Marked and conditional determinantal point processes

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MSRI workshop Integrable structures in random matrix theory and beyond

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Based on joint work with Gabriel Glesner

MSRI, September 2002:
Recent Progress In Random Matrix Theory And Its Applications

Dyson, "Random matrices, neutron capture levels, quasicrystals and zeta-function".
How to detect missing or spurious eigenvalues of a random matrix?

See https://www.msri.org/workshops/220/schedules/1385.

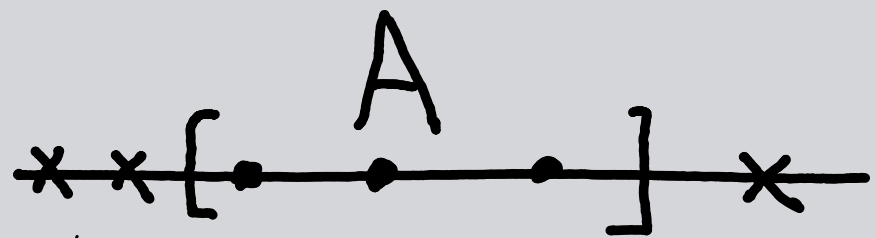
About his work in the 1960-1970s, comparing nuclear physics data with random matrix eigenvalues: "But we had to expect that a small percentage of genuine levels would be missed and a small percentage of spurious levels would be included in the data. [...] It would be a great triumph for mathematics if an error-correcting code could correct Nature's mistakes as well as our own. [...] Now, thirty years later, I am no longer interested in the nuclear level data. I am interested in the general question, whether error-correcting codes for random matrix eigenvalues are possible in principle. I suspect that they might lead you to some interesting mathematics."

Randomly incomplete spectra

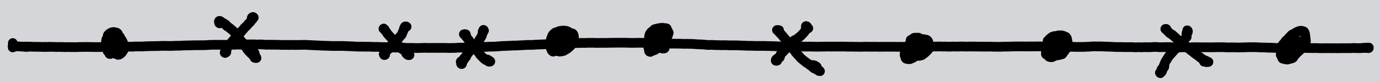
Question

Suppose that we see an incomplete sample of a determinantal point process (DPP) on $\mathbb{R}$, what can we say about the missing part?

The rule to obtain the incomplete sample may be



- ✓ deterministic, e.g., we see all points in $A \subset \mathbb{R}$
- ✓ random, e.g., we see all points with probability $1/2$ (cf. Bohigas-Pato '06).



Independent position-dependent thinning of the DPP

We see each point x in the random point configuration independently with probability $\theta(x)$ for some Borel measurable $\theta : \mathbb{R} \to [0,1]$.

Correlation functions and average multiplicative statistics of a DPP

A DPP on $\mathbb{R}$ is a random point process on $\mathbb{R}$ such that

1. Correlation functions are expressed in terms of a correlation kernel $K(x,y)$:

$$\rho_m(x_1,\ldots,x_m) = \det\left(K(x_i,x_j)\right)_{i,j=1}^m.$$

2. Average multiplicative statistics are Fredholm determinants:

$$\begin{aligned}\mathbb{E}\prod(1-\phi(x_i)) &= \det(1-\mathrm{M}_\phi \mathrm{K}) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\mathbb{R}^n}\det\left(K(x_i,x_j)\right)_{i,j=1}^n\prod_{j=1}^n\phi(x_j)dx_j.\end{aligned}$$

(See e.g. Macchi '75, Soshnikov '00, Lyons '03, Shirai-Takahashi '03, Johansson '06, Hough-Krishnapur-Peres-Virag '06, Borodin '11.)

DPPs: main examples

1. Orthogonal Polynomial Ensembles

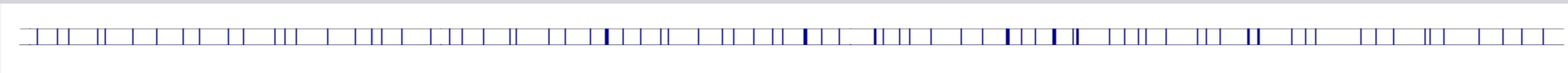
N points with symmetric joint probability distribution

$$\frac{1}{Z_N} \prod_{1\leq i<j\leq N} (x_j - x_i)^2 \prod_{j=1}^{N} w(x_j) dx_j.$$

Eigenvalue jpdf of unitary invariant random matrices if $w(x) = e^{-NV(x)}$.

Correlation kernel expressed in terms of orthonormal polynomials with respect to w:

$$K_N(x,y) = \sqrt{w(x)w(y)} \sum_{j=0}^{N-1} p_j(x)p_j(y).$$



2. DPPs induced by orthogonal projections

Correlation kernel K for which the associated integral operator

$$\mathrm{K}f(x) = \int K(x,y)f(y)w(y)dy$$

is an orthogonal projection (of finite or infinite rank).

Particular cases: OPEs and scaling limits of OPEs:

$$K^{\sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}, \quad K^{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y},$$

Bessel kernels, Painlevé kernels, ...

General property (Soshnikov '00): Number of points is a.s. equal to $\mathrm{rank}(\mathrm{K})$.

3. Integrable kernels

A kernel is k-integrable (in the sense of Its-Izergin-Korepin-Slavnov '93, Deift-Its-Zhou '97) if it is of the form

$$K(x,y)=\frac{\sum_{j=1}^{k} f_j(x)h_j(y)}{x-y} \qquad \text{with } \sum_{j=1}^{k} f_j(x)h_j(x)=0.$$

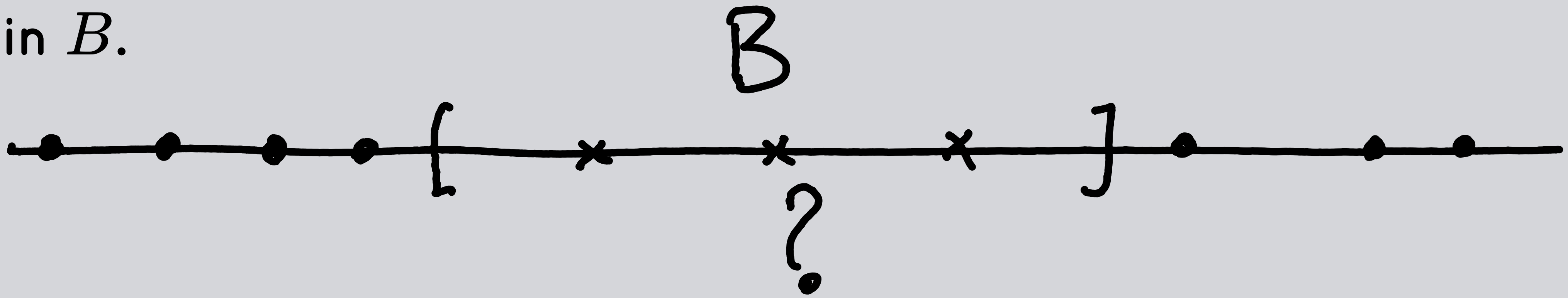
For instance: OPEs (by the Christoffel-Darboux identity), Airy, Sine, Bessel, Painlevé kernel DPPs.

A random thinning of a DPP with k-integrable kernel K, in which each point x is observed independently with probability $\theta(x)$, is also k-integrable, with kernel (Lavancier-Moller-Rubak '15)

$$\theta(x)K(x,y), \quad \theta:\mathbb{R}\to[0,1].$$

Definition (Ghosh '16, Ghosh-Peres '17)

A point process is number rigid if for any bounded set $B \subset \mathbb{R}$, the configuration of points outside B almost surely determines the number of points in B.



Properties

DPPs induced by finite rank projections are trivially number rigid, since number of points is a.s. equal to the rank of the projection.

DPPs can only be number rigid if they are induced by a projection operator (Ghosh-Krishnapur '15).

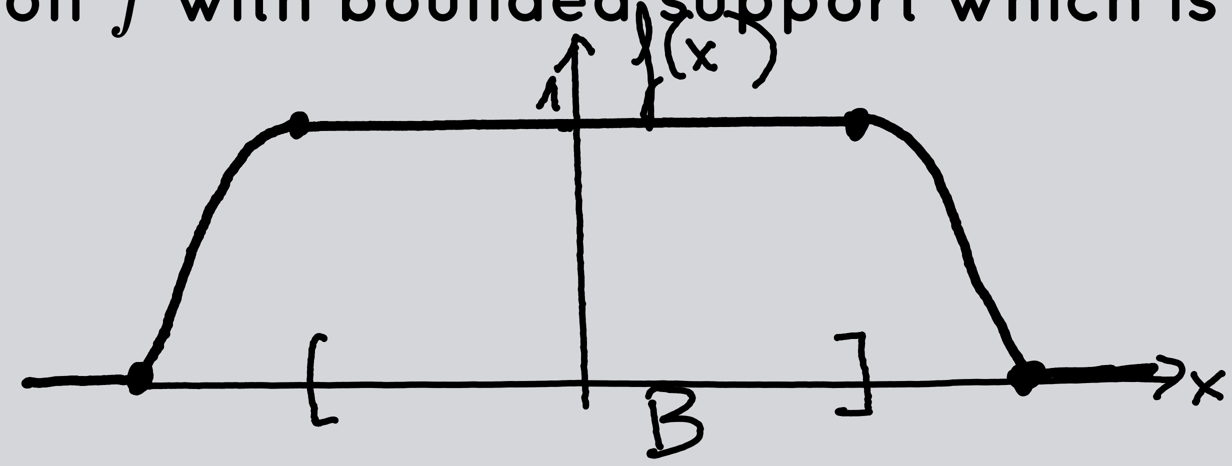
What about DPPs induced by infinite rank projections?

Number rigidity

Sufficient condition for number rigidity (Ghosh '16, Ghosh-Peres '17)

A point process is number rigid if for any $\epsilon > 0$ and for any bounded set $B \subset \mathbb{R}$, there exists a bounded function f with bounded support which is such that

$$f|_B = 1, \quad \mathrm{Var} \sum f(x_i) < \epsilon.$$



Theorem (Bufetov '16)

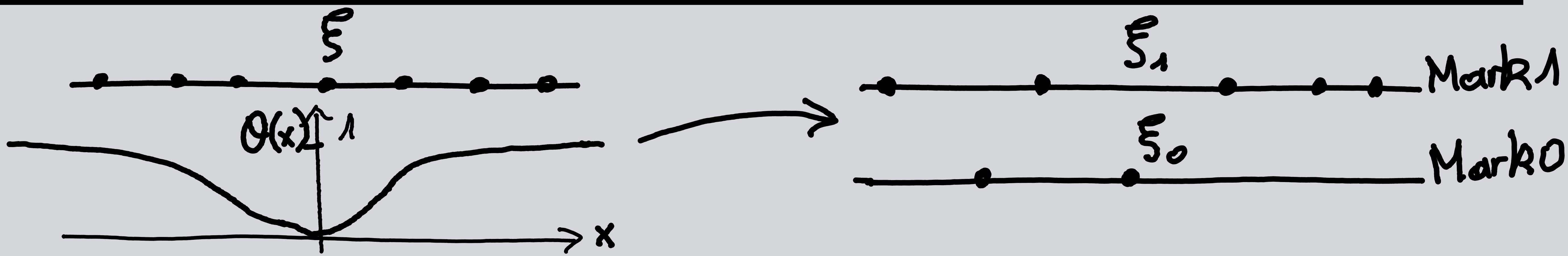
DPPs induced by orthogonal projections with sufficiently regular 2-integrable kernels

$$K(x,y) = \frac{f_1(x)g_1(y) + f_2(x)g_2(y)}{x-y}$$

are number rigid (e.g., Sine, Airy, Bessel).

Marked point process

Given a point process $\mathbb{P}$ on $\mathbb{R}$ and $\theta : \mathbb{R} \to [0,1]$, we define a marked point process $\mathbb{P}^\theta$ on $\mathbb{R} \times \{0,1\}$ by assigning to each point x independently mark 1 with probability $\theta(x)$ and mark 0 with probability $1-\theta(x)$.



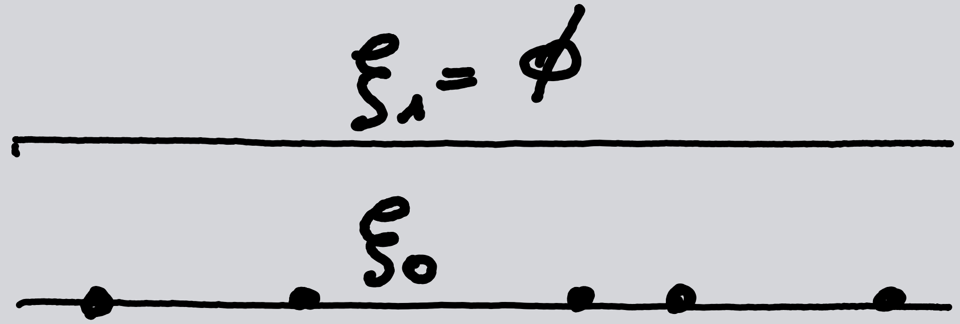
Property

If $\mathbb{P}$ is the DPP with kernel $K(x,y)$ and reference measure μ, then so is $\mathbb{P}^\theta$, but with reference measure

$$d\mu^\theta(x,b) = \theta(x)d\mu(x)d\delta_{\{b=1\}} + (1-\theta)(x)d\mu(x)d\delta_{\{b=0\}} \text{ on } \mathbb{R} \times \{0,1\}.$$

No observed particles

Given a configuration ξ, we write ξ_j for the configuration of mark j points.



The probability to observe no particles is

$$\mathbb{P}^{\theta}(\xi_1 = \emptyset) = \det(1 - \mathrm{M}_{\theta}\mathrm{K}).$$

If this is non-zero, we can define the conditional ensemble $\mathbb{P}^{\theta}_{|\emptyset}$ (on $\mathbb{R}$) by conditioning $\mathbb{P}^{\theta}$ on the event $\xi_1 = \emptyset$.

Conditional ensemble

If $\mathbb{P}$ is the DPP with kernel K of the operator K, then $\mathbb{P}^{\theta}_{|\emptyset}$ is the DPP with kernel of the operator

$$\mathrm{M}_{1-\theta}\mathrm{K}(1 - \mathrm{M}_{\theta}\mathrm{K})^{-1} \quad \text{on } L^2(\mathbb{R}, d\mu), \text{ or}$$

$$\mathrm{K}(1 - \mathrm{M}_{\theta}\mathrm{K})^{-1} \quad \text{on } L^2(\mathbb{R}, (1-\theta)d\mu).$$

Finite number of observed particles

The probability to observe particles (only) at positions $\mathbf{v} = \{v_1, v_2, \ldots, v_k\}$ will typically be 0. Conditional ensembles can still be defined via disintegration.

Palm measures

The reduced Palm DPP represents the conditioning of $\mathbb{P}$ on particles (among others) at $v_1, \ldots, v_k$, and then removing these particles, and is the DPP with kernel (Shirai-Takahashi '03)

$$K_{\mathbf{v}}(x, y) = \det \begin{pmatrix} K(x,y) & K(x,\mathbf{v}) \\ K(\mathbf{v},y) & K(\mathbf{v},\mathbf{v}) \end{pmatrix}.$$

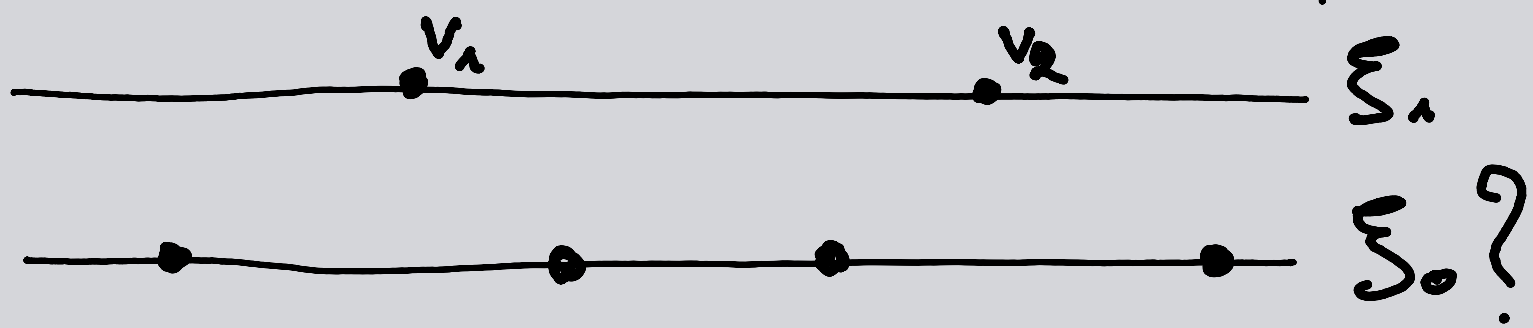
We want to define a conditional ensemble $\mathbb{P}^{\theta}_{|\mathbf{v}}$ which represents the conditioning of $\mathbb{P}$ on particles at $v_1, \ldots, v_k$, and on no other mark 1 particles.

Conditional ensemble

Let $\mathbb{P}$ be a DPP with kernel K of the operator K, and suppose that $\mathbb{P}^\theta(\#\xi_1 = k) > 0$. For $\mathbb{P}^\theta$-a.e. k-point mark 1 configuration $\mathbf{v} = \{v_1, \dots, v_k\}$, $\mathbb{P}^\theta_{|\mathbf{v}}$ is the DPP with kernel of the operator

$$\mathrm{K}_\mathbf{v}(1 - \mathrm{M}_\theta \mathrm{K}_\mathbf{v})^{-1} \text{ on } L^2(\mathbb{R}, (1-\theta)d\mu).$$

Compare this to the Poisson process $\mathbb{P}$ with intensity ρ on $(\mathbb{R}, d\mu)$: then $\mathbb{P}^\theta_{|\emptyset}$ is the Poisson point process with intensity ρ on $(\mathbb{R}, (1-\theta)d\mu)$, which is the same as the unconditioned distribution of mark 0 points.

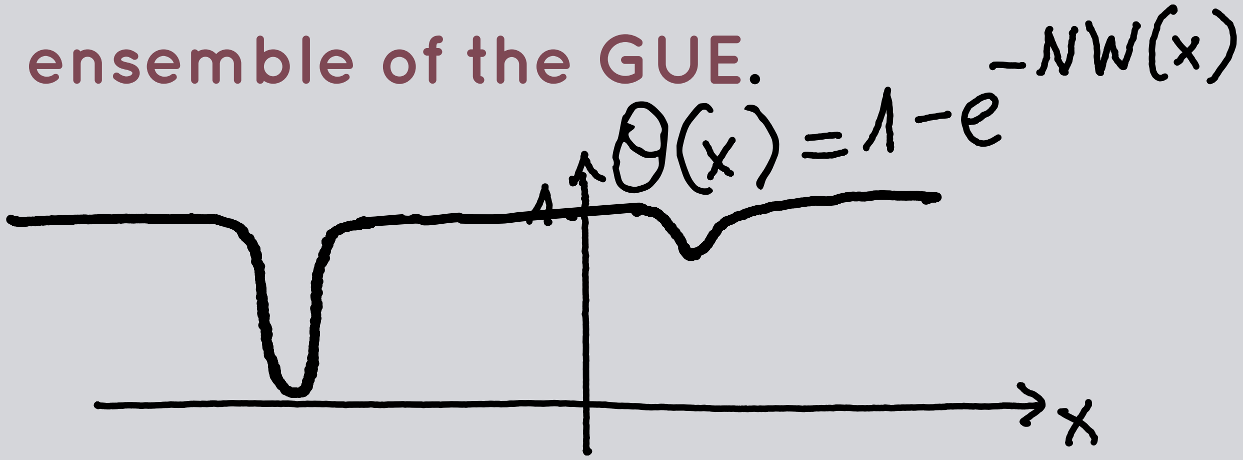


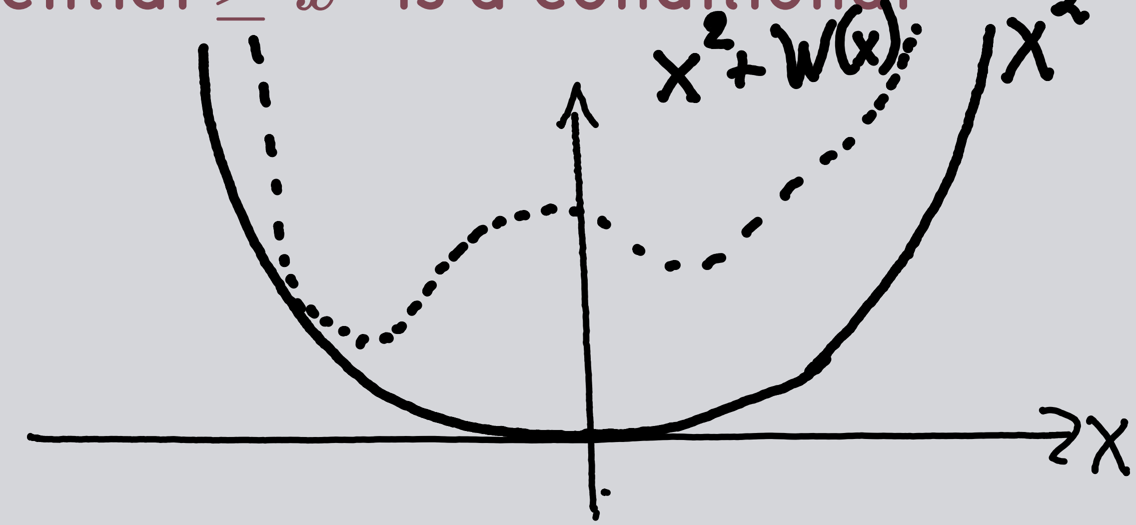
Conditional ensembles of OPEs

If $\mathbb{P}$ is an N-point OPE with density $\frac{1}{Z_n}\prod_{1\le i<j\le N}(x_j-x_i)^2\ \prod_{j=1}^N w(x_j)dx_j$, then $\mathbb{P}^\theta_{|\mathbf{v}}$ is an $n=(N-k)$-point OPE with density

$$\frac{1}{Z_n'}\prod_{1\le i<j\le n}(x_j-x_i)^2\ \prod_{j=1}^{n}\left(\prod_{\ell=1}^{k}(x_j-v_\ell)^2\right)(1-\theta)(x_j)w(x_j)dx_j.$$

For $w(x)=e^{-Nx^2}$, $k=0$, $1-\theta(x)=e^{-NW(x)}$ for $W\ge 0$, $\mathbb{P}^\theta_{|\mathbf{v}}$ is the OPE with confining potential x^2 replaced by $x^2+W(x)$. so any unitary invariant ensemble with confining potential $\ge x^2$ is a conditional ensemble of the GUE.





Conditional ensembles of orthogonal projection DPPs

Let $\mathbb{P}$ be a DPP with kernel K of an orthogonal projection operator K onto a closed L^2-subspace H.

$\mathbb{P}^{\theta}_{|\mathbf{v}}$ is the DPP with kernel $K^{\theta}_{|\mathbf{v}}$ of the orthogonal projection onto the closure of

$$\mathrm{M}_{1-\theta} H_{\mathbf{v}},\ H_{\mathbf{v}} = \{h \in H : h|_{\mathbf{v}} = 0\}.$$

(There is a similar result for skew projection DPPs.)

Caveat: this is not true in general for infinite mark 1 configurations $\mathbf{v}$!

Conditional ensembles of integrable kernel DPPs

If $\mathbb{P}$ is a DPP with k-integrable kernel

$$K(x,y) = \frac{\sum_{j=1}^{k} f_j(x)g_j(y)}{x-y}, \quad \sum_{j=1}^{k} f_j(x)g_j(x) = 0,$$

then the method of Its-Izergin-Korepin-Slavnov '93, Deift-Its-Zhou '97, Bertola-Cafasso '14 implies that $\mathbb{P}^{\theta}_{|\emptyset}$ is also a DPP with k-integrable kernel, which can be characterized in terms of a Riemann-Hilbert problem.

The same is true for $\mathbb{P}^{\theta}_{|\mathbf{v}}$, but then the Riemann-Hilbert problem has singularities at $v_1, \ldots, v_k$.

This opens the door for asymptotic analysis and for deriving integrable differential equations.

Jacobi's identity in terms of the conditional ensembles

For a smooth deformation θ_t, Jacobi's variational formula implies that

$$\partial_t \log\det(1-\mathrm{M}_{\theta_t}\mathrm{K}) = -\mathbf{Tr}\left[\mathrm{M}_{\partial_t\theta_t}\mathrm{K}(1-\mathrm{M}_{\theta_t}\mathrm{K})^{-1}\right] = -\mathbf{Tr}\left[\mathrm{M}_{\frac{\partial_t\theta_t}{1-\theta}}\mathrm{K}^{\theta_t}_{|\emptyset}\right].$$

In probabilistic terms,

$$\partial_t \log\mathbb{E}\prod(1-\theta_t(x_i)) = \mathbb{E}^{\theta_t}_{|\emptyset}\sum\partial_t\left(\log(1-\theta_t(x_i))\right),$$

so the logarithmic derivative of an average multiplicative statistic in $\mathbb{P}$ is equal to the average of a linear statistic in $\mathbb{P}^{\theta_t}_{|\emptyset}$.

If one allows for continuous marks in $[0,1]$, one can interpret this in terms of the hazard rate function.

Infinite rank projection operators

If $\mathrm{Tr}\, \mathrm{M}_\theta \mathrm{K} = \infty$, we have $\mathbb{P}^\theta(\#\xi_1 = \infty) = 1$.

Theorem (Bufetov-Qiu-Shamov '16)

If $\mathbb{P}$ is a DPP defined by an orthogonal projection K and $\mathrm{Tr}\, \mathrm{M}_\theta \mathrm{K} = \infty$, one can still a.s. define $\mathbb{P}^\theta_{|\mathbf{v}}$, and $\mathbb{P}^\theta_{|\mathbf{v}}$ is a DPP defined by a locally trace class Hermitian operator $\mathrm{K}^\theta_{|\mathbf{v}}$ (but not necessarily a projection!).

Question

Under which conditions is $\mathrm{K}^\theta_{|\mathbf{v}}$ a projection, or under which conditions is the number of points in the conditional ensemble deterministic?

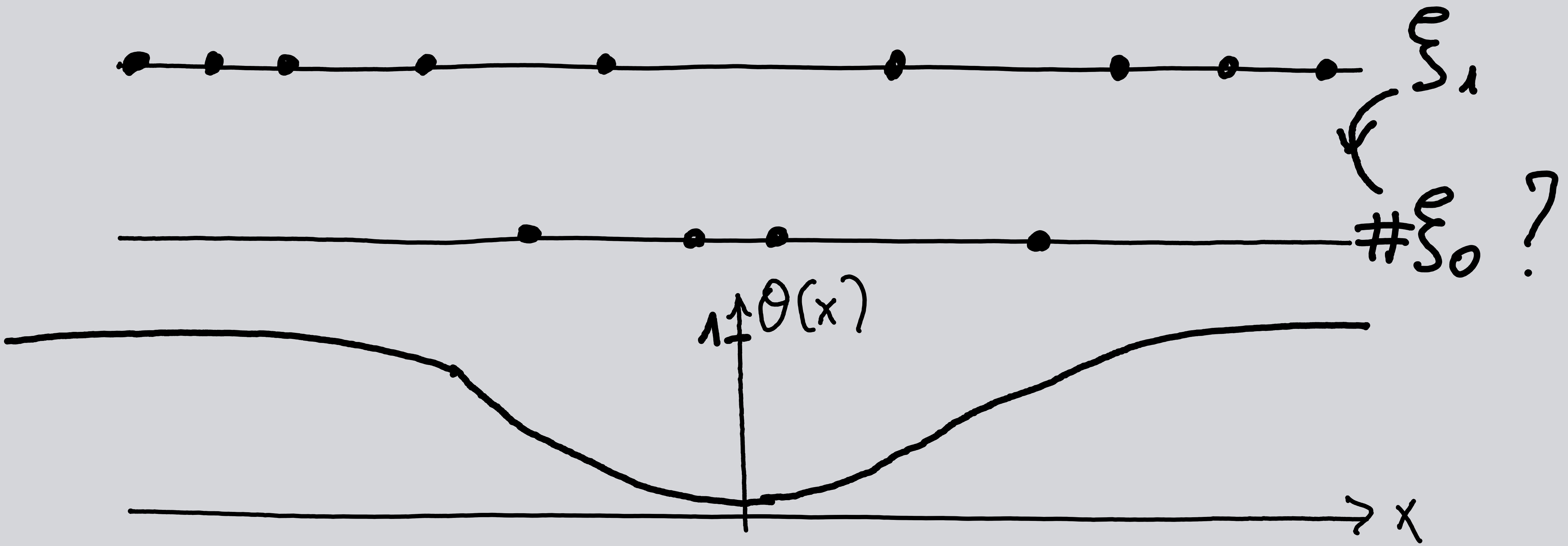
Rigidity

Definition

A point process $\mathbb{P}$ is marking rigid if for any measurable $\theta : \mathbb{R} \to [0,1]$ and for $\mathbb{P}_1^\theta$-a.e. mark 1 configuration $\mathbf{v}$, there exists $\ell_{\mathbf{v}} \in \mathbb{N} \cup \{0, \infty\}$ such that

$$\mathbb{P}_{|\mathbf{v}}^\theta(\#\xi_0 = \ell_{\mathbf{v}}) = 1.$$

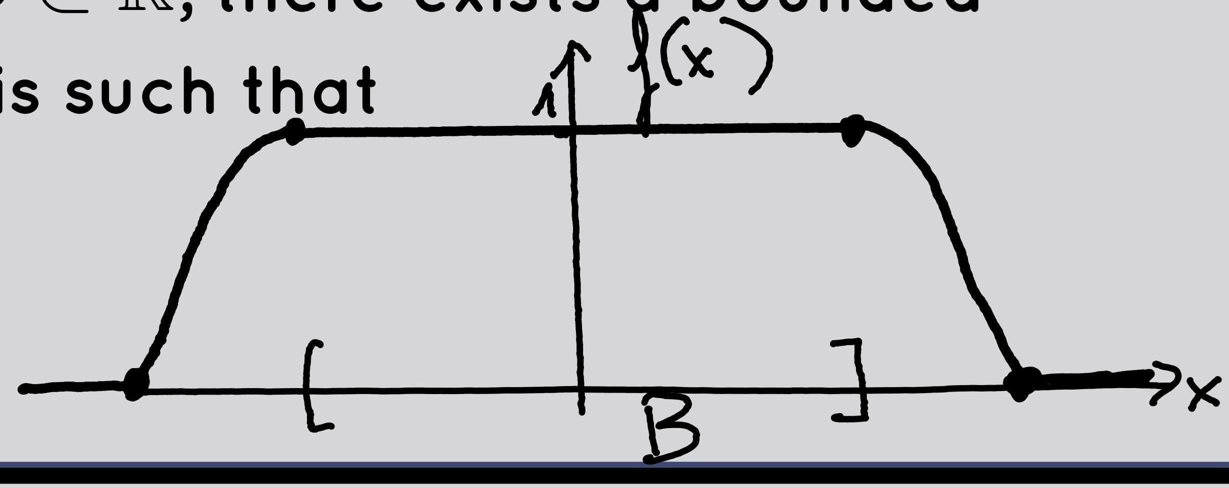
(Note that marking rigidity implies number rigidity, by setting $\theta = 1_{B^c}$.)



Recall: Sufficient condition for number rigidity

For any $\epsilon > 0$ and for any bounded set $B \subset \mathbb{R}$, there exists a bounded function f with bounded support which is such that

$$f|_B = 1, \quad \mathrm{Var} \sum f(x_i) < \epsilon.$$

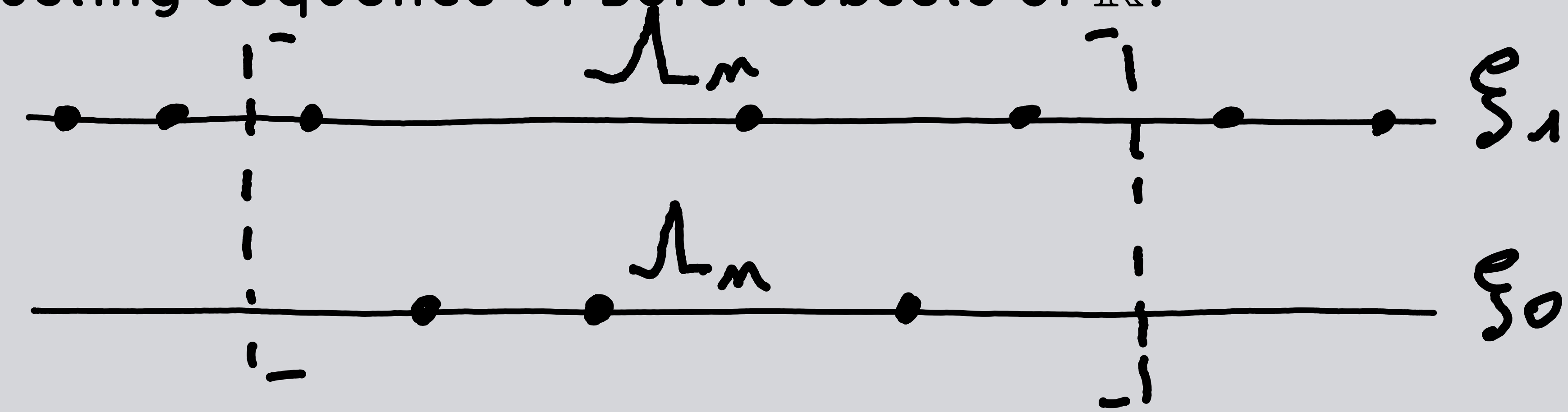


Theorem (C-Glesner '21)

If $\mathbb{P}$ satisfies the sufficient condition for number rigidity and if for any θ, $\mathbb{P}^\theta(\xi_0(\mathbb{R}) < \infty)$ is either 0 or 1, then $\mathbb{P}$ is marking rigid.

Sketch of the proof (cf. Ghosh '16, Ghosh-Peres '17)

1. If $\mathbb{P}^\theta(\xi_0(\mathbb{R}) < \infty) = 0$, then $\xi_0(\mathbb{R}) = \infty$ a.s.
2. If $\mathbb{P}^\theta(\xi_0(\mathbb{R}) < \infty) = 1$, let $\Lambda_1 \subset \Lambda_2 \subset \cdots$ be an exhausting sequence of Borel subsets of $\mathbb{R}$.



First observation: $\xi_0(\mathbb{R} \setminus \Lambda_n) \to 0$ as $n \to \infty$.

Second observation: sufficient condition implies existence of bounded measurable functions $f_1, f_2, \ldots$ with bounded support such that $f_n|_{\Lambda_n} = 1$ and

$$\lim_{n\to\infty} \operatorname{Var} \sum f_n(x_i) = 0, \quad \lim_{k\to\infty} \left(\sum f_{n_k}(x_i) - \mathbb{E} \sum f_{n_k}(x_i) \right) = 0 \text{ a.s.}$$

Sketch of the proof (cf. Ghosh '16, Ghosh-Peres '17)

3. Then,

$$\begin{aligned}\xi_0(\mathbb{R}) &= \sum_0 f_{n_k}(u_i) + \sum_1 f_{n_k}(v_i) + \sum_0 (1 - f_{n_k}(u_i)) - \sum_1 f_{n_k}(v_i) \\ &= \left(\sum_{0,1} f_{n_k}(x_i) - \mathbb{E} \sum_{0,1} f_{n_k}(x_i) \right) + \mathbb{E} \sum_{0,1} f_{n_k}(x_i) \\ &\quad + \sum_0 (1 - f_{n_k}(u_i)) - \sum_1 f_{n_k}(v_i).\end{aligned}$$

4. Taking the limit $k \to \infty$, we obtain (using the disintegration property)

$$\mathbb{P}^{\theta}_{|\mathbf{v}} \left(\xi_0(\mathbb{R}) = \lim_{k\to\infty} \left(\mathbb{E} \sum_{0,1} f_{n_k}(x_i) - \sum_1 f_{n_k}(v_i) \right) \right) = 1 \text{ for } \mathbb{P}^{\theta}_1\text{-a.e. } \mathbf{v}.$$

Marking rigidity of Sine, Airy, Bessel point processes

If $\mathbb{P}$ is a DPP, then $\mathbb{P}^\theta(\xi_0(\mathbb{R}) < \infty)$ is 0 or 1 (Soshnikov '00). As a consequence, DPPs satisfying the sufficient condition for number rigidity, including Sine, Airy, Bessel point processes, are marking rigid.

Dependence on θ

Depending on the marking function θ, the number of invisible points is either ∞ or equal to

$$\lim_{k\to\infty}\left(\mathbb{E}\sum_{0,1} f_{n_k}(x_i) - \sum_{1} f_{n_k}(v_i)\right).$$

This does not depend explicitly on θ.

Marked and conditional DPPs

Already appeared implicitly in:

- ✓ IIKS method to study Fredholm determinants,
- ✓ unitary invariant random matrix ensembles,
- ✓ study of number rigidity.

Why study conditional ensembles?

- ✓ Natural in view of the search for error-correcting codes/spectrum completion codes,
- ✓ useful in asymptotic analysis of Fredholm determinants of the form $\det(1 - \mathrm{M}_\theta \mathrm{K})$ via the IIKS method, where it helps to guess a convenient g-function,
- ✓ allows to study refined notion of number rigidity,
- ✓ well-behaving transformation of sufficiently regular point processes and DPPs in particular.

The end

Thank you for your attention!