# Fractal Geometry of the KPZ equation

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Integrable Structures in Random Matrix Theory and Beyond Workshop

Based on joint works with Sayan Das and Jaeyun Yi

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# Outline

Preliminaries

Law of Iterated Logarithms

Fractality

Proof Ideas and Tools

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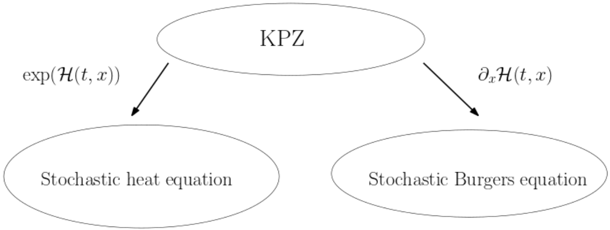
# Kardar-Parisi-Zhang equation

$$\partial_t \mathcal{H} = \frac{1}{2} \underbrace{\partial_x^2 \mathcal{H}}_{smoothing} + \frac{1}{2} \underbrace{(\partial_x \mathcal{H})^2}_{lateral-speed} + \underbrace{\xi}_{white-noise}$$

- Introduced by Kardar, Parisi and Zhang in 1986.
- KPZ equation is one of the cornerstones of the KPZ universality class.
- Immense improvement in understanding the models in the KPZ universality class and the universal limit in last 35 years.
- Goal is to understand macroscopic fractal geometry of the KPZ equation.
- Underlying theme: interplay between integrability and probability.

# Background

- KPZ is a paradigm for modeling interface fluctuation of the random growth models.




- **Stochastic Heat Equation (SHE)**:

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_x^2 \mathcal{Z} + \xi \mathcal{Z}.$$

The Cole-Hopf solution of the KPZ equation is $\log \mathcal{Z}(t,x)$.

- The Cole-Hopf solution is a physically relevant solution (Bertini and Giacomin' 97) and arises naturally in various renormalization and regularization scheme.

# Motivations & Goals

Denote the Cole-Hopf solution by $\mathcal{H}^{\mathbf{nw}}$ when $\mathcal{Z}(0,x) = \delta_{x=0}$.

- Amir, Corwin & Quastel '11 showed that

$$\mathfrak{h}_t(x) := \frac{\mathcal{H}^{\mathbf{nw}}(t, t^{2/3}x) + \frac{t}{24}}{t^{1/3}} \xrightarrow{d} 2^{-\frac{1}{3}}\mathrm{TW}_{\mathrm{GUE}} - \frac{x^2}{2}.$$

- Quastel, Sarkar '20 showed $\mathfrak{h}_t(\cdot)$ weakly converges to the $\mathrm{Airy}_2$ process.

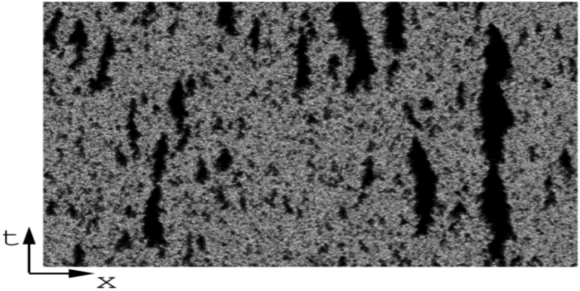
- Broad Question: In the spatio-temporal profile of $\mathfrak{h}_t(x)$, how do the 'tall buildings' (aka tall peaks) and 'deep tunnels' (aka deep valleys) look like and how often they occur?

# Goal of this talk

1. Fix $x = 0$. What are the scaling of the tall peaks and deep valleys of the $\mathfrak{h}_t(0)$? How frequent they occur?

2. How frequent the peaks and valleys in the spatio-temporal profile of $\mathcal{Z}(t, x)$?

3. Why should we be interested?




@ Zimmermann et al. '00, PRL.

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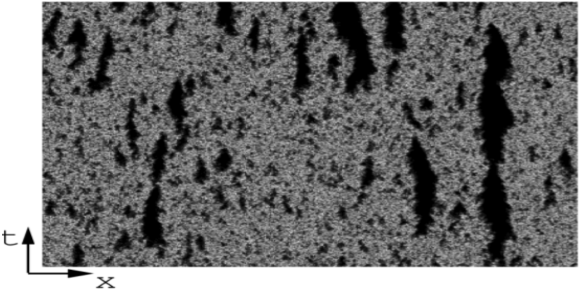


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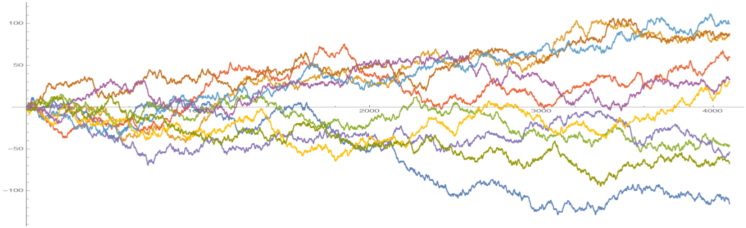
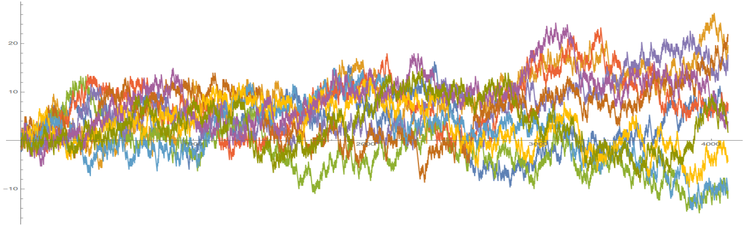
# Peaks and Valleys of Brownian motion and $\mathfrak{h}_t(0)$




@ Sayan Das.

# Limsup LIL

## Upper tail: Brownian motion and KPZ

- $\frac{\mathfrak{B}_t}{\sqrt{t}} \xrightarrow{d} N(0,1)$.
- $\mathbb{P}\big(\frac{\mathfrak{B}}{\sqrt{t}} > s\big) \sim e^{-\frac{1}{2}s^2}$.
- $\mathfrak{h}_t(0) \to 2^{-\frac{1}{3}} \mathrm{TW}_{\mathrm{GUE}}$.
- $\mathbb{P}\big(\mathrm{TW}_{\mathrm{GUE}} > s\big) \sim e^{-\frac{4}{3}s^{\frac{3}{2}}}$.

LIL (Limsup) of Brownian motion: $\limsup_{t\to\infty} \frac{\mathfrak{B}_t}{(t \log\log t)^{\frac{1}{2}}} \overset{a.s.}{=} (2)^{\frac{1}{2}}$.

**Theorem (Das & G., 2021)**

*With probability* 1, *we have*

$$\limsup_{t\to\infty} \frac{\mathfrak{h}_t(0)}{(\log\log t)^{\frac{2}{3}}} = \Big(\frac{3}{4\sqrt{2}}\Big)^{\frac{2}{3}}.$$

# Liminf LIL

## Lower tail: Brownian motion and KPZ

- $\frac{\mathfrak{B}_t}{\sqrt{t}} \xrightarrow{d} N(0,1)$.
- $\mathbb{P}\big(\frac{\mathfrak{B}}{\sqrt{t}} < -s\big) \sim e^{-\frac{1}{2}s^2}$.
- $\mathfrak{h}_t(0) \to 2^{-\frac{1}{3}}\mathrm{TW}_{\mathrm{GUE}}$.
- $\mathbb{P}\big(\mathrm{TW}_{\mathrm{GUE}} < -s\big) \sim e^{-\frac{1}{12}s^3}$.

LIL (Liminf) of Brownian motion: $\liminf_{t\to\infty} \frac{\mathfrak{B}_t}{(t \log\log t)^{\frac{1}{2}}} \overset{a.s.}{=} -(2)^{\frac{1}{2}}$.

**Theorem (Das & G., 2021)**

*With probability 1, we have*

$$\liminf_{t\to\infty} \frac{\mathfrak{h}_t(0)}{(\log\log t)^{\frac{1}{3}}} = -(6)^{\frac{1}{3}}.$$

# Previous Works

1. Chen '15 showed for any $t > 1$,

$$\limsup_{x\to\infty} \frac{\mathcal{H}(t,x)}{(\log x)_+^{2/3}} \overset{a.s.}{=} t^{\frac{1}{3}} \left(\frac{3}{4\sqrt{2}}\right)^{\frac{2}{3}}$$

   when $\mathcal{Z}(0,x)$ is positive and uniformly bounded $\forall x \in \mathbb{R}$.

2. Paquette and Zeitouni '15 proved a law of fractional logarithm for the GUE minor process:

$$\limsup_{n\to\infty} \frac{\hat{\lambda}_n}{(\log n)^{2/3}} \overset{a.s.}{=} \left(\frac{1}{4}\right)^{\frac{2}{3}}, \quad -c_1 < \liminf_{n\to\infty} \frac{\hat{\lambda}_n}{(\log n)^{1/3}} < -c_2,$$

   where

$$\hat{\lambda}_n := (\lambda_n - \sqrt{2n})\sqrt{2}n^{1/6} \overset{d}{\Rightarrow} \mathrm{TW}_{\mathrm{GUE}}.$$

3. Ledoux '15, Basu-Ganguly-Manjunath-Hegde '18 showed the law of iterated logarithms for the last passage percolation.

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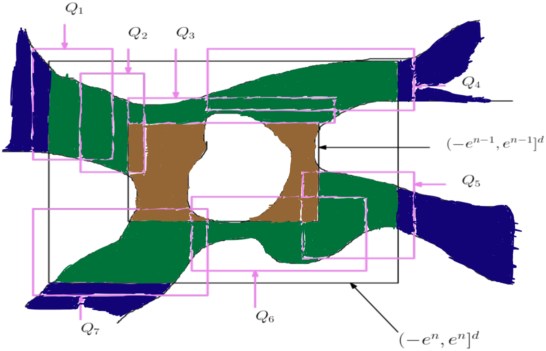
# Macroscopic Fractality

- Barlow-Taylor '91 introduced the notion of macroscopic Hausdorff dimension.
- How does one define it? Via Hausdorff content.




$$\nu_{\rho,n}(E) = \inf_{Q_1,\ldots,Q_m} \sum_{i=1}^{m} \left(\frac{\mathrm{MaxSide}(Q_i)}{e^n}\right)^{\rho}, \quad \mathrm{MinSide}(Q_i) > 1.$$

# Multi- and Mono-fractality

- $\rho$-dimensional Hausdorff content $:= \sum_n \nu_{\rho,n}$.
- Macroscopic Hausdorff dimension of any set $E$ is defined as

$$\mathrm{Dim}_{\mathbb{H}}(E) = \inf\Big\{\rho > 0 : \sum_n \nu_{\rho,n}(E) < \infty\Big\}.$$

- Stoch. process $X$ is multifractal w.r.t a gauge function $g$ if there are infinitely many scales $\gamma_1 > \gamma_2 > \dots$

$$\mathrm{Dim}_{\mathbb{H}}\left(\left\{t \geq 1 : \frac{X(t)}{g(t)} \geq \gamma_i\right\}\right) < \mathrm{Dim}_{\mathbb{H}}\left(\left\{t \geq 1 : \frac{X(t)}{g(t)} \geq \gamma_{i+1}\right\}\right).$$

Otherwise it is called monofractal.

# Fractality of the KPZ equation

**Theorem (Fractality of peaks, Das & G. '21)**

$\mathfrak{h}_t(0)$ *is* *monofractal* *w.r.t. the gauge function* $(\log\log t)^{2/3}$, *i.e.,*

$$\mathrm{Dim}_{\mathbb{H}}\left(\left\{t \geq e^e : \frac{\mathfrak{h}_t(0)}{(\log\log t)^{2/3}} \geq \gamma\right\}\right) \stackrel{a.s.}{=} \begin{cases} 1 & \textit{when } \gamma \leq \left(\frac{3}{4\sqrt{2}}\right)^{\frac{2}{3}}, \\ 0 & \textit{when } \gamma > \left(\frac{3}{4\sqrt{2}}\right)^{\frac{2}{3}}. \end{cases}$$

*However,* $\mathfrak{h}_{e^t}(0)$ *is* *multifractal* *w.r.t. the gauge function* $(\log t)^{2/3}$, *i.e.,*

$$\mathrm{Dim}_{\mathbb{H}}\left(\left\{t \geq e : \frac{\mathfrak{h}_{e^t}(0)}{(\log t)^{2/3}} \geq \gamma\left(\frac{3}{4\sqrt{2}}\right)^{\frac{2}{3}}\right\}\right) \stackrel{a.s.}{=} 1 - \gamma^{3/2}.$$

**Remarks:**

- Similar transition occurs for the valleys.
- Brownian motion also exhibits transition from mono- to multifractality under exponential time change (Khoshnevisan-Kim-Xiao '17).

# Previous work

Khoshnevisan, Kim and Xiao '18 proved multifractality of the spatio-temporal peaks of the SHE.

**Theorem (Khoshnevisan, Kim and Xiao '18)**

*For $\mathcal{Z}(0,\cdot) \in \mathbb{L}^\infty(\mathbb{R})$, the peaks of the spatio-temporal profile of the SHE is multifractal, i.e., there exists constants $A, B > 0$ and $\varepsilon > 0$ such that*

$$A\beta^{3/2}\gamma \leq 2 - \mathrm{Dim}_{\mathbb{H}}\Big(\big\{(t,x) : \mathcal{Z}(\gamma \log t, x) \geq \exp(\beta\gamma \log t)\big\}\Big) \leq B\beta^{2/3}\gamma,$$

*almost surely for all $\beta > -\frac{1}{24}$ and $\gamma \in (0, \varepsilon\beta^{-3/2})$.*

**Question:** Do the valleys behave similarly?

# Fractality of Valleys

**Theorem (G. & Yi '21)**

*For $\mathcal{Z}(0,\cdot) \in \mathbb{L}^\infty(\mathbb{R})$, the valleys of the spatio-temporal profile of the SHE is monofractal, i.e.,*

$$\mathrm{Dim}_{\mathbb{H}}\Big(\big\{(t,x) : \mathcal{Z}(\gamma \log t, x) \leq \exp(\beta\gamma \log t)\big\}\Big) \overset{a.s.}{=} \begin{cases} 2 & 0 > \beta > -\frac{1}{24}, \\ 1 & \beta < -\frac{1}{24}. \end{cases}$$

**Conjecture:** In the same setting as above, at $\beta = \frac{1}{24}$,

$$\mathrm{Dim}_{\mathbb{H}}\Big(\big\{(t,x) : \mathcal{Z}(e^t, e^{2t/3}x) \leq \exp\big(-\frac{1}{24}e^t - \alpha(e^t \log t)^{1/3}\big)\big\}\Big) \overset{a.s.}{=} 2 - C\alpha^3$$

for $\alpha \in [0, C^{-1/3}]$ where $C > 0$ depends on initial data.

# Fractality of Valleys

**Theorem (G. & Yi '21)**

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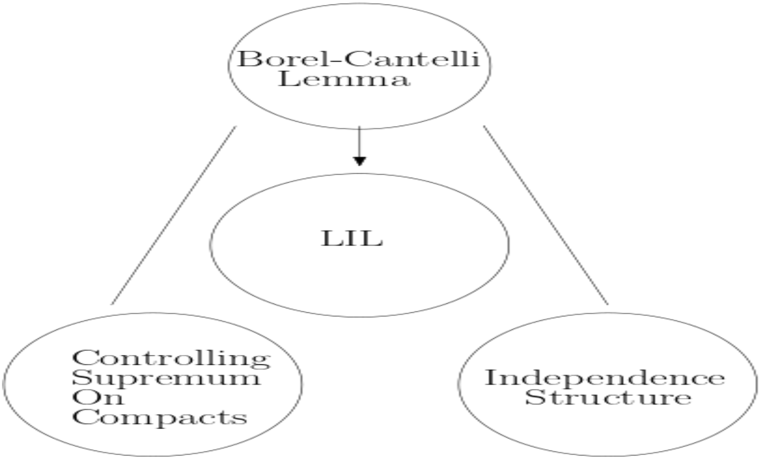
# Key Proof Ideas for LIL

$$\boxed{\limsup_{t\to\infty} \frac{\mathfrak{h}_t(0)}{(\log\log t)^{\frac{2}{3}}} \overset{a.s.}{=} \left(\frac{3}{4\sqrt{2}}\right)^{\frac{2}{3}}} \qquad \boxed{\liminf_{t\to\infty} \frac{\mathfrak{h}_t(0)}{(\log\log t)^{\frac{1}{3}}} \overset{a.s.}{=} -(6)^{\frac{1}{3}}}.$$



**Requirements:**

- $\mathfrak{h}_{t_1}$ is approximately independent of $\mathfrak{h}_{t_2} - \mathfrak{h}_{t_1}$ when $t_1 \ll t_2$.
- The law of $\mathfrak{h}_{t_2} - \mathfrak{h}_{t_1}$ is approximately same as $\mathfrak{h}_{t_2 - t_1}$.

**Theorem (Das & G., '21)**

*For any* $0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_m$, *there exist independent random variables* $Y_1, Y_2, \ldots, Y_m$ *such that for all* $1 \le i \le m$ *and large* $x$,

$$Y_i \stackrel{d}{=} (1 - e^{-(t_i - t_{i-1})})^{1/3} \mathfrak{h}_{e^{t_i} - e^{t_{i-1}}}(0),$$

*and*

$$\Pr\left(|\mathfrak{h}_{e^{t_i}}(0) - Y_i| \ge x\right) \le \exp(-cx^{3/2}).$$

# Starting point: Composition Law of KPZ

**Proposition (Multipoint composition law, Das & G. '21)**

*For any* $t_k > t_{k-1} > \ldots > t_1 > t_0 = 0$, *there exist independent spatial process* $\mathcal{H}^{\mathbf{nw}}_{t_i \downarrow t_{i-1}}(\cdot)$ *independent of* $\mathcal{H}^{\mathbf{nw}}(t_{i-1}, \cdot)$ *for* $k \geq i \geq 2$ *such that*

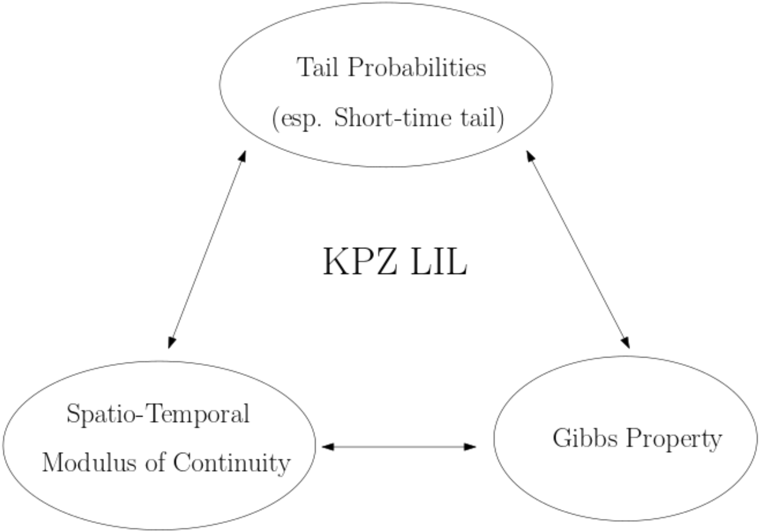
$$\mathcal{H}^{\mathbf{nw}}_{t_i \downarrow t_{i-1}}(\cdot) \stackrel{d}{=} \mathcal{H}^{\mathbf{nw}}(t_i - t_{i-1}, \cdot)$$

*and,*

$$\mathcal{H}^{\mathbf{nw}}(t_i, 0) = \log\Big(\int_{-\infty}^{\infty} \exp\big(\mathcal{H}^{\mathbf{nw}}(t_{i-1}, x) + \mathcal{H}^{\mathbf{nw}}_{t_i \downarrow t_{i-1}}(-x)\big)\, dx\Big).$$

- Our proof of the composition law relies on the linearity and time reversal property of the solution of the SHE.
- Two point composition was proved before in the KPZ line ensemble paper by Corwin and Hammond'14.

# Other Tools


Tail Probabilities
(esp. Short-time tail)
KPZ LIL
Spatio-Temporal
Modulus of Continuity
Gibbs Property

# Tail Probabilities

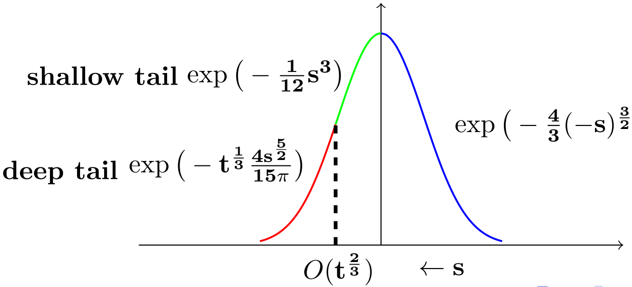
**Theorem (Corwin & G.' 18)**

*There exists $s_0$ such that for all $t > 1$ and $s > s_0$,*

$$\mathbb{P}\big(2^{\frac{1}{3}}\mathfrak{h}_{2t}(0) \leq -s\big) = \Theta\big(\exp\big(-\frac{4t^{\frac{1}{3}}s^{\frac{5}{2}}}{15\pi}\big)\big) + \Theta\big(\exp\big(-\frac{s^3}{12}\big)\big)$$

*and*

$$\mathbb{P}\big(2^{\frac{1}{3}}\mathfrak{h}_{2t}(0) > s\big) = \Theta\big(\exp\big(-\frac{4}{3}s^{\frac{3}{2}}\big)\big).$$



# Tail probabilities via RMT

- We show how lower tail probabilities are obtained from RMT.
- We used Borodin & Gorin's formula:

$$\mathbb{E}\Big[\exp\big(-\exp(t^{1/3}(2^{\frac{1}{3}}\mathfrak{h}_{2t}(0)+s))\big)\Big] = \mathbb{E}\Big[\prod_{k=1}^{\infty}\frac{1}{1+\exp(t^{1/3}(\mathbf{a}_k+s))}\Big]$$

for any $s \in \mathbb{R}$. Here $\mathbf{a}_1 > \mathbf{a}_2 > \ldots$ are the Airy point process.

- We note

$$\mathbf{LHS} = \mathbb{P}(2^{\frac{1}{3}}\mathfrak{h}_{2t}(0) + t^{-1/3}G \leq -s) \approx \mathbb{P}\big(2^{\frac{1}{3}}\mathfrak{h}_{2t}(0) \leq -s\big)$$

when $s$ is a large number and $t > 1$. $G$ is an independent Gumbel r.v.

# Continued..

- Why $4t^{1/3}s^{\frac{5}{2}}/15\pi$? $\{\mathbf{a}_k\}_{k\geq 1}$ are very close to the **zeros** of the **Airy function** which are located at $-(\mathbf{3\pi k/2})^{\frac{2}{3}}$.

$$\prod_{k=1}^{\infty} \frac{1}{1+\exp\left(t^{1/3}\left(-(\mathbf{3\pi k/2})^{\frac{2}{3}}+s\right)\right)} \approx \exp\left(-\mathbf{t^{1/3}}\frac{\mathbf{4s^{\frac{5}{2}}}}{\mathbf{15\pi}}\right).$$

- Why $s^3/12$? If $\mathbf{a}_1 < -s$, then,

$$\prod_{k=1}^{\infty} \frac{1}{1+\exp(t^{1/3}(\mathbf{a}_k+s))} \approx \mathbf{1}.$$

  But, the corresponding penalty is $\mathbb{P}(\mathbf{a}_1 < -s)$ which is $\exp(-\mathbf{s^3/12})$.

- These heuristics are made rigorous by exploring the connections of the Airy point process with the **stochastic Airy operators** and the Ablowitz-Segur solution of the **Painlevé II**.

# Outlook

**Summary:**

- Obtained law of iterated logarithms for the KPZ equation under narrow wedge initial data. General initial data case is still open.
- Macroscopic fractal dimension of the peaks and valleys of the temporal process of KPZ.
- Macroscopic fractal dimension of the valleys of the KPZ equation.

**Future directions:**

- Fractality of the peaks of the spatial process of the KPZ class models?
- What happens for the KPZ fixed point? Is there a short time LIL like as in Brownian motion?
- What happens to intermittency and multifractality when dimension increases?