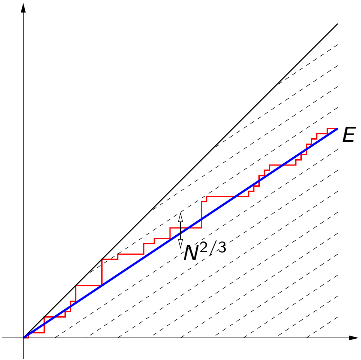
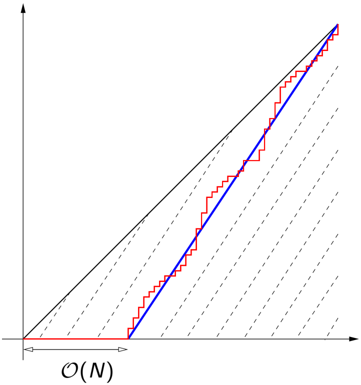
Characteristics are lines with slope $((\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha))^2$

Case $\alpha < 0$



Case $\alpha > 0$



- Critical scaling: $\alpha = 2^{-4/3}\delta N^{-1/3}$
- End point: $(N, N-n)$ with $n = u2^{5/3}N^{2/3}$
- Law of large numbers:

$$L_{N,N-n} \simeq 4N - 2u2^{5/3}N^{2/3} + \delta(2u+\delta)2^{4/3}N^{1/3}$$

pictures by P. Ferrari

Limit distribution

Theorem (Betea–Ferrari–O. '19)
Let $\delta \in \mathbb{R}$, $u > 0$. Let

$$\alpha = 2^{-4/3}\delta N^{-1/3}, \quad n = u 2^{5/3} N^{2/3}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,N-n} - 4N + 4u(2N)^{2/3}}{2^{4/3}N^{1/3}} \le S \right) = F_{0,\, half}^{(\delta,u)}(S)$$

where

$$F_{0,\, half}^{(\delta,u)}(S) = \partial_S \left\{ \mathrm{pf}(J - \overline{\mathcal{A}}) G_{\delta,u}(S) \right\}$$

with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *and*

$$G_{\delta,u}(S) = e^{\delta,u}(S) - \left\langle -g_1^{\delta,u} \quad g_2^{\delta,u} \middle| (\mathbb{1} - J^{-1}\overline{\mathcal{A}})^{-1} \begin{pmatrix} -h_1^{\delta,u} \\ h_2^{\delta,u} \end{pmatrix} \right\rangle$$

- $\overline{\mathcal{A}} = \lim_{N\to\infty} \overline{K}$ is the limit kernel of Sasamoto–Imamura '04 and Baik–Barraquand–Corwin–Suidan '18 interpolating between the GOE, GSE, GUE and Gaussian distributions
- Analogue result for the half-line stationary KPZ equation by Barraquand–Krajenbrink–Le Doussal '21

1. A Pfaffian model

Consider the half-space LPP $\tilde{L}_{N,N-n}$ with weights

$$\tilde{\omega}_{i,j} \sim \begin{cases} Exp(\frac{1}{2}+\alpha) & i=j>1 \\ Exp(\frac{1}{2}+\beta) & j=1, i>1 \\ Exp(\alpha+\beta) & i=j=1 \\ Exp(1) & \text{otherwise} \end{cases}$$

where $\alpha \in (-1/2, 1/2)$, $\beta \in (0, 1/2)$ and $\alpha+\beta>0$

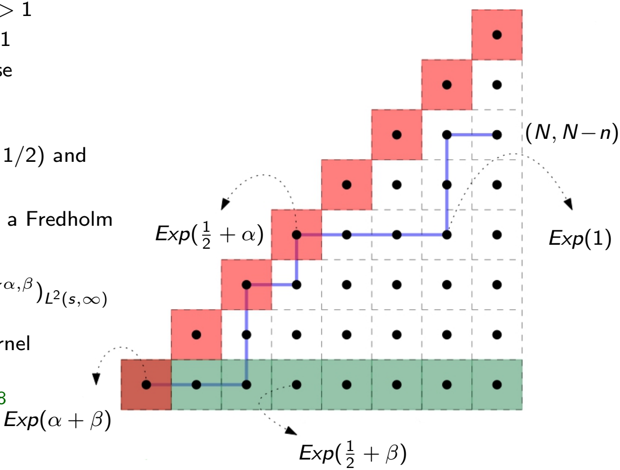
$\Rightarrow$ the distribution of $\tilde{L}_{N,N-n}$ is a Fredholm pfaffian

$$\mathbb{P}(\tilde{L}_{N,N-n} \leq s) = \mathrm{pf}(J - K^{\alpha,\beta})_{L^2(s,\infty)}$$

where $K^{\alpha,\beta}$ is a 2×2 matrix kernel

Rains '00

Baik–Barraquand–Corwin–Suidan '18



Geometric LPP

Consider the half-space LPP $L^{geo}_{N,N}$ with weights

$$w_{i,j} \sim \begin{cases} Geom(ax_i) & i = j \\ Geom(x_i x_j) & i > j \end{cases}$$

where $0 < x_1, \dots, x_N < 1$ and $0 \le a < \min_i \frac{1}{x_i}$

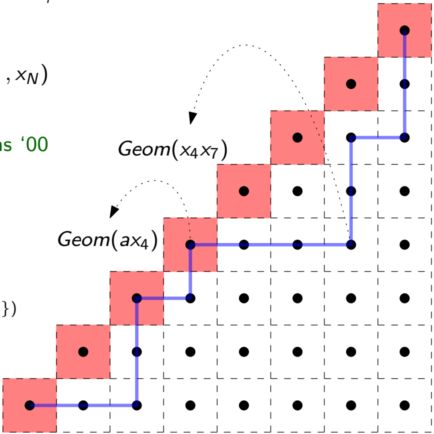
$$\mathbb{P}(L^{geo}_{N,N} \le l) = Z^{-1} \sum_{\lambda:\lambda_1 \le l} a^{oc(\lambda)} s_\lambda(x_1, \dots, x_N)$$

Knuth '70, Greene '74, Rains '00

$oc(\lambda)$ = number of odd colums of λ

$$\mathbb{P}(L^{geo}_{N,N} \le l) = \mathrm{pf}(J - K)_{\ell^2(\{l+1, l+2, \dots\})}$$

Rains '00



From geometric to exponential

We consider the case

$$x_1 = b \in (0,1)$$

$$x_2 = \cdots = x_N = \sqrt{q} \text{ for } q \in (0,1)$$

and $a > 0$ such that $a\sqrt{q} < 1$ and $ab < 1$

We take the limit for $\epsilon \to 0$ of

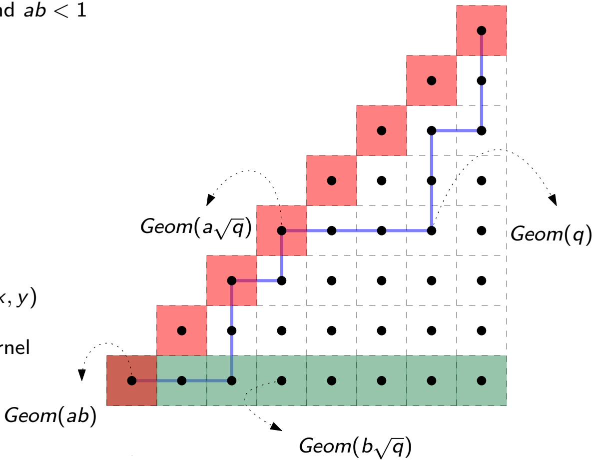
$q = 1 - \epsilon$, $(k, \ell) = \epsilon^{-1}(x, y)$

$(a, b) = (1 - \epsilon\alpha, 1 - \epsilon\beta)$

$$\Downarrow$$

$$\lim_{\epsilon \to 0} \epsilon^{\mu_{ij}} K_{ij}^{a,b}(k,\ell) = K_{ij}^{\alpha,\beta}(x,y)$$

where $K^{a,b}$ is a 2×2 matrix kernel



2. Shift argument

- $\tilde{L}_{N,N-n}$ last passage percolation for weights $\tilde{\omega}_{i,j}$ (integrable case)

  $L_{N,N-n}$ last passage percolation for weights $\omega_{i,j}$ (stationary case)

- Let $L^0_{N,N-n} = \tilde{L}_{N,N-n} - \tilde{\omega}_{1,1}$. For $\alpha + \beta > 0$,

$$\mathbb{P}(L^0_{N,N-n} \leq s) = \left(\mathbb{1} + \frac{1}{\alpha+\beta}\partial_s\right)\mathbb{P}(\tilde{L}_{N,N-n} \leq s)$$

- GOAL: obtain $L_{N,N-n} = \lim_{\alpha+\beta\to 0} L^0_{N,N-n}$

3. Kernel decomposition

To isolate the vanishing contribution ($\alpha + \beta \to 0$) we split the kernel K of $\tilde{L}_{N,N-n}$ as

$$K = \overline{K} + (\alpha + \beta) R$$

where

$$R = \begin{pmatrix} |g_1\rangle\langle f_\beta| - |f_\beta\rangle\langle g_1| & |f_\beta\rangle\langle g_2| \\ -|g_2\rangle\langle f_\beta| & 0 \end{pmatrix}$$

with $f_\beta(x) \sim e^{-\beta x}$

$$\Downarrow$$

$$\mathbb{P}(L_{N,N-n} \leq s) = \lim_{\alpha+\beta\to 0} \partial_S \left\{ \mathrm{pf}(J - \overline{K}) \left(\frac{1}{\alpha+\beta} - \langle Y, (\mathbb{1} - \overline{G})^{-1} X \rangle \right) \right\}$$

with $X = \left| \begin{matrix} 0 \\ f_\beta \end{matrix} \right\rangle$ and $Y = \langle -g_1 \quad g_2 |$ and $\overline{G} = J^{-1}\overline{K}$

4. Analytic continuation

- **Problem**: $f_\beta(x) \sim e^{-\beta x}$ is diverging for $\beta < 0$

$$\frac{1}{\alpha+\beta} - \langle Y, (\mathbb{1} - \overline{G})^{-1} X \rangle = \frac{1}{\alpha+\beta} - \langle Y, X \rangle - \langle Y, (\mathbb{1} - \overline{G})^{-1} \overline{G} X \rangle$$

- $\langle Y, \overline{G} X \rangle$ contains terms with $f_\beta \Rightarrow$ taking the limit $\beta \to -\alpha$, $\alpha \geq 0$ is not possible for each term

- **Solution**: Decompose $\overline{G} = \hat{G} + O$ with $\hat{G}$ without diverging terms and O orthogonal to Y

$$\Downarrow$$

$$\langle Y, (\mathbb{1} - \overline{G})^{-1} \overline{G} X \rangle = \langle Y, (\mathbb{1} - \overline{G})^{-1} \hat{G} X \rangle$$

- The result is analytic for $\alpha, \beta \in (-1/2, 1/2)^2$

5. Large time asymptotics

- We recall that $\overline{K}$ (or equivalently $\overline{G}$) is a 2×2 matrix kernel whose entries are expressed as double contour integrals
- Under the scaling $(x, y) = 4N - 2u2^{5/3}N^{2/3} + (X, Y)2^{4/3}N^{1/3}$ the rescaled kernel entries have Airy-like decay in both variables X, Y
- We perform the large time asymptotics of the kernels via steepest descent analysis (similar to the full-space case)

$$\Downarrow$$

$$\lim_{N\to\infty} \mathrm{pf}(J - \overline{K}^{resc})_{L^2(S,\infty)\times L^2(S,\infty)} = \mathrm{pf}(J - \overline{\mathcal{A}})_{L^2(S,\infty)\times L^2(S,\infty)}$$

(∂_s produces only polynomial factors)

(The inverse operator in $G_{\delta,u}$ can be written as linear combination of two Fredholm pfaffians)

Summary of the result

- Two-parameters family of distributions:

  u = distance of the end point from the diagonal

  δ = limit strength of the diagonal weights

- The distribution has a Pfaffian structure

- A similar strategy leads to the multi-point distribution and the definition of the *half-space Airy$_{stat}$ process*

  Betea–Ferrari–O. '21

The half-space Airy stationary process

Theorem (Betea–Ferrari–O. '21)

Let $m \geq 1$ and $\delta \in \mathbb{R}$. Fix m real numbers $u_1 > u_2 > \cdots > u_m \geq 0$ and m real numbers S_k, $1 \leq k \leq m$. Consider the stationary last passage times L_{N,j_k} in the $N \to \infty$ limit with $N - j_k = u_k 2^{5/3} N^{2/3}$, $\alpha = \delta 2^{-4/3} N^{-1/3}$.

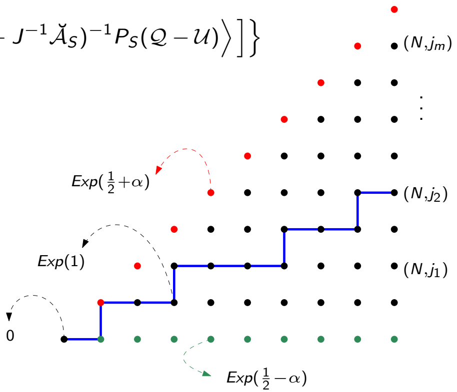
We have that

$$\lim_{N\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \left\{ \frac{L_{N,j_k} - 4N + 4u_k(2N)^{2/3}}{2^{4/3}N^{1/3}} \leq S_k \right\} \right)$$

$$= \sum_{k=1}^{m} \partial_{S_k} \left\{ \mathrm{pf}(J - \breve{\mathcal{A}}_S) \cdot \left[e^{\delta,\, u_1}(S_1) - \left\langle P_S \mathcal{Y} \middle| (\mathbb{1} - J^{-1}\breve{\mathcal{A}}_S)^{-1} P_S(\mathcal{Q} - \mathcal{U}) \right\rangle \right] \right\}$$

where $\mathcal{Y}, \mathcal{Q}$ and $\mathcal{U}$ are column vectors; J is the $2m \times 2m$ matrix with 2×2 block $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal; $\breve{\mathcal{A}}_S = P_S \breve{\mathcal{A}} P_S$ with $\breve{\mathcal{A}}$ the $2m \times 2m$ matrix kernel having 2×2 block at position (k, ℓ) given by an anti-symmetric extended Airy-like kernel

$$\begin{pmatrix} \breve{\mathcal{A}}_{11}^{u_k u_\ell}(X,Y) & \breve{\mathcal{A}}_{12}^{u_k u_\ell}(X,Y) \\ \breve{\mathcal{A}}_{21}^{u_k u_\ell}(X,Y) & \breve{\mathcal{A}}_{22}^{u_k u_\ell}(X,Y) \end{pmatrix}$$



Limit to Baik–Rains distribution

What happens if we look far away from the diagonal?

As $\delta \to -\infty$, the characteristic line moves from the diagonal

$\Rightarrow$ The path touches the diagonal rarely outside the $N^{2/3}$-neighborhood of the origin

Theorem (Betea–Ferrari–O. '19)

Let $S = s + \delta(2u + \delta)$ and $u + \delta = w$ fixed. Then

$$\lim_{u \to \infty} F^{(\delta,u)}_{0,\,half}(S) = F_{BR,w}(s)$$

where $F_{BR,w}(s)$ is the extended Baik–Rains distribution

$$F_{BR,w}(s) = \partial_s \left[F_{GUE}(s + w^2) \left(\mathcal{R}_w - \left\langle \Psi_w \middle| (\mathbb{1} - \mathcal{K}_{Ai,w})^{-1} \Phi_w \right\rangle \right) \right]$$

with $\mathcal{K}_{Ai,w}$ the (shifted) Airy kernel.

Thank you
for your attention!