a soliton interacting with a regular gas of solitons (The Soliton vs. the Gas)

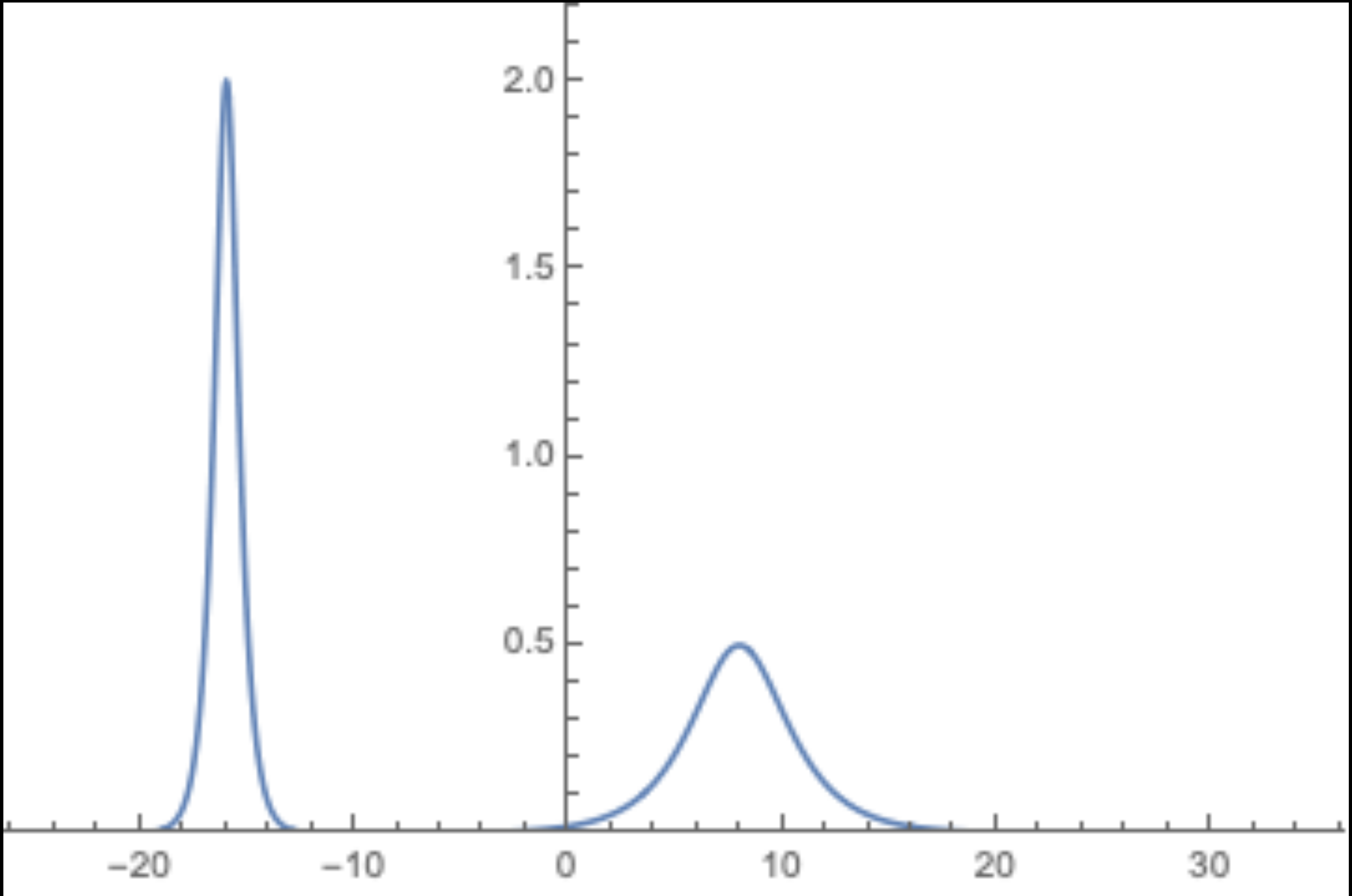
Joint work with: Manuela Girotti (Saint Mary's University), Bob Jenkins (University of Central Florida), Tamara Grava (SISSA and University of Bristol), and Alexander Minakov (University of Prague)

Solitons: fundamental solutions of nonlinear evolution equations

Integrable nonlinear PDEs possess solutions with many solitons

Example: the two-soliton solution

$q_t + 6q^2 q_x + q_{xxx} = 0$



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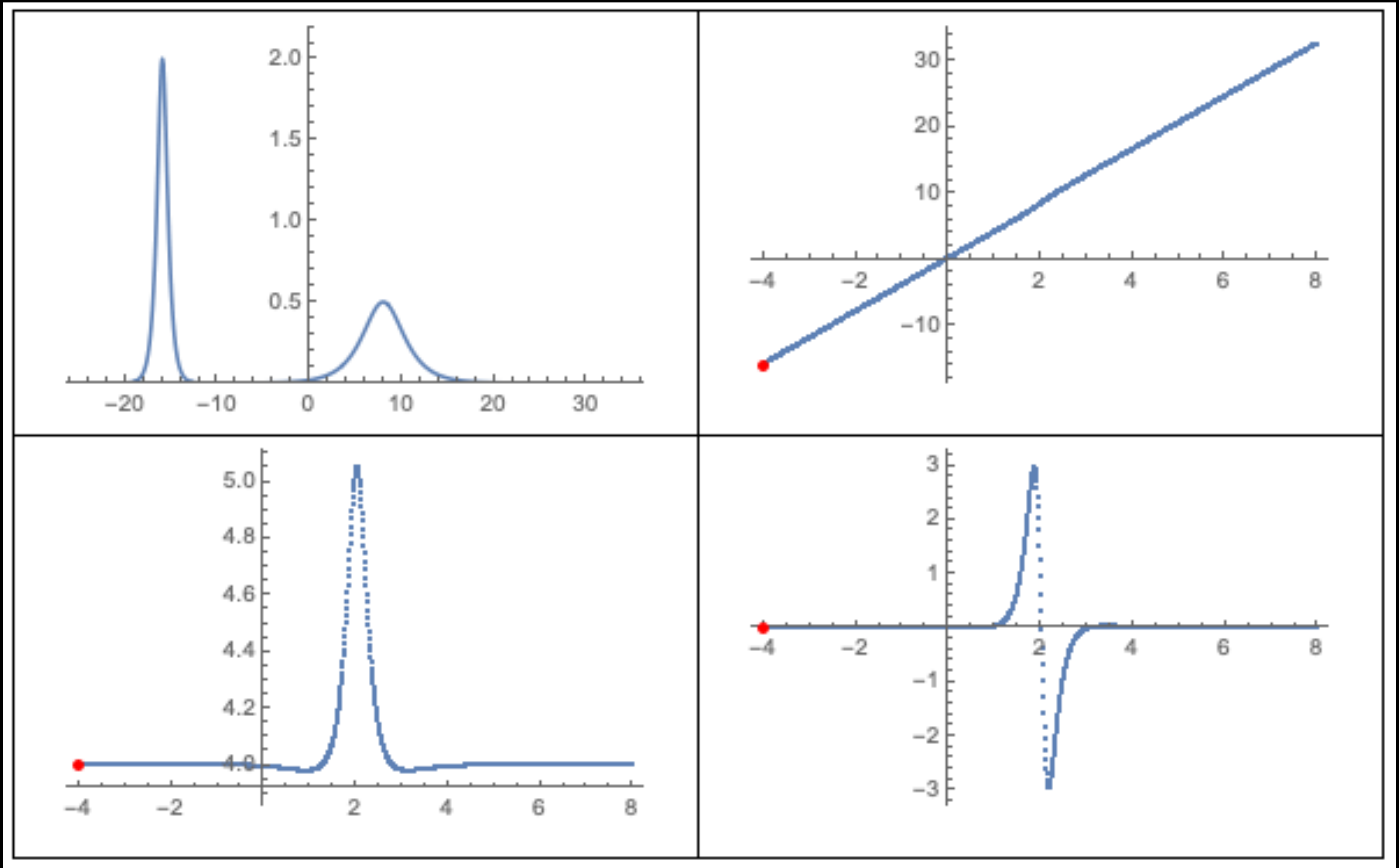


The solution procedure for many integrable PDEs:

- Discover effective Lax pair, spectral data linearizes the flow
- Analyze direct spectral transform to identify key spectral features of initial data
- Evolve forward in time or consider singular limits of parameters
- Analyze inverse spectral transform to obtain phenomena (often asymptotic) about the solution.

Modified focusing KdV equation:

$$u_t + 6u^2 u_x + u_{xxx} = 0 \ , x \in \mathbb{R} \ .$$

Lax pair

$$\Phi_x = \begin{pmatrix} -ik & u(x,t) \\ -u(x,t) & ik \end{pmatrix} \Phi$$

$$\Phi_t = \begin{pmatrix} -4ik^3 + 2iku^2 & 4k^2 u + 2iku_x - 2u^3 - u_{xx} \\ -4k^2 u + 2iku_x + 2u^3 + u_{xx} & 4ik^3 - 2iku^2 \end{pmatrix} \Phi$$

KdV Equation

$$u_t - 6uu_x + u_{xxx} = 0$$

Lax pair

$$-\psi_{xx} + u\psi = E\psi,$$

$$\psi_t - 4\psi_{xxx} + 6u\psi_x + 3u_x\psi = 0$$

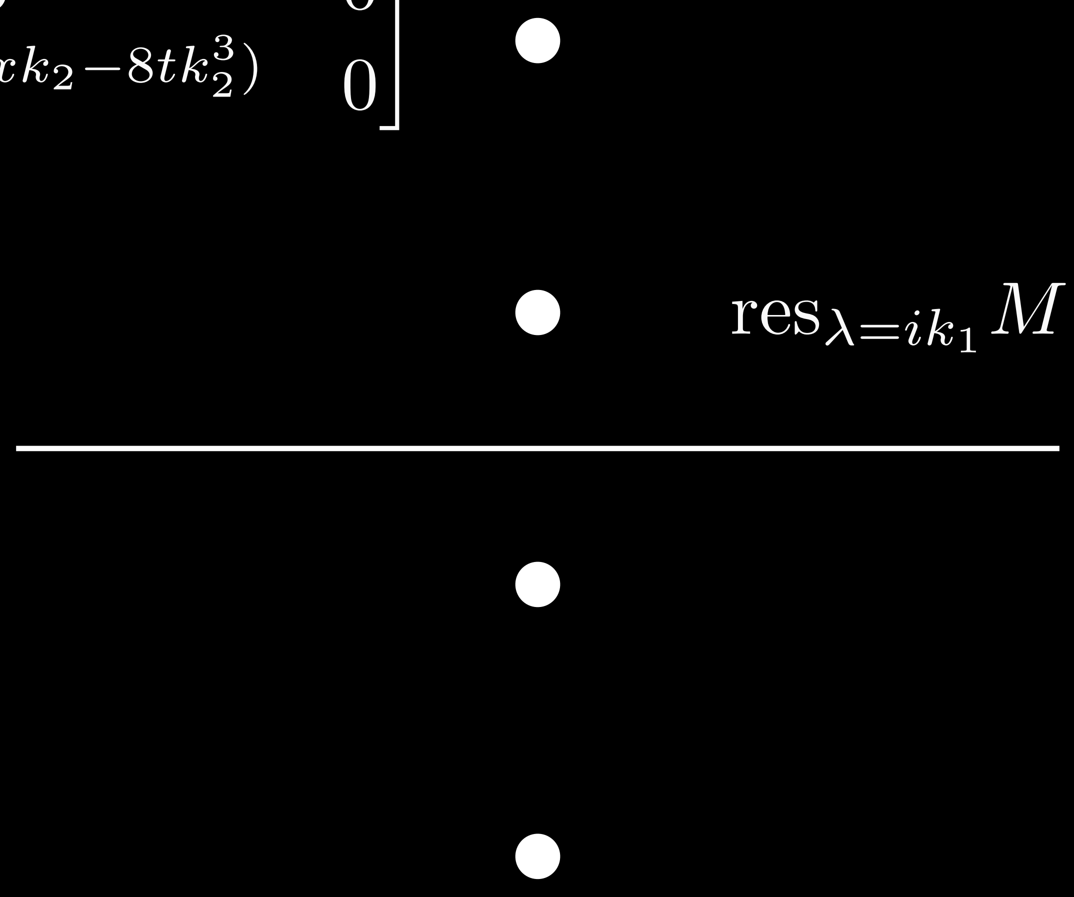
The two soliton solution

The two soliton solution

$$u(x,t) = -\frac{9\left(25e^{\frac{129t}{8}+\frac{x}{2}+5} + 18\sqrt[4]{2}e^{\frac{33t}{4}+2x+10} + 100\ 2^{3/4}e^{8t+3x} + 9e^{\frac{t}{8}+\frac{9x}{2}+5}\right)}{1250\ 2^{3/4}e^{16t+x} + 576e^{\frac{65t}{8}+\frac{5x}{2}+5} + 81\sqrt[4]{2}e^{\frac{t}{4}+4x+10} + 81\sqrt[4]{2}e^{\frac{65t}{4}+10} + 162\ 2^{3/4}e^{5x}}$$

The two soliton solution

$$\mathrm{res}_{\lambda=ik_2} M(\lambda) = \lim_{\lambda \to ik_2} M(\lambda) \begin{bmatrix} 0 & 0 \\ 2ik_2 e^{-(2xk_2 - 8tk_2^3)} & 0 \end{bmatrix}$$



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$$M = (M_1(\lambda), M_2(\lambda)) = (1,1) + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

$$M(-\lambda) = (M_2(\lambda), M_1(\lambda))$$

$$u(x,t) = 2\frac{d}{dx}\left(\lim_{\lambda \to \infty} \frac{\lambda}{i}\left(M_1(\lambda) - 1\right)\right)$$

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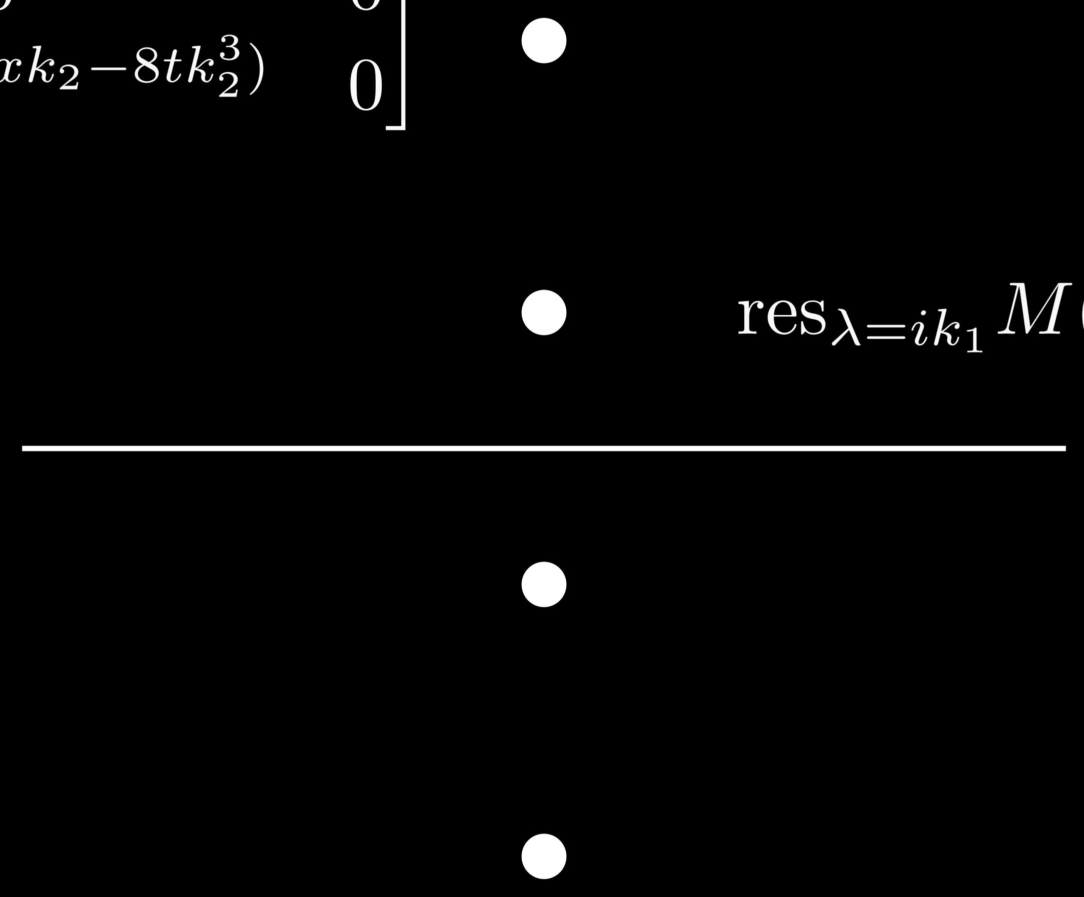

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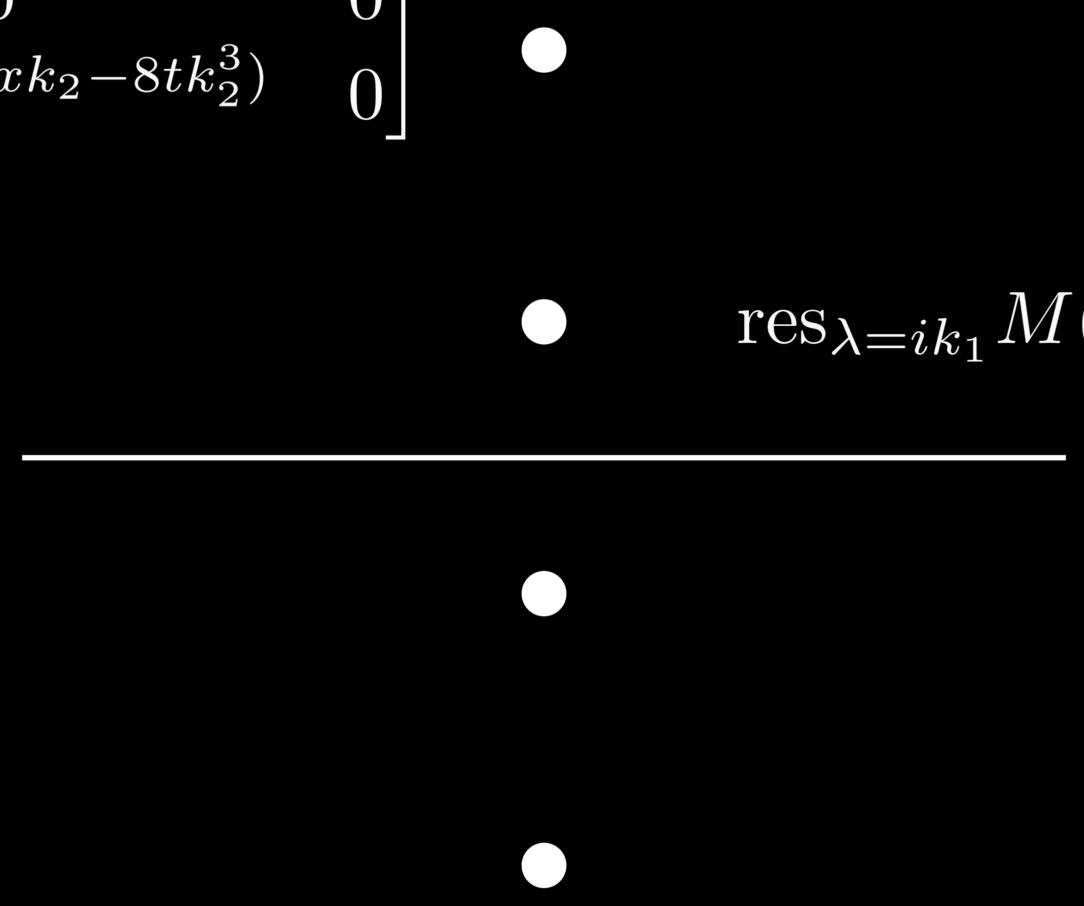

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$\mathcal{O}(1)$

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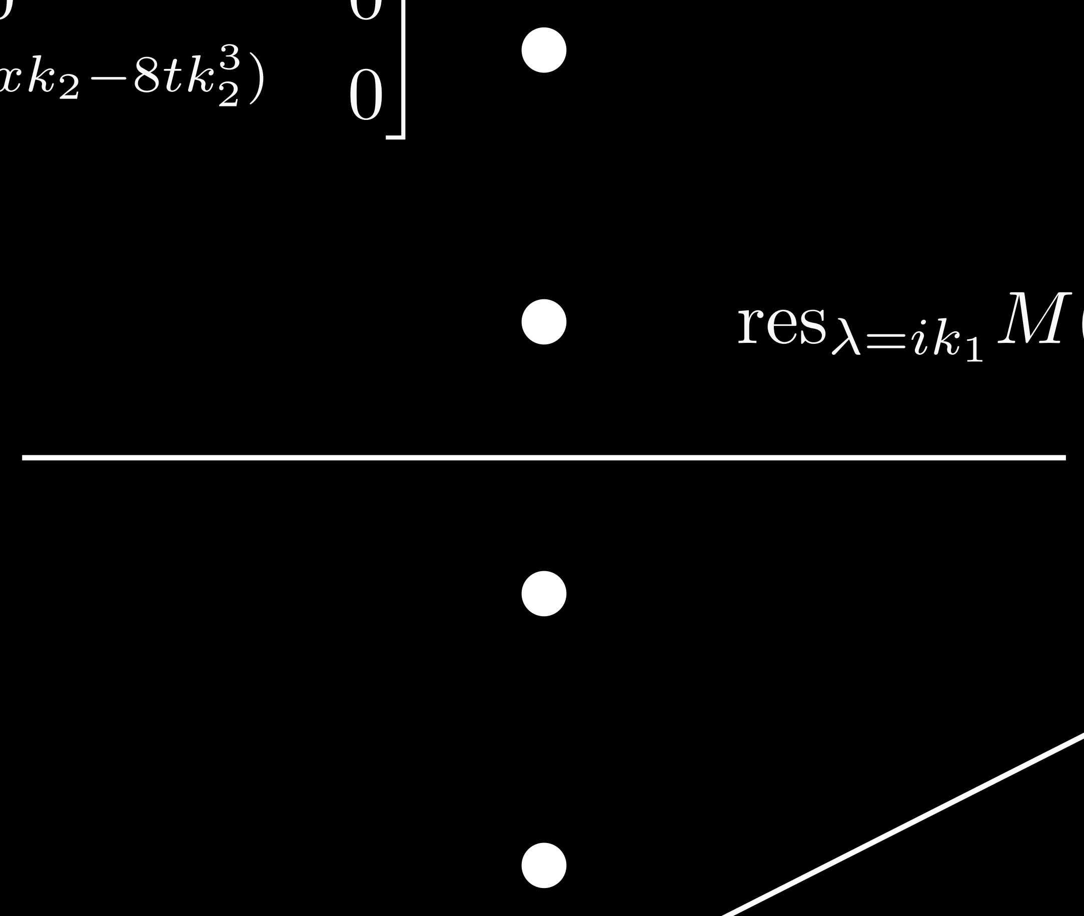

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$t \ll -1$: $\mathcal{O}(1)$

$x \approx 4k_2^2 t$ Exponentially small!

$$\Longrightarrow \quad u \approx 2k_2^2 \mathrm{sech}^2\left(k_2\left(x-4k_2^2 t\right)\right)$$

For x near $4k_2^2 t$

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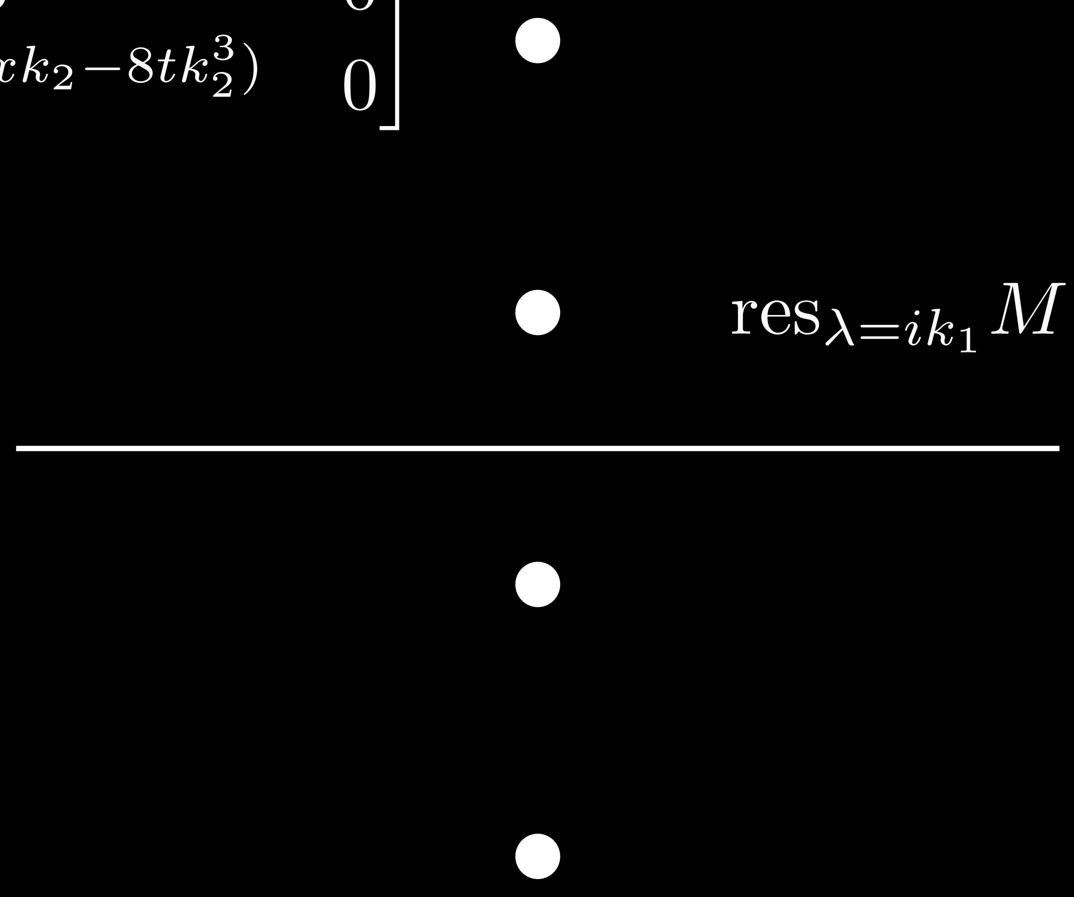

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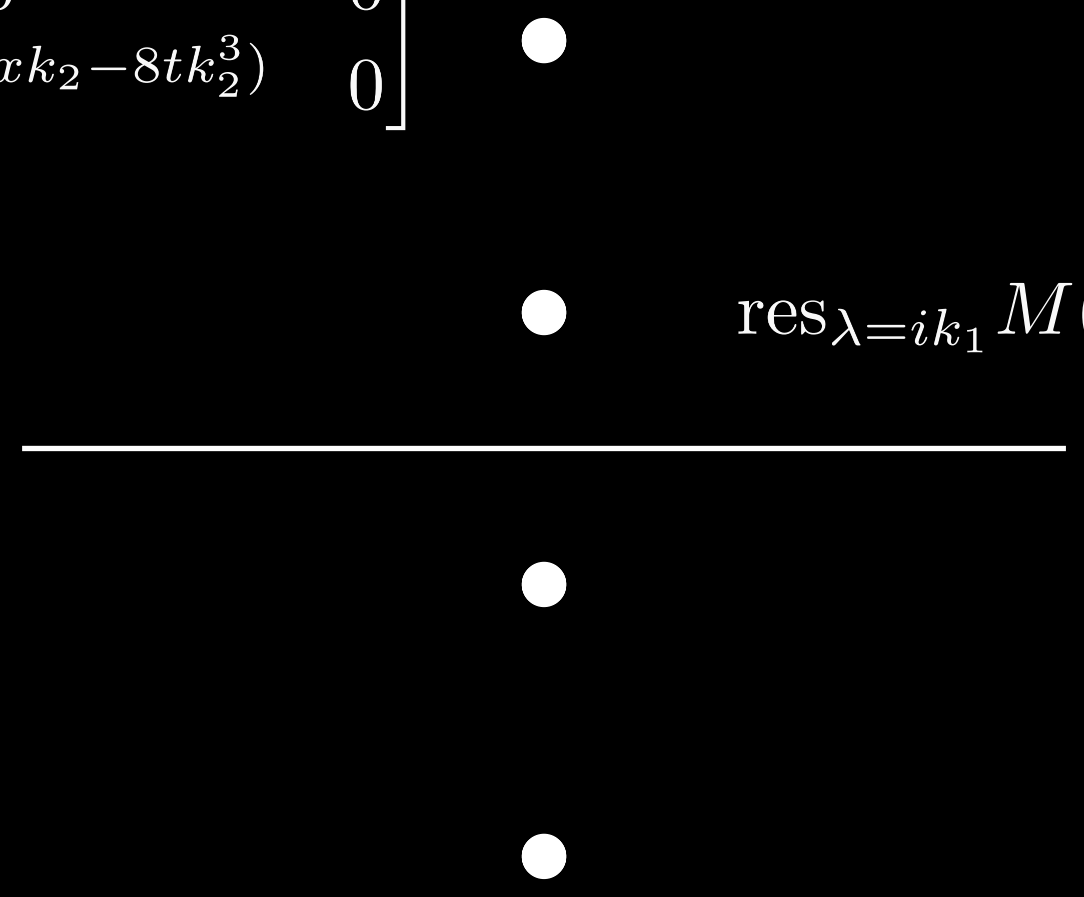

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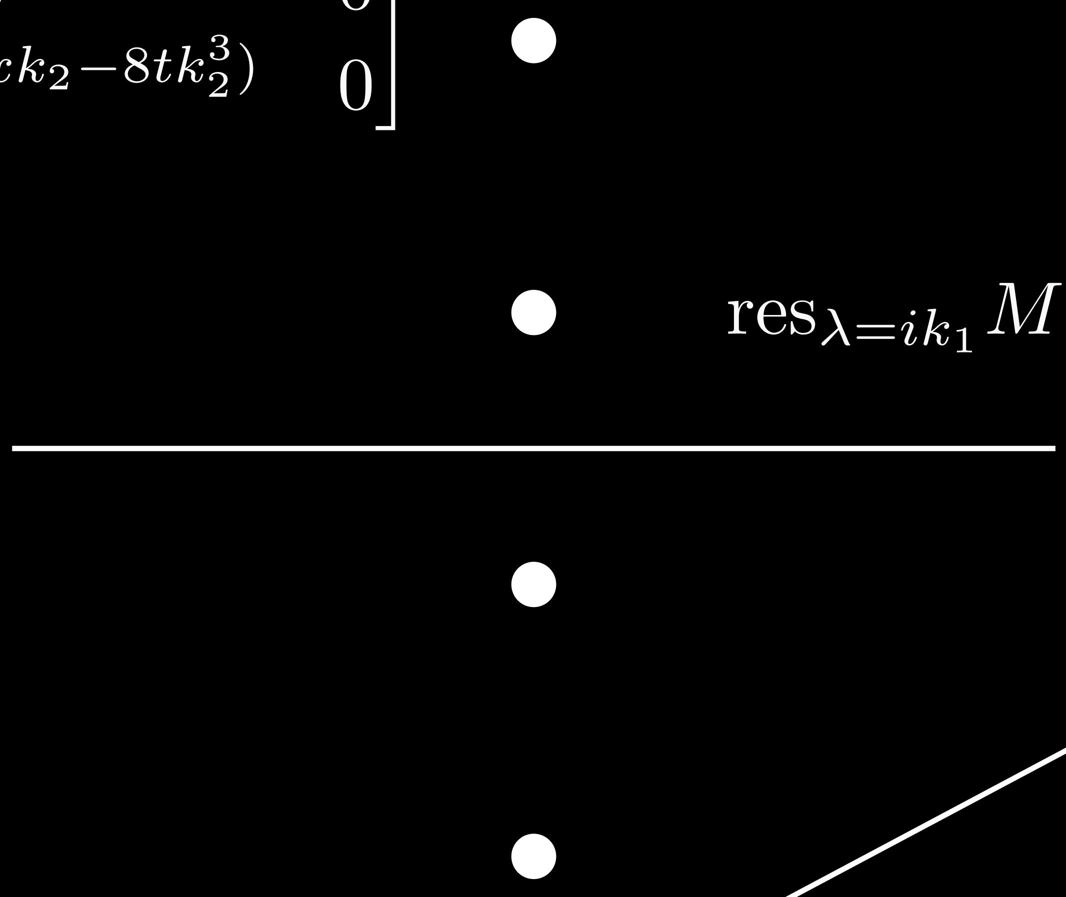

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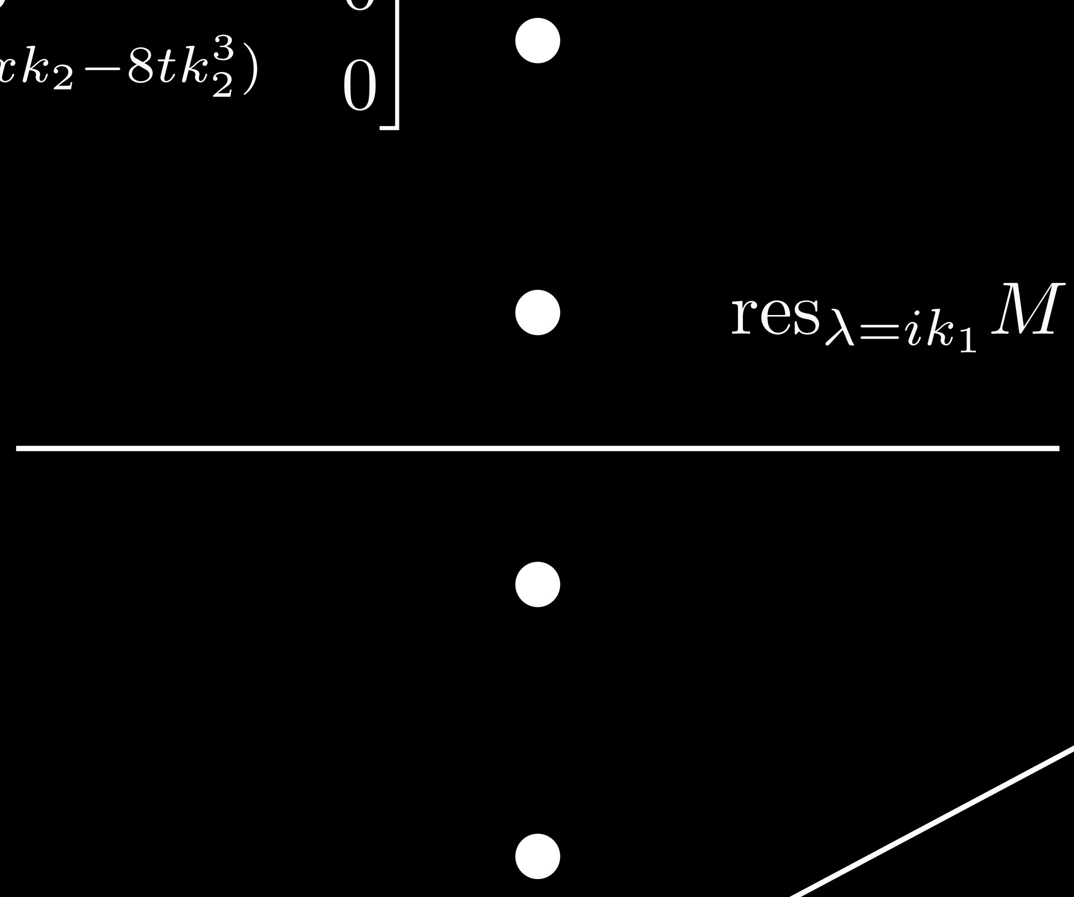

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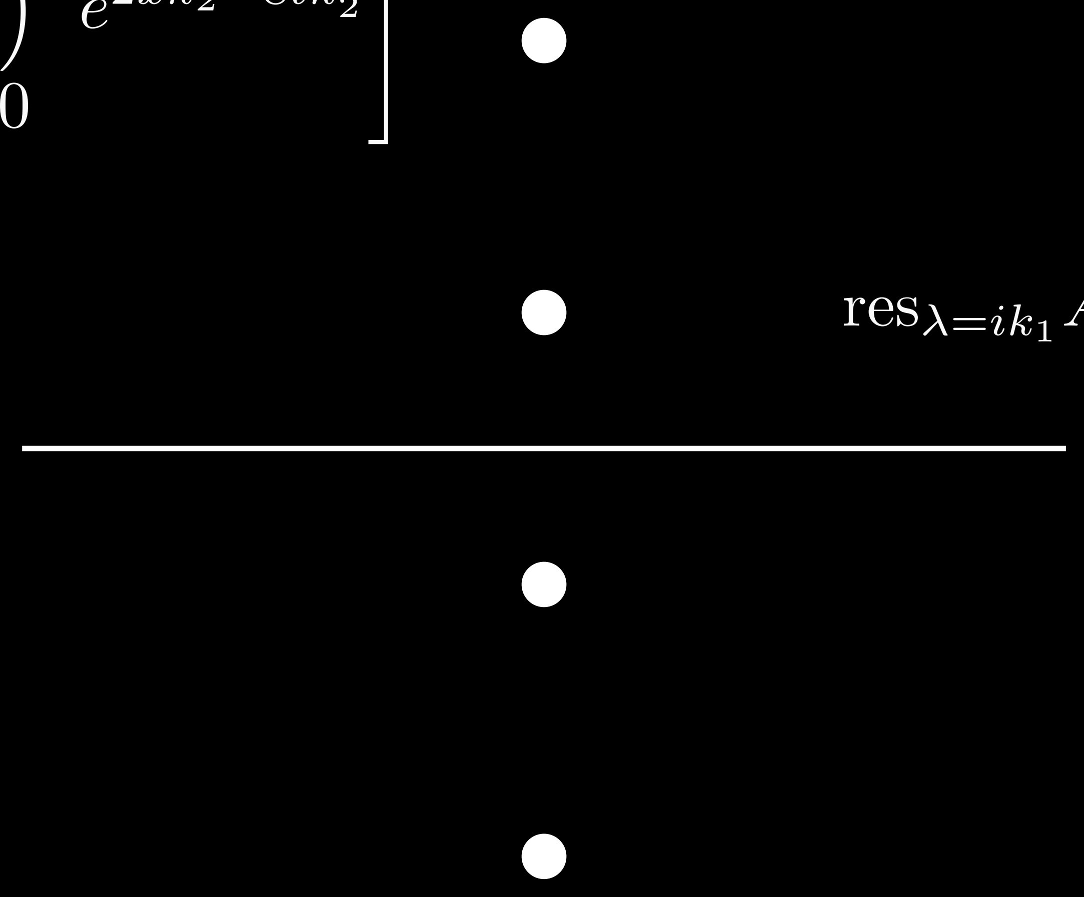
Flip triangularity of the residue conditions

$$A = M \begin{pmatrix} \frac{\lambda-ik_2}{\lambda+ik_2}\frac{\lambda+ik_1}{\lambda-ik_1} & 0 \\ 0 & \frac{\lambda+ik_2}{\lambda-ik_2}\frac{\lambda-ik_1}{\lambda+ik_1} \end{pmatrix}$$

A new meromorphic RHP for A...

The two soliton solution

$$\mathrm{res}_{\lambda=ik_2} A(\lambda) = \lim_{\lambda \to ik_2} A(\lambda) \begin{bmatrix} 0 & 2ik_1 \left(\frac{k_2-k_1}{k_2+k_1}\right)^2 e^{2xk_2-8tk_2^3} \\ 0 & 0 \end{bmatrix}$$



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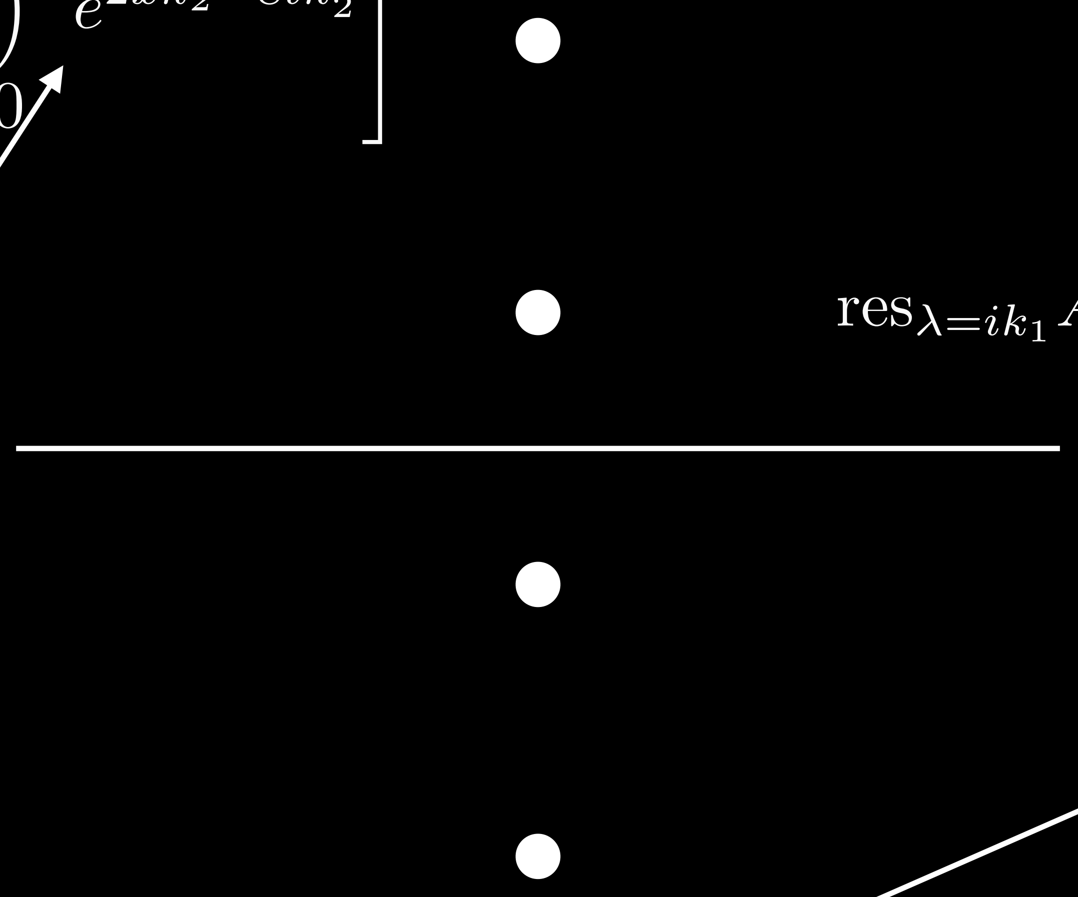

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For x near $4k_2^2t$

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N- Soliton solution

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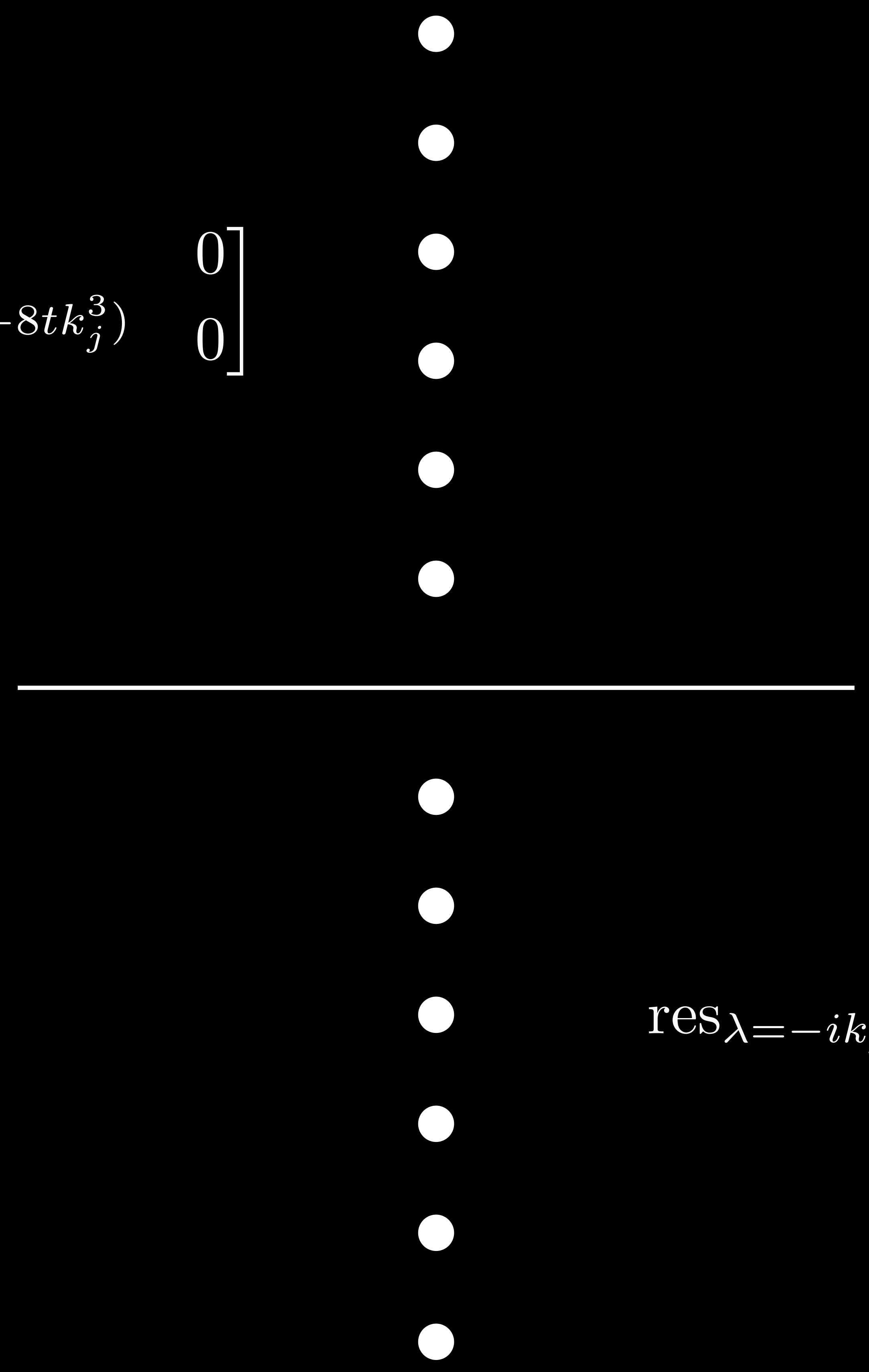

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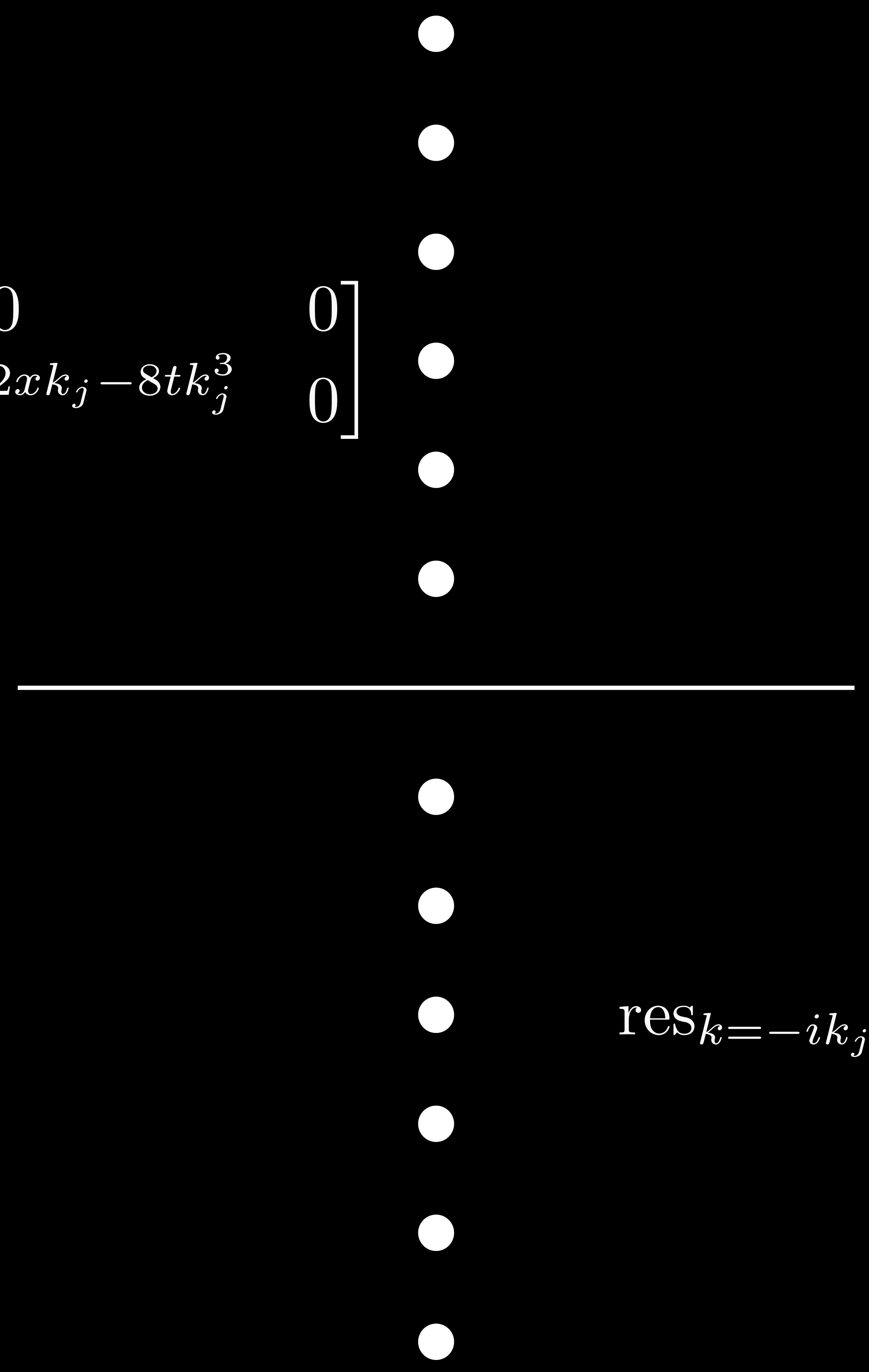
$u_t + 6u^2 u_x + u_{xxx} = 0 \ , x \in \mathbb{R} \ .$

N- Soliton solution MKdV case

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\mathrm{res}_{k=ik_j} M(k) = \lim_{k \to ik_j} M(k) \begin{bmatrix} 0 & 0 \\ N^{-1} c_j e^{2xk_j - 8tk_j^3} & 0 \end{bmatrix}$$



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$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

$$M(k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M(-k) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$u(x,t) = \frac{i}{2} \frac{\partial}{\partial x} \log \det(\mathbb{I} + M) - \frac{i}{2} \frac{\partial}{\partial x} \log \det(\mathbb{I} + M^{\dagger}),$$

$$M_{j\ell} = \sqrt{\chi_j \chi_\ell} \frac{e^{-2i(\theta(\kappa_j,x,t) + \theta(\kappa_\ell,x,t))}}{i(\kappa_j + \kappa_\ell)} \ .$$

$$u(x,t) = 2\imath \lim_{k \to \infty} k(M(k;x,t))_{21}$$

In 1971, Zakharov considered the interaction of a dilute gas of solitons.

Solitons with different velocities interact by exchanging their velocities, but: they emerge after the interaction with slightly shifted positions.

The fundamental calculation: prepare 2 solitons:

$$u(x,t) \approx 12\eta_1^2 \mathrm{sech}^2\left(\eta_1(x-4\eta_1^2 t)\right) + 12\eta_2^2 \mathrm{sech}^2\left(\eta_2(x-4\eta_2^2 t)\right), \quad t \to -\infty ,$$

Then:

$$u(x,t) \approx 12\eta_1^2 \mathrm{sech}^2\left[\eta_1\left(x-4\eta_1^2 t + \frac{1}{\eta_1}\log\left|\frac{\eta_2+\eta_1}{\eta_2-\eta_1}\right|\right)\right] + \\ +12\eta_2^2 \mathrm{sech}^2\left[\eta_2\left(x-4\eta_2^2 t - \frac{1}{\eta_2}\log\left|\frac{\eta_2+\eta_1}{\eta_2-\eta_1}\right|\right)\right], \quad t \to \infty ,$$

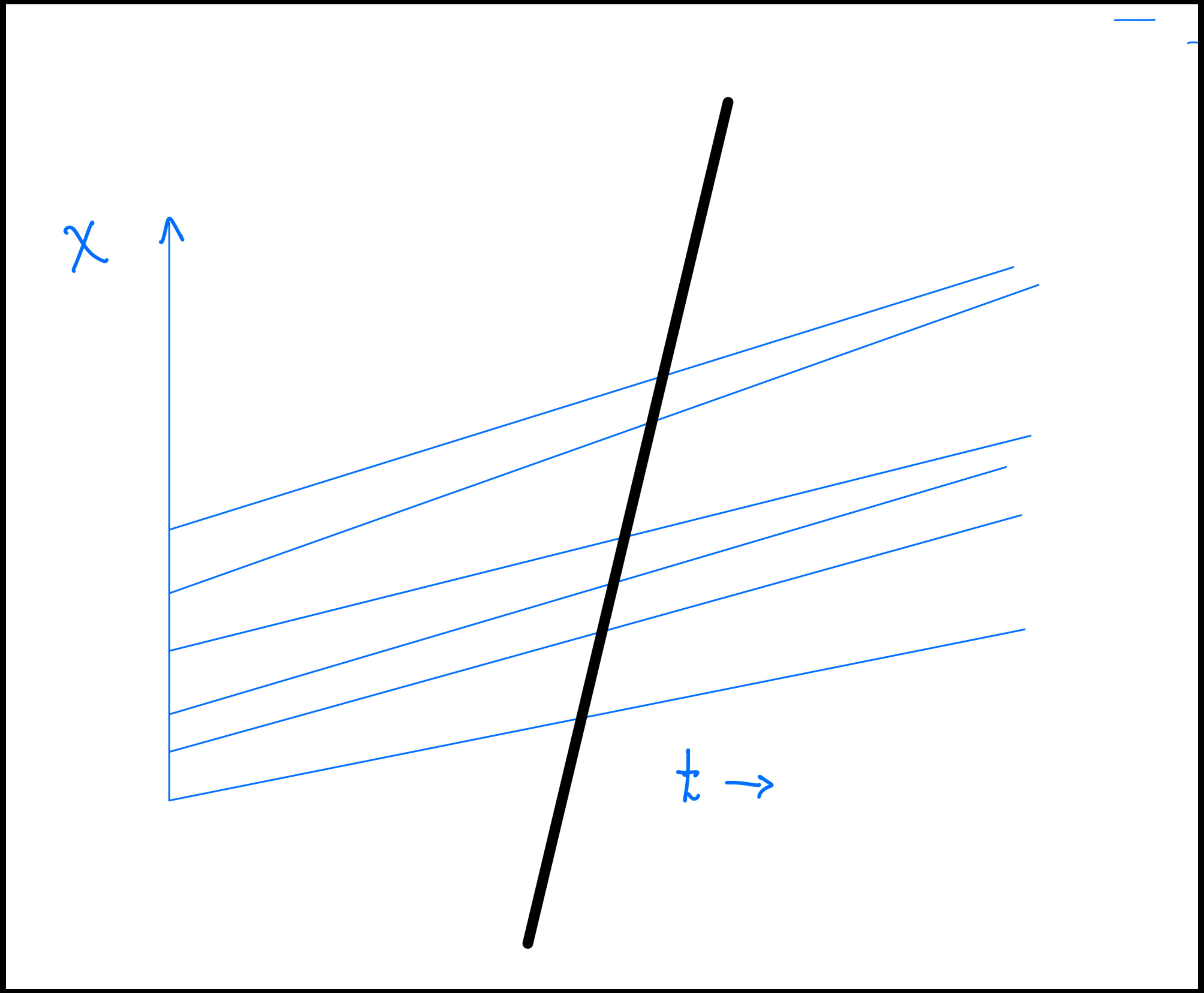
Dilute gas of solitons with "density" $f(\eta, x, t)$. Launch a "tracer":

isolated pairwise interactions: accumulated shifts alter tracer velocity.

Derives an equation for tracer velocity:

$$s(\eta) = 4\eta^2 + \frac{4}{\eta}\int_0^\infty \ln\left|\frac{\eta_1+\eta}{\eta_1-\eta}\right|(\eta^2-\eta_1^2) f \, d\eta_1$$

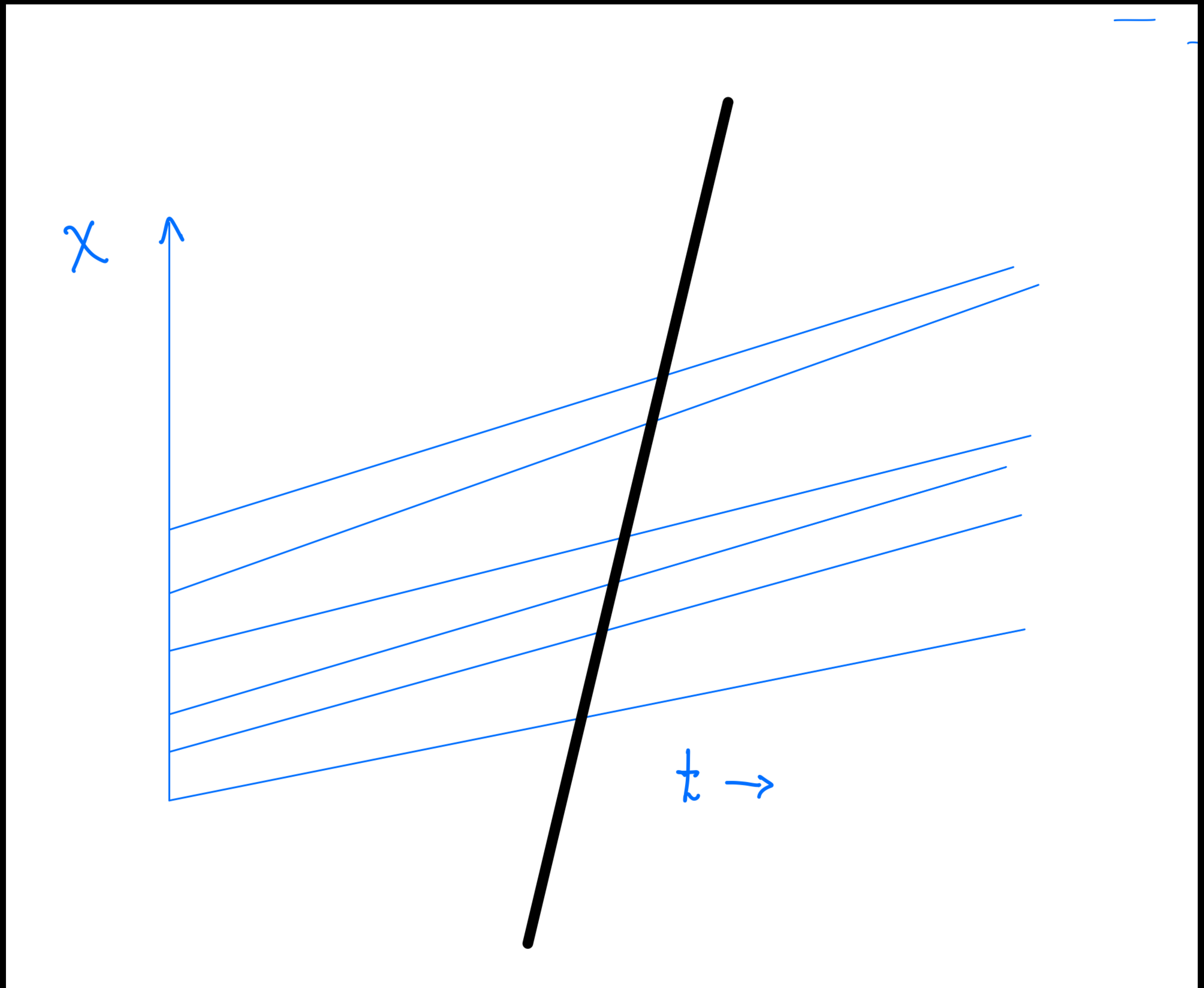
Continuity equation for f: $f_t + (s(\eta) f)_x = 0$.



Fast soliton is cruising...

Encounters a dilute gas of slow solitons

Gas: well-separated solitons, described by a density $f(\eta, X)$



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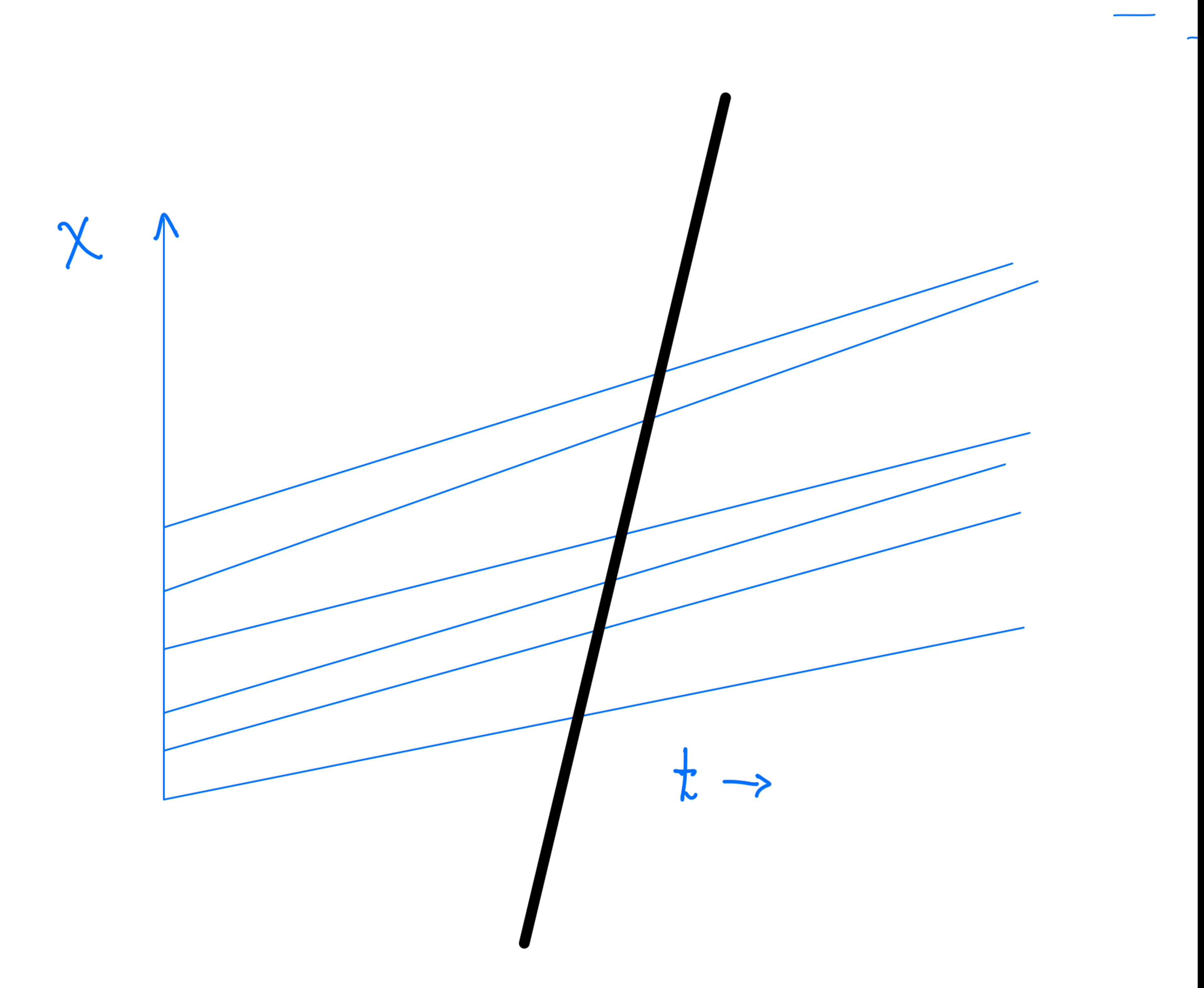
In time ΔT: accumulated shift in the position of the fast soliton:

$$\frac{1}{\eta_0}\sum \ln\left(\frac{\eta_0+\eta_j}{\eta_0-\eta_j}\right) \approx \sum_{\eta} \frac{1}{\eta_0} \ln\left(\frac{\eta_0+\eta}{\eta_0-\eta}\right) \cdot \#$$



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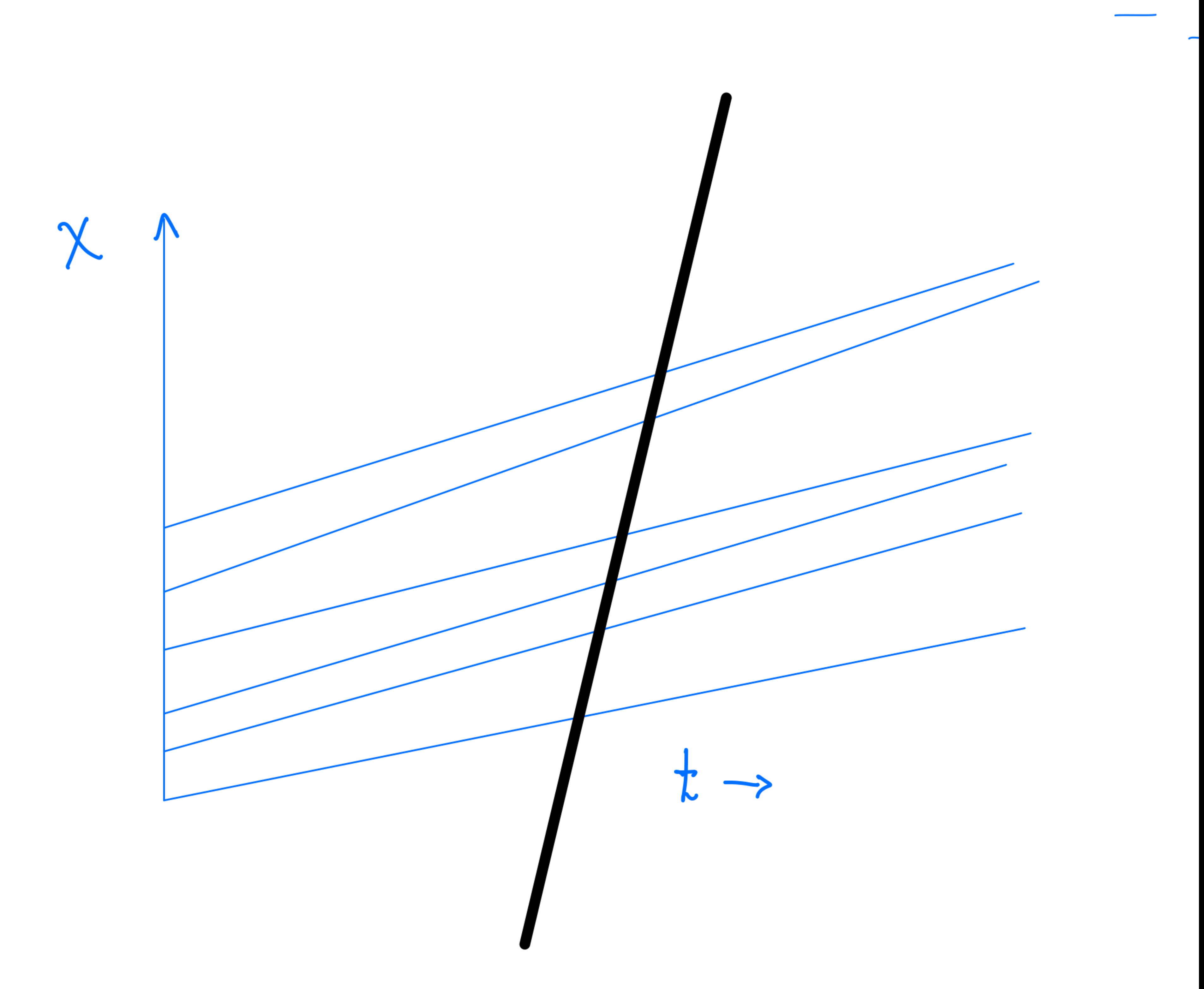

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Yielding a shift in the velocity of the fast soliton

$$v_{\text{tracer}} \approx 4\eta_0^2 + \int \frac{1}{\eta_0}\ln\left(\frac{\eta_0+\eta}{\eta_0-\eta}\right)\cdot(4\eta_0^2-4\eta^2) f(\eta, X)\, d\eta$$

Gennady El (2003) extracted integral eqn for tracer velocity for a dense gas:

$$s(\eta) = 4\eta^2 + \frac{4}{\eta}\int_0^\infty \ln\left|\frac{\eta_1+\eta}{\eta_1-\eta}\right| f(\eta_1)(s(\eta)-s(\eta_1))d\eta_1$$
$$f_t + (sf)_x = 0\ .$$

Extended in several directions, by El & Kamchatnov, El & Tovbis, Pelinovsky & Shurgalina, Pelinovsky & Dutykh, and others...

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These same equations describe a *regular* gas of solitons.

1983: Lax and Levermore described the small dispersion limit of KdV:

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0 \ , \quad u(x,0) = u_0(x) \ .$$

Fact: the (infinite) solitonic component of the spectral data drives the dynamics.

The point: a well-prepared "gas" of solitions yields similar kinetic equations.

2016: Dyachenko, Zakharov, and Zakharov: constructed new class of potentials

A continuum limit of solitons with a spectral gap:

$$\mathrm{res}_{k=i\kappa_j} M(k) = \lim_{k\to i\kappa_j} M(k) \begin{bmatrix} 0 & 0 \\ \frac{-\imath\chi_j}{N} e^{-2\imath\theta(k,x,t)} & 0 \end{bmatrix}$$

N Poles accumulating here, $N \to \infty$

$$\mathrm{res}_{k=-i\kappa_j} M(k) = \lim_{k\to -i\kappa_j} M(k) \begin{bmatrix} 0 & \frac{-\imath\chi_j}{N} e^{2\imath\theta(k,x,t)} \\ 0 & 0 \end{bmatrix}$$

χ_j: discretization of a smooth function.

Actually one case in Dyachenko, Zakharov, and Zakharov, adapted to MKdV

Subsequent work by P. Nabelek, and Nabelek, Zakharov, and Zakharov (2020-21)

A Riemann-Hilbert problem emerges in the limit

$$\boldsymbol{X}_+(k) = \boldsymbol{X}_-(k) \begin{bmatrix} 1 & 0 \\ ir(k)e^{-2i\theta(k,x,t)} & 1 \end{bmatrix}$$

$\theta(k,x,t) = 4tk^3 + xk$

$$\boldsymbol{X}_+(k) = \boldsymbol{X}_-(k) \begin{bmatrix} 1 & i\overline{r(\bar{k})}e^{2i\theta(k,x,t)} \\ 0 & 1 \end{bmatrix} \quad k \in \Sigma_2$$

This is the MKdV case.

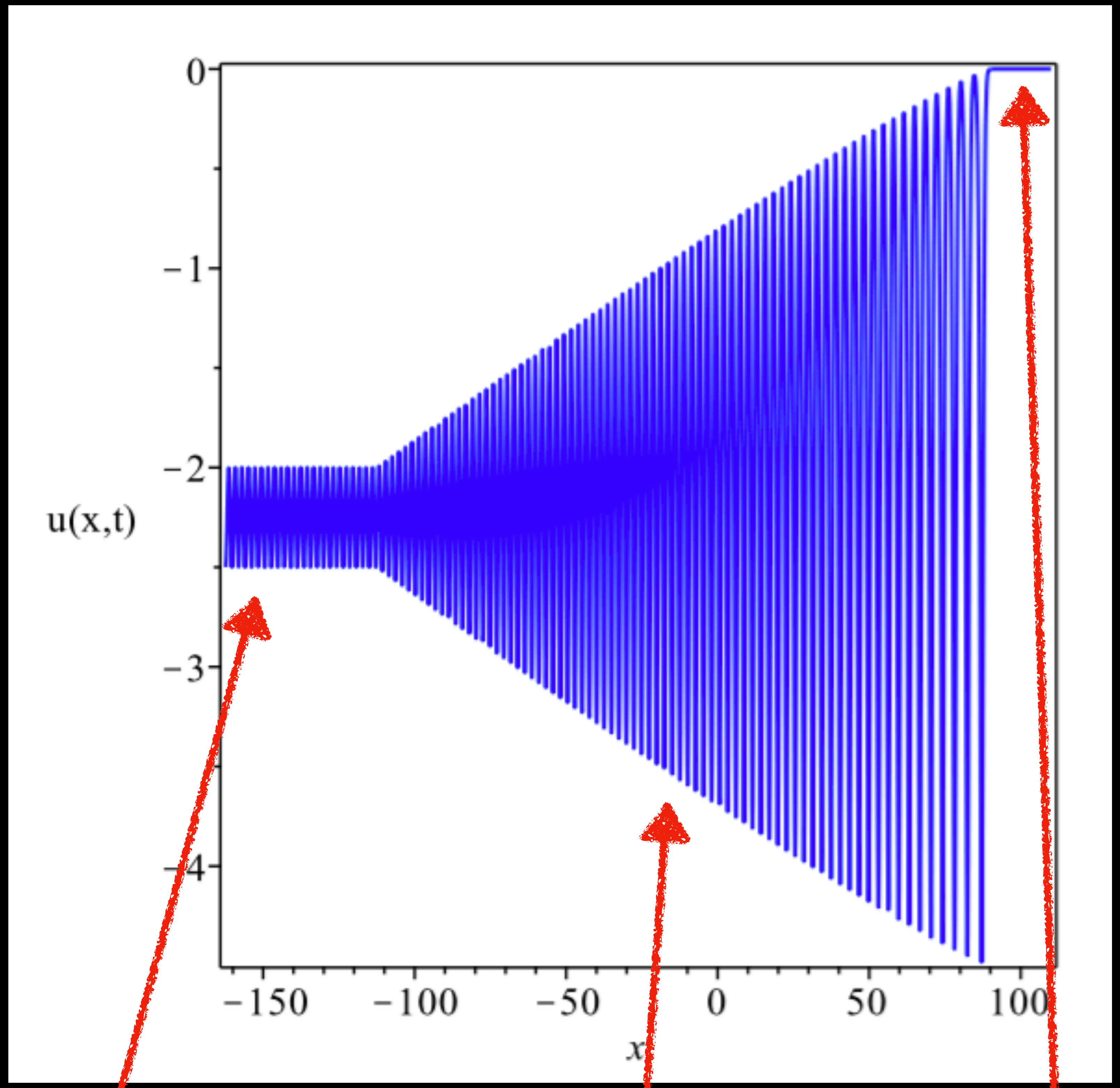
KdV is very similar

Claim: these are really soliton gasses.

Justification: long-time asymptotics produces densities solving kinetic equation.

Long time behavior of $u(x,t)$ solving KdV, obtained from above RHP

Dynamics driven by g-function.



Quiescent region

Modulating cnoidal wave

elliptic wave

(Girotti, Grava, Jenkins, KM, ’20)

(MKdV case: Girotti, Grava, Jenkins, KM, Minakov, ’21)

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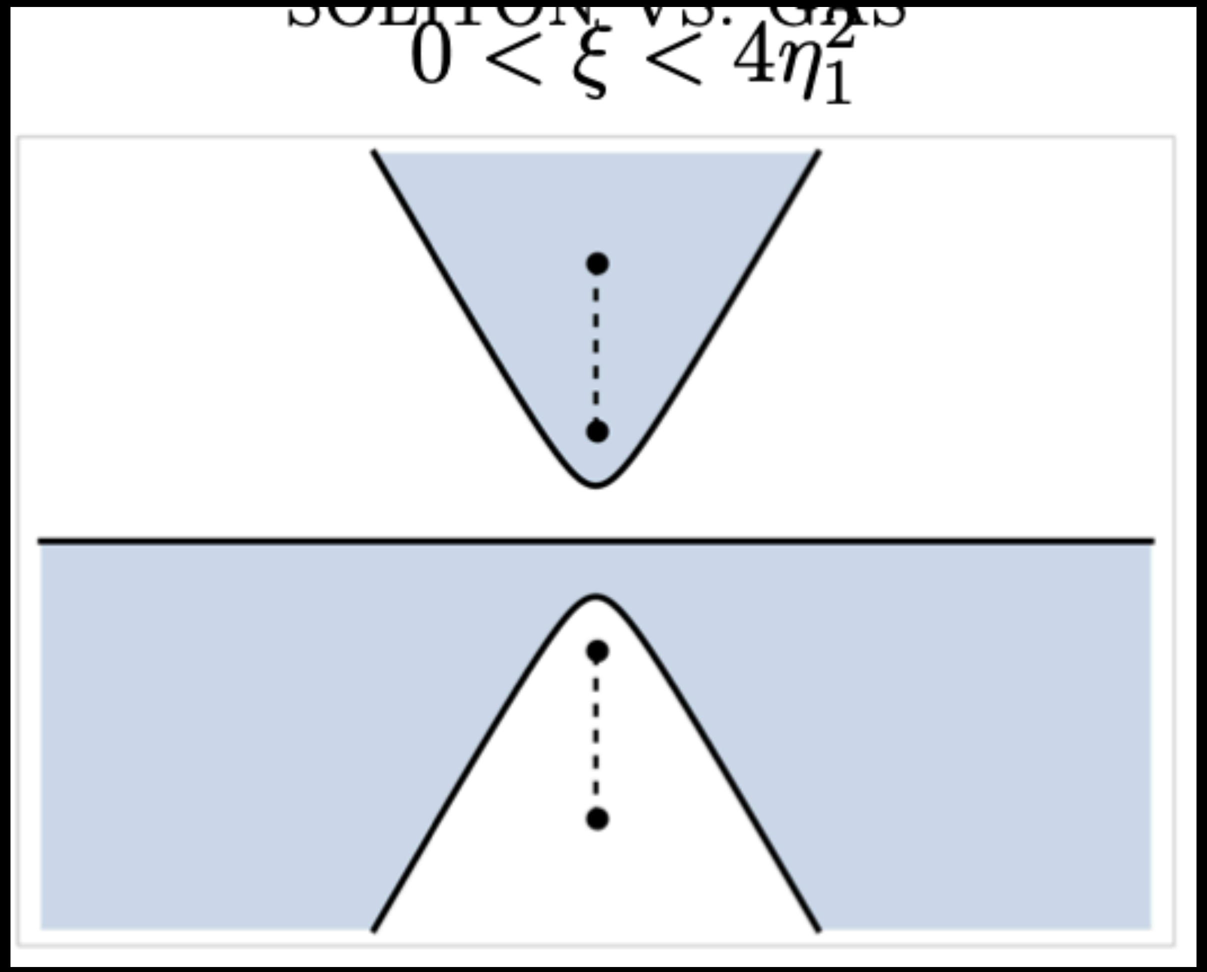
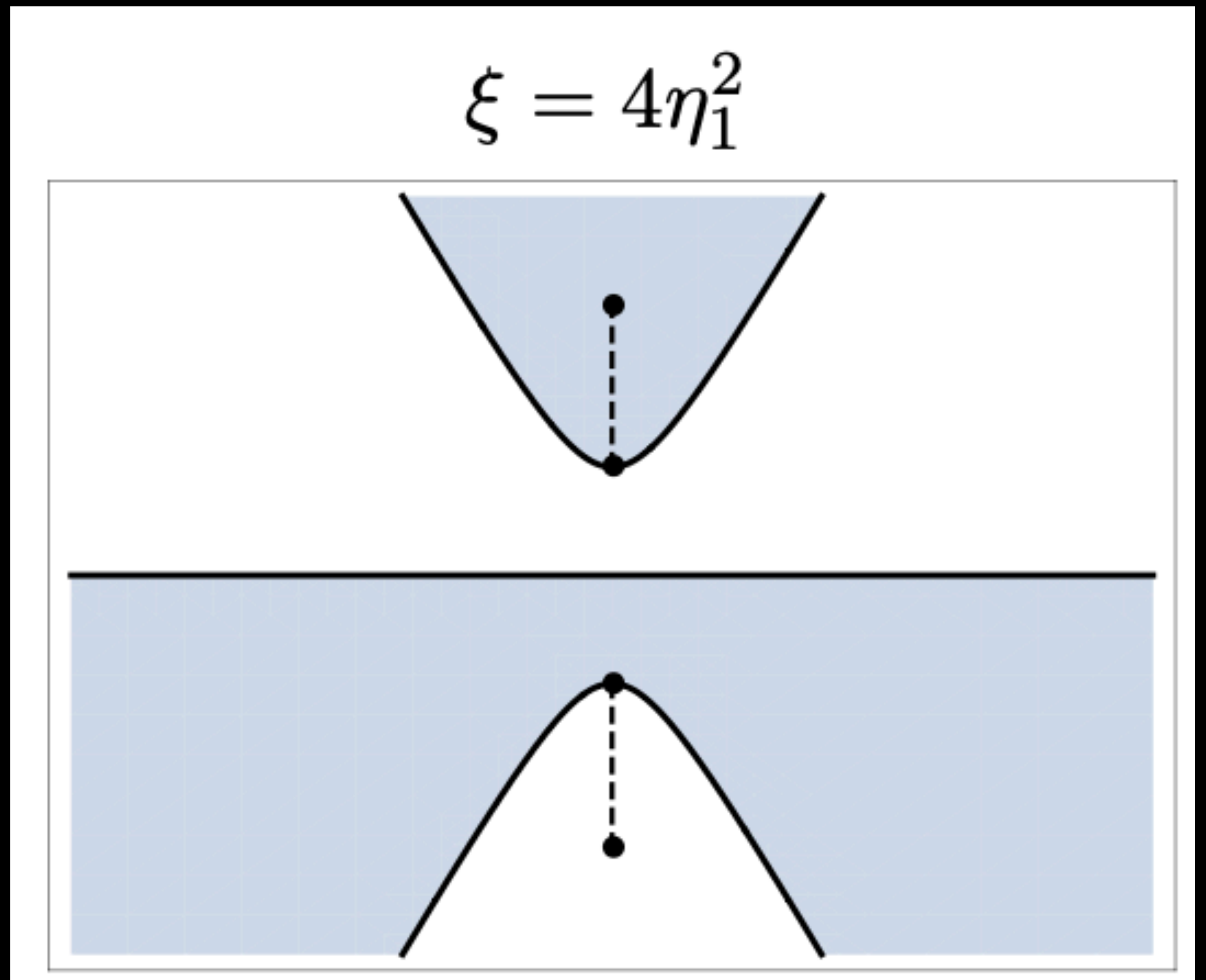
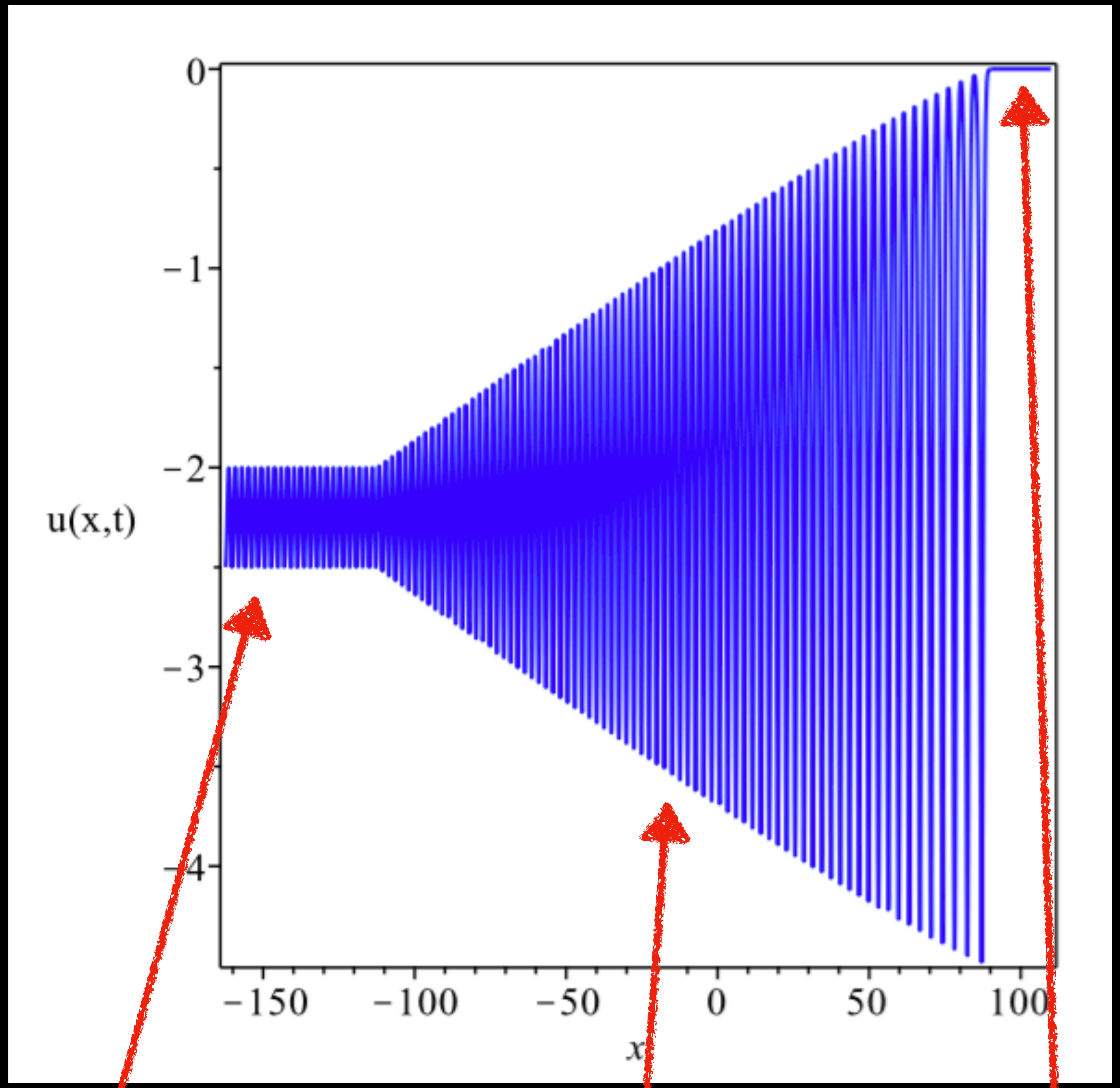


Zero level of Im $(\theta) =$ Im $(4tk^3 + xk)$





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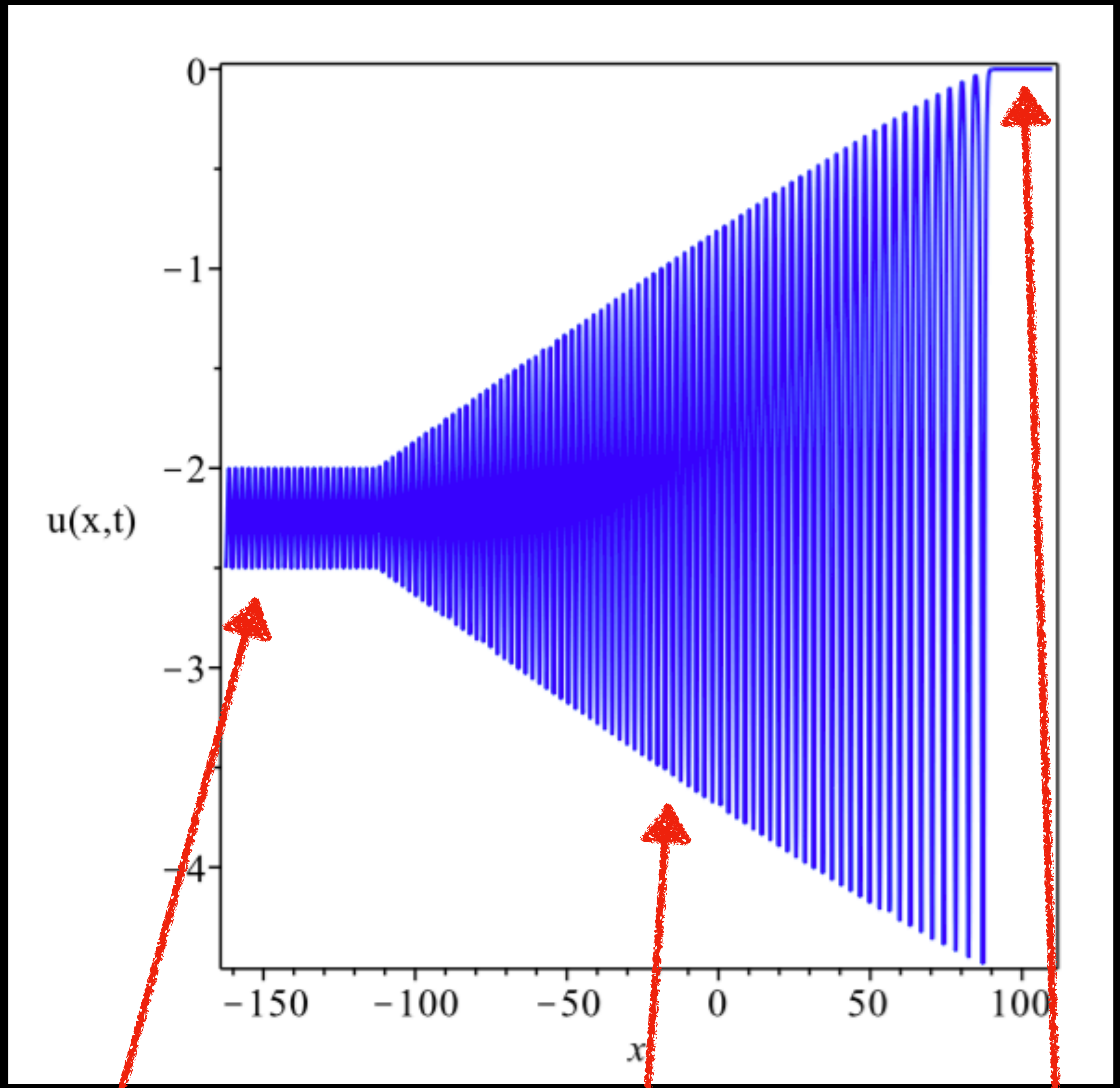
Dynamics driven by g-function.

$$g = \int_{\Sigma_1} \log\left(\frac{k-s}{k+s}\right)\rho(s)ds$$

Determined by the usual variational equations:

$$2\int_{\Sigma_1} \log\left|\frac{k-s}{k+s}\right|\rho(s)ds = -8tk^3 - 2xk\ , \quad k \in \mathcal{S}$$

Along with appropriate variational inequalities



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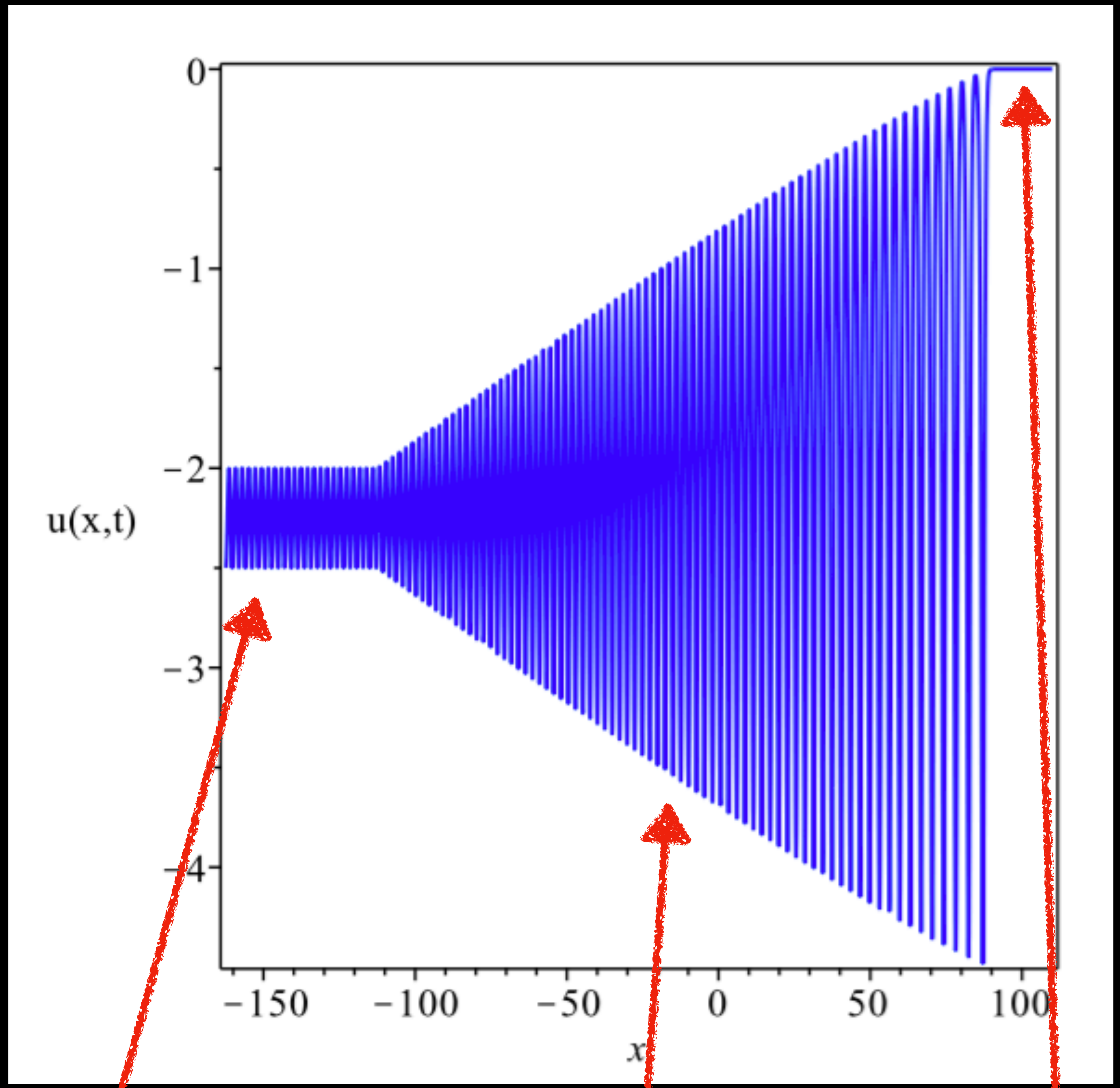
Fact: $v = -\frac{\rho_t(s)}{\rho_x(s)}$ solves the kinetic equation

$$v(k) = -4k^2 + \frac{1}{k}\int_{\Sigma_1} \log\left|\frac{k-s}{k+s}\right|(v(s)-v(k))u(s)ds\ .$$



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Soliton vs Gas

$i\kappa_0$

$$\mathrm{Res}_{k=i\kappa_0}\boldsymbol{X}(k,x,t) = \lim_{k\to i\kappa_0}\boldsymbol{X}(k,x,t)\begin{bmatrix}0 & 0\\ -i\chi e^{-2i\theta(k,x,t)} & 0\end{bmatrix}$$

$$\boldsymbol{X}_+(k) = \boldsymbol{X}_-(k)\begin{bmatrix}1 & 0\\ ir(k)e^{-2i\theta(k,x,t)} & 1\end{bmatrix}$$

$$\theta(k,x,t) = 4tk^3 + xk$$

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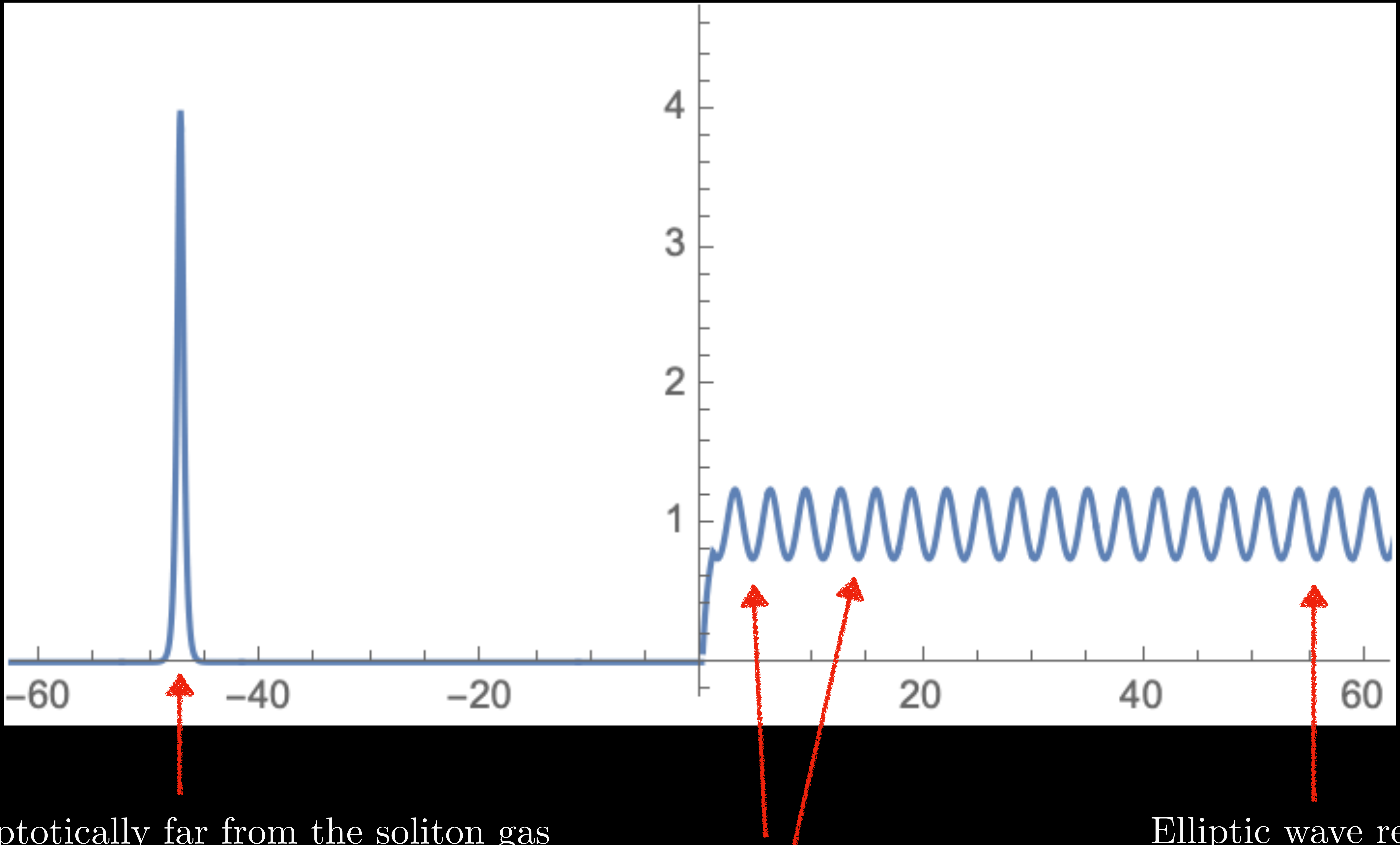
$$\mathrm{Res}_{k=-i\kappa_0}\boldsymbol{X}(k,x,t) = \lim_{k\to -i\kappa_0}\boldsymbol{X}(k,x,t)\begin{bmatrix}0 & -i\chi e^{2i\theta(k,x,t)}\\ 0 & 0\end{bmatrix}$$

$-i\kappa_0$

Soliton vs Gas



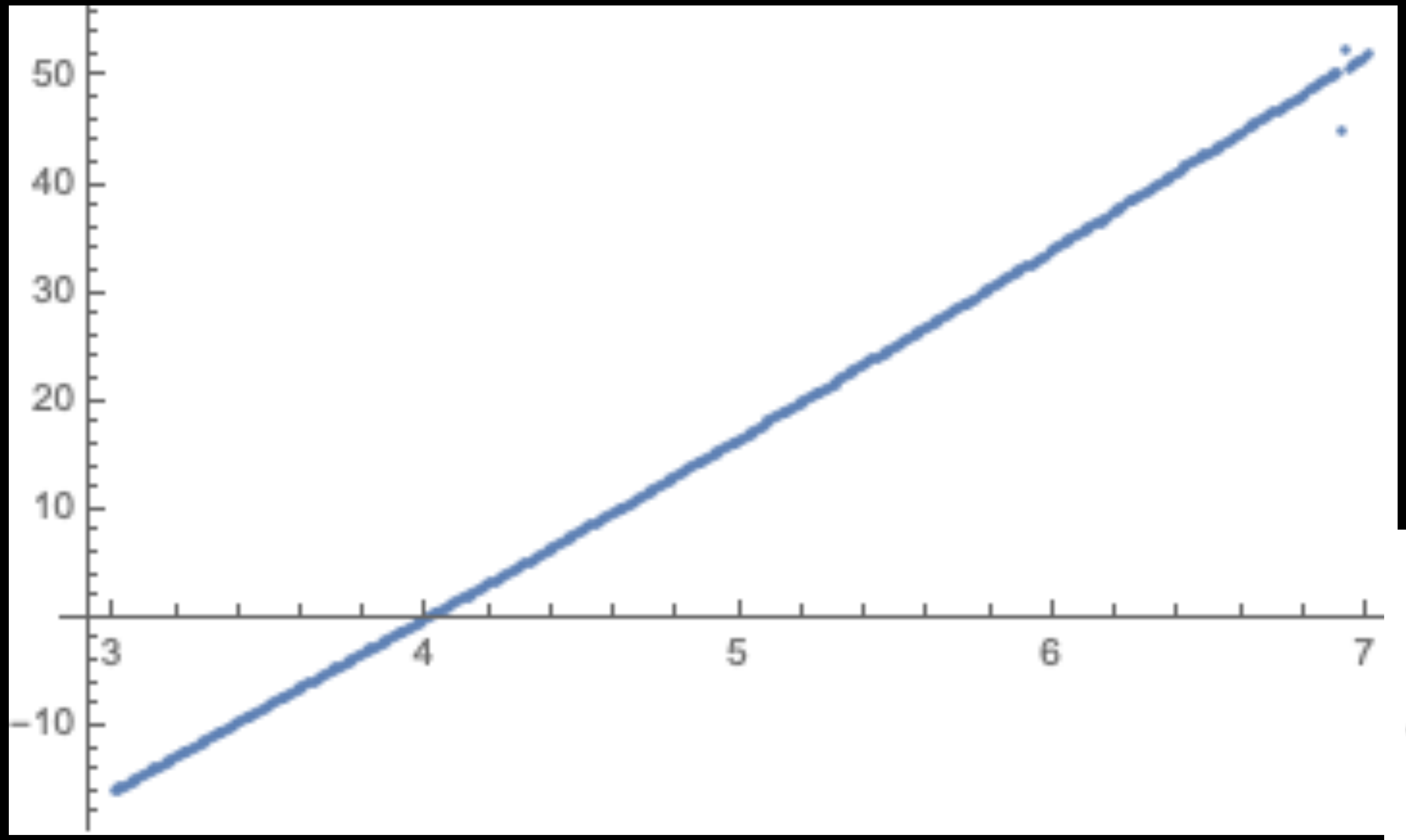
Soliton initialized asymptotically far from the soliton gas

Elliptic wave region

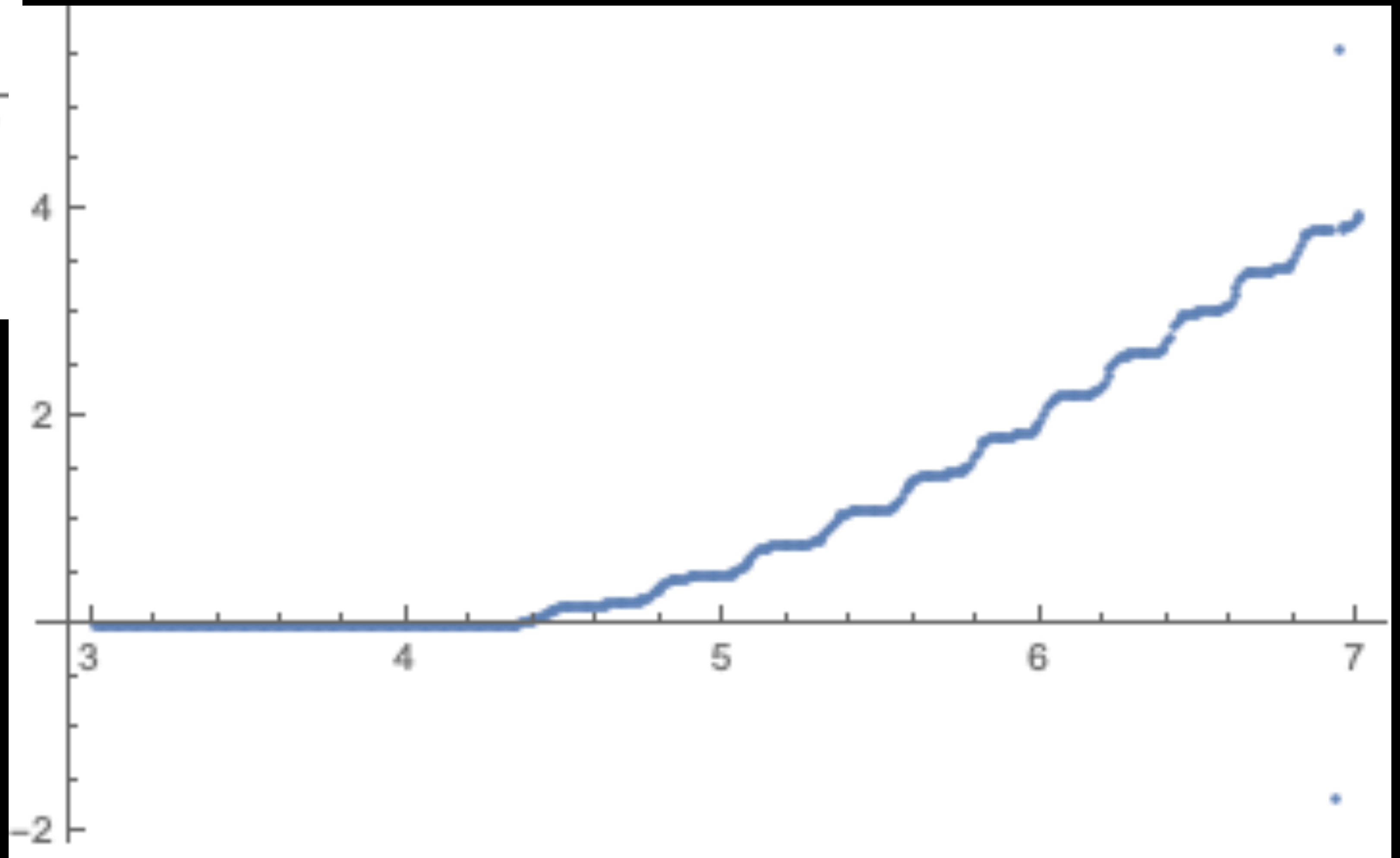
Modulated elliptic wave region

Soliton vs Gas

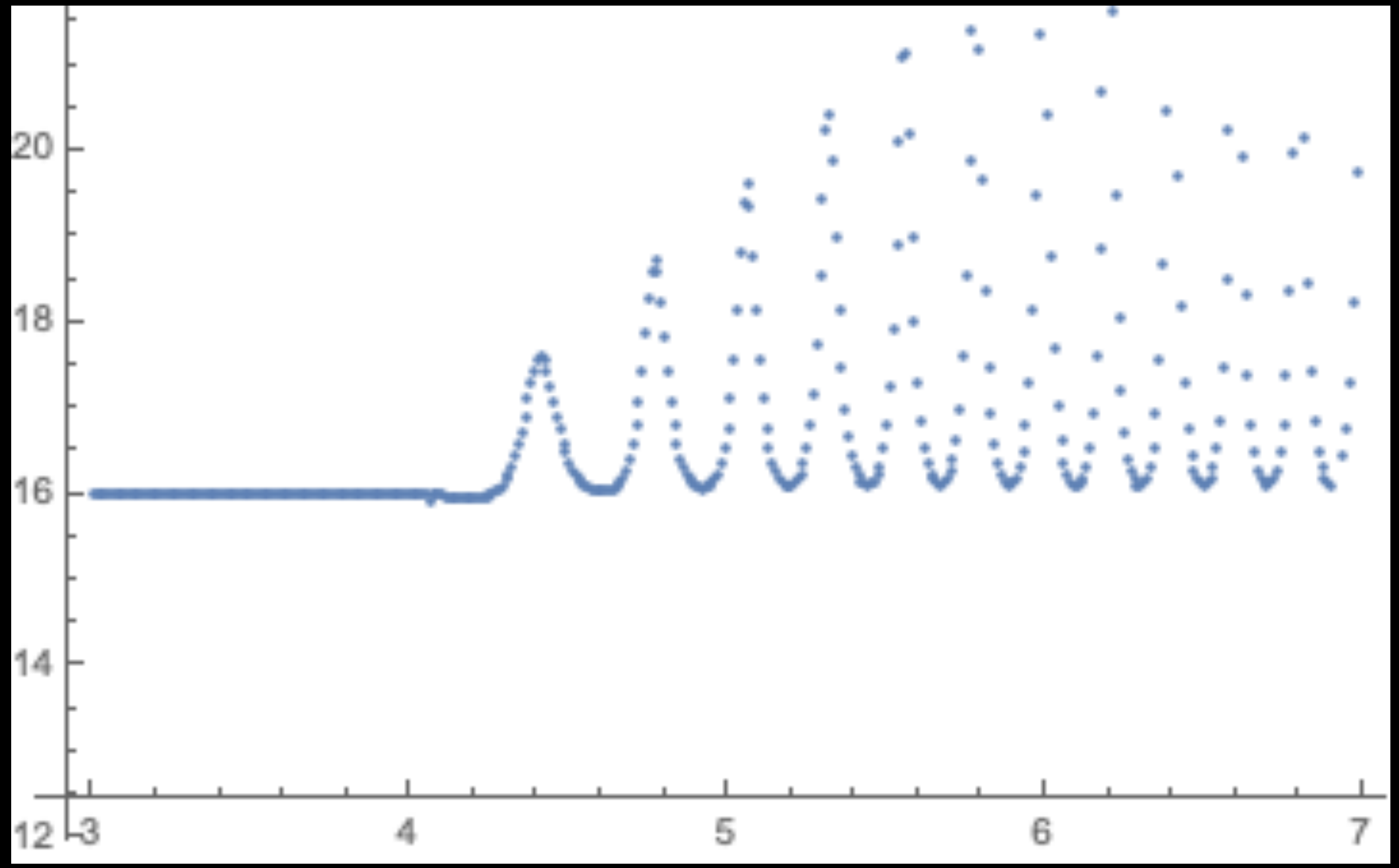
Soliton peak location



after subtracting location of vacuum soliton

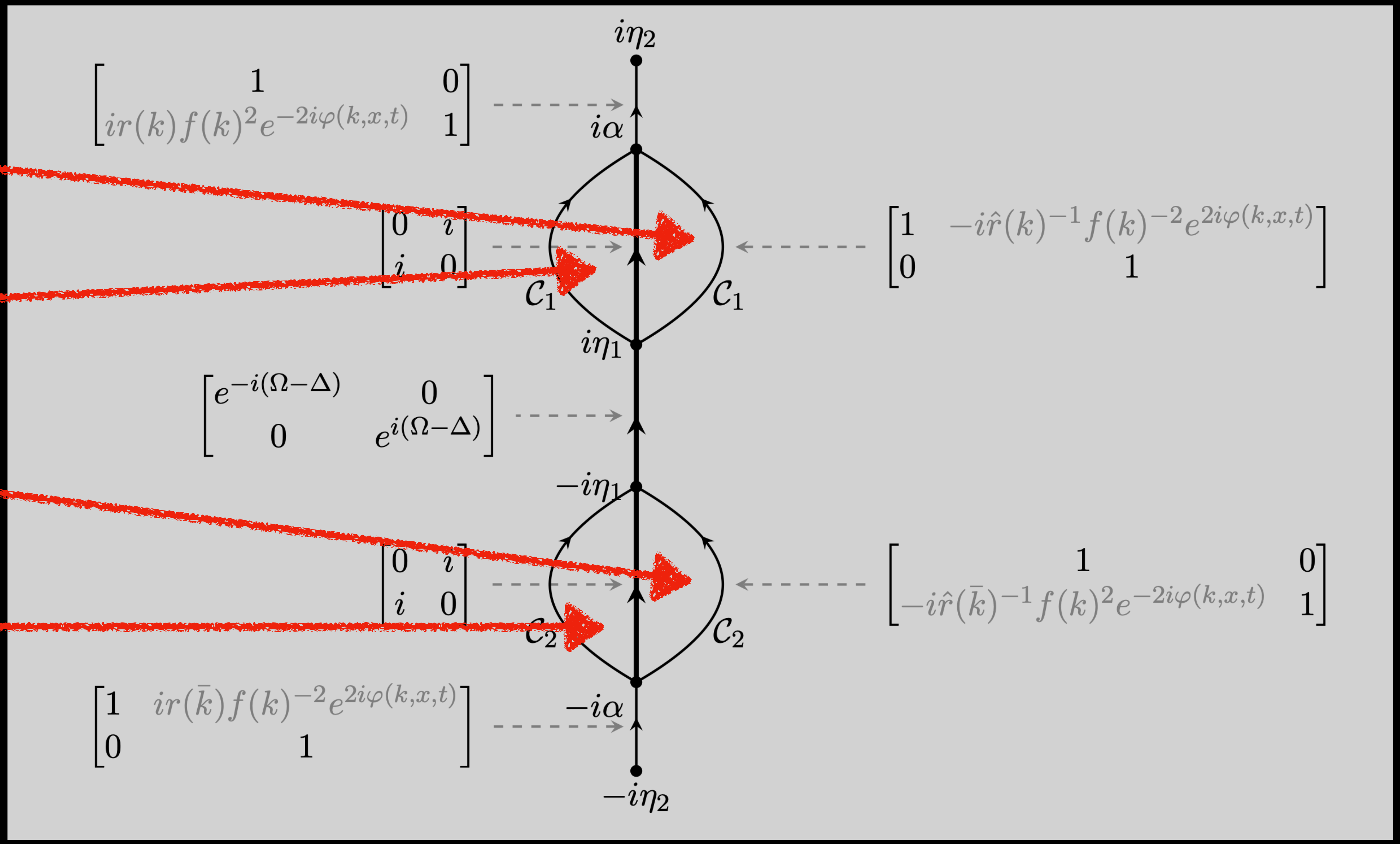


Velocity of peak location



$$\boldsymbol{T} = \boldsymbol{X} e^{-ig(k,x,t)\sigma_3} f(k)^{\sigma_3}$$

$$\boldsymbol{S}(k,x,t) = \begin{cases} \boldsymbol{T}(k,x,t) \begin{bmatrix} 1 & -i\hat{r}(k)^{-1} f(k)^{-2} e^{2i\varphi(k,x,t)} \\ 0 & 1 \end{bmatrix} \\ \boldsymbol{T}(k,x,t) \begin{bmatrix} 1 & i\hat{r}(k)^{-1} f(k)^{-2} e^{2i\varphi(k,x,t)} \\ 0 & 1 \end{bmatrix} \\ \boldsymbol{T}(k,x,t) \begin{bmatrix} 1 & 0 \\ -ir(\bar{k})^{-1} f_-(k)^2 e^{-2i\varphi_-(k,x,t)} & 1 \end{bmatrix} \\ \boldsymbol{T}(k,x,t) \begin{bmatrix} 1 & 0 \\ ir(\bar{k})^{-1} f_-(k)^2 e^{-2i\varphi_-(k,x,t)} & 1 \end{bmatrix} \\ \boldsymbol{T}(k,x,t) \end{cases}$$



$$\boldsymbol{T} = \boldsymbol{X} e^{-ig(k,x,t)\sigma_3} f(k)^{\sigma_3}$$ ← Only this affects the residue conditions at $\pm i\kappa_0$

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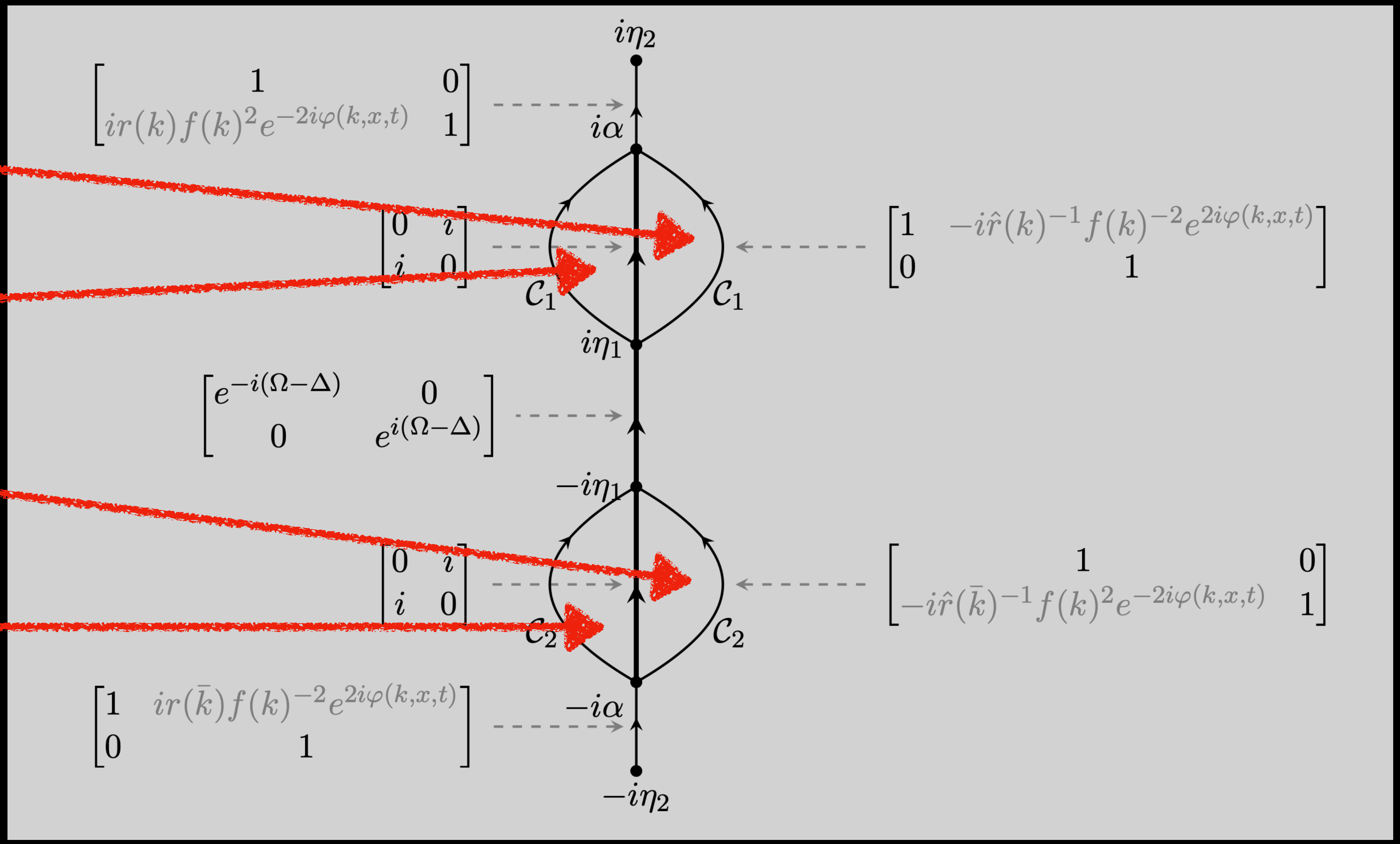

$$\boldsymbol{T} = \boldsymbol{X} e^{-ig(k,x,t)\sigma_3} f(k)^{\sigma_3}$$ ← Only this affects the residue conditions at $\pm i\kappa_0$

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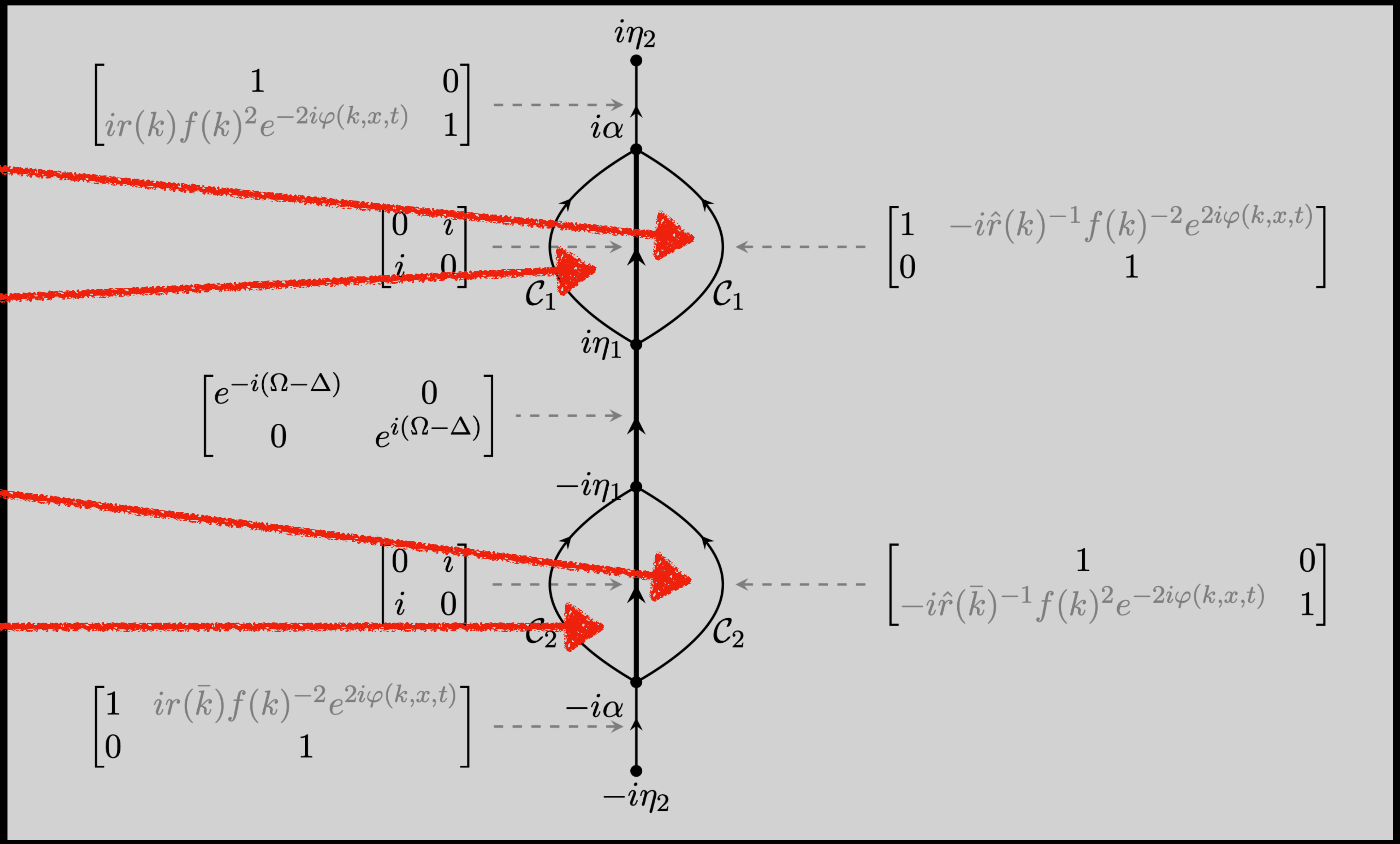


$$\mathrm{Res}_{k=i\kappa_0} \boldsymbol{S}(k,x,t) = \lim_{k \to i\kappa_0} \boldsymbol{S}(k,x,t) \begin{bmatrix} 0 & 0 \\ -i\chi f(i\kappa_0,x,t)^2 e^{-2i\varphi(k,x,t)} & 0 \end{bmatrix}$$

$$\varphi(k,x,t) = g(k,x,t) + 4tk^3 + xk$$

$$\mathrm{Res}_{k=i\kappa_0}\boldsymbol{S}(k,x,t) = \lim_{k\to i\kappa_0}\boldsymbol{S}(k,x,t)\begin{bmatrix} 0 & 0 \\ -i\chi f(i\kappa_0,x,t)^2 e^{-2i\varphi(k,x,t)} & 0 \end{bmatrix} \qquad \varphi(k,x,t) = g(k,x,t) + 4tk^3 + xk$$

Asymptotic parameters: χ, x, t.

Intuition: small residue conditions are negligible. Large ones are negligible.

Soliton peak can be isolated by refining this intuition with hard analysis.

Averaged soliton velocity is obtained by the equation

$$\varphi(i\kappa_0,x,t) = \mathrm{Const.}$$

Averaged soliton velocity:

$$v_{\mathrm{sol}} = -\frac{\partial_t\varphi(i\kappa_0,x,t)}{\partial_x\varphi(i\kappa_0,x,t)} = 4\kappa_0^2\frac{K(\eta_1^2/\alpha^2)}{\Pi(\eta_1^2/\kappa_0^2,\eta_1^2/\alpha^2)}$$

Averaged soliton velocity:

$$v_{\text{sol}} = -\frac{\partial_t \varphi(i\kappa_0, x, t)}{\partial_x \varphi(i\kappa_0, x, t)} = 4\kappa_0^2 \frac{K(\eta_1^2/\alpha^2)}{\Pi(\eta_1^2/\kappa_0^2, \eta_1^2/\alpha^2)}$$

satisfies the tracer equation

$$v_{\text{sol}}(\kappa_0, x, t) = -4\kappa^2 + \frac{1}{k} \int_{\Sigma_1} \log\left(\frac{k-s}{k+s}\right) \left(\frac{-\rho_t}{\rho_x}(s) - v_{\text{sol}}(\kappa_0)\right) u(s) ds$$

Interaction with a *regular* gas of solitons...same velocity equation

RMT

$$\frac{1}{Z_N} e^{-N\sum_{j=1}^N V(\lambda_j)} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod d\lambda_j$$

$$V = \frac{1}{2}\lambda^2 + x\lambda + t\lambda^2 + \cdots + t_{2\nu}\lambda^{2\nu}$$

Equilibrium measure

$$\psi(\lambda, x, t)$$

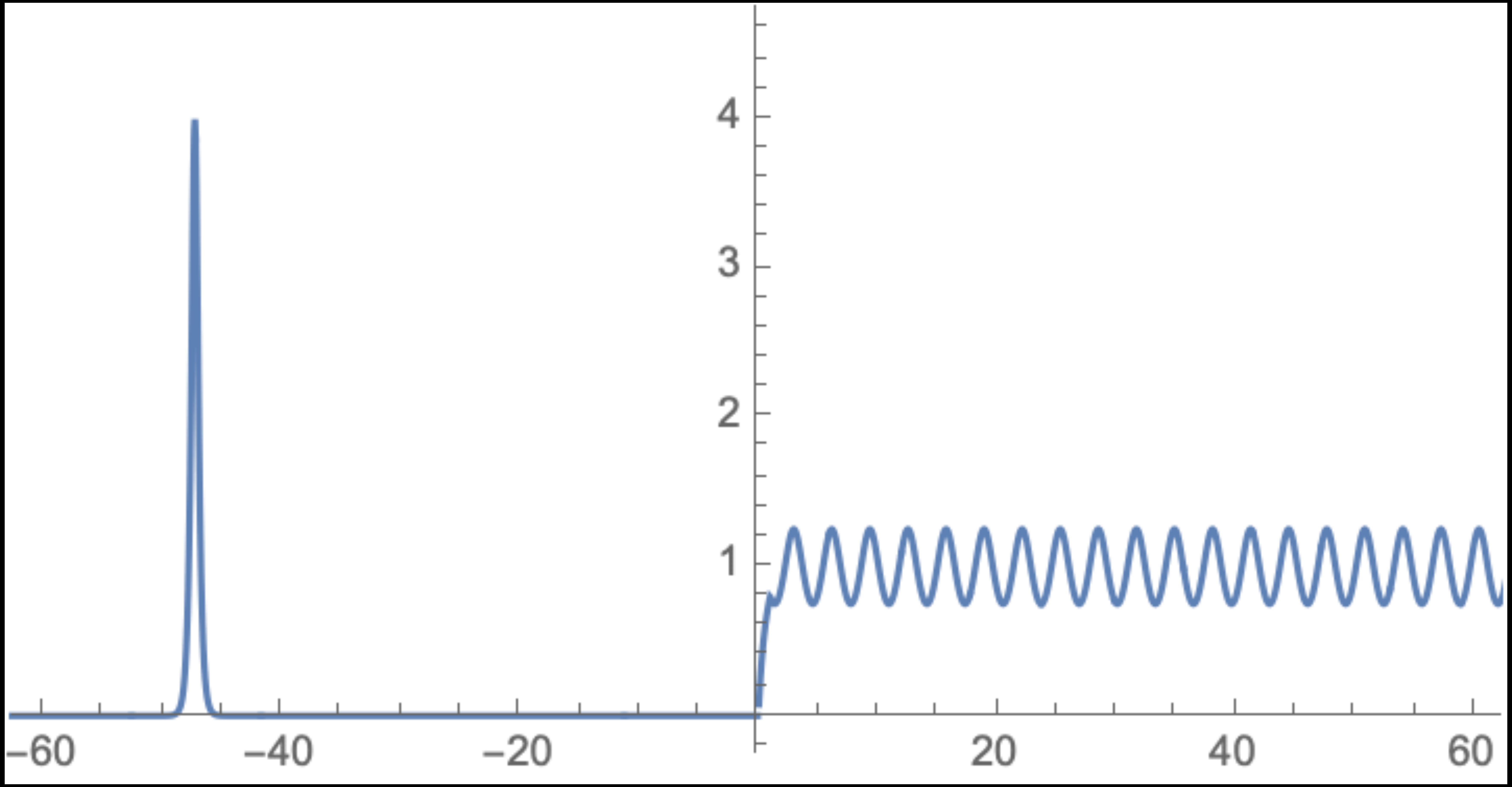
$$\psi(\lambda, x(t,\lambda), t) \equiv \text{const}$$

$$\frac{d}{dt}x(t,\lambda) = -\frac{\psi_t(\lambda, x, t)}{\psi_x(\lambda, x, t)}$$

$\frac{d}{dt}x(t,\lambda)$ solves the analogous kinetic equation

$$\frac{d}{dt}x(t,\lambda) = \lambda^2 - \frac{2}{\lambda}\int \log|\lambda - s| \left(\frac{d}{dt}x(t,s) - \frac{d}{dt}x(t,\lambda)\right)\psi_x(s,x,t)ds$$

THANK YOU!



THANK YOU!

Solitons are characterized via a meromorphic Riemann Hilbert problem.

KdV Equation

$M = (M_1, M_2)$ satisfies

1. $M(\lambda)$ is meromorphic in $\mathbb{C}$, with simple poles at $\{\lambda_j \text{ and } \overline{\lambda_j}\}_{j=1}^N$;
2. M satisfies the residue conditions

$$\operatorname{res}_{\lambda=\lambda_j} M(\lambda) = \lim_{\lambda\to\lambda_j} M(\lambda) \begin{bmatrix} 0 & 0 \\ \dfrac{c_j e^{8i\lambda_j^3 t + 2i\lambda_j x}}{N} & 0 \end{bmatrix}, \quad \operatorname{res}_{\lambda=\overline{\lambda_j}} M(\lambda) = \lim_{\lambda\to\overline{\lambda_j}} M(\lambda) \begin{bmatrix} 0 & \dfrac{-c_j e^{-8i\overline{\lambda_j}^3 t - 2i\overline{\lambda_j} x}}{N} \\ 0 & 0 \end{bmatrix}$$

   where $c_j \in i\mathbb{R}_+$;

3. $M(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\dfrac{1}{\lambda}\right)$ as $\lambda \to \infty$,
4. M satisfies the symmetry

$$M(-\lambda) = M(\lambda) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

N-soliton solution $u(x,t)$ is determined from M via

$$u(x,t) = 2\frac{d}{dx}\left(\lim_{\lambda\to\infty} \frac{\lambda}{i}\left(M_1(\lambda) - 1\right)\right).$$

Solitons are characterized via a meromorphic Riemann Hilbert problem.

Modified focusing KdV equation:

Find 2×2 matrix $M(k;x,t)$:

1. M is meromorphic in $\mathbb{C}$, with simple poles at $\{\pm\imath\kappa_j\}_{j=1}^N$;
2. M satisfies the residue conditions

$$\operatorname{res}_{k=i\kappa_j} M(k) = \lim_{k\to i\kappa_j} M(k) \begin{bmatrix} 0 & 0 \\ \frac{-\imath\chi_j}{N} e^{-2\imath\theta(k,x,t)} & 0 \end{bmatrix}, \quad \operatorname{res}_{k=-i\kappa_j} M(k) = \lim_{k\to -i\kappa_j} M(k) \begin{bmatrix} 0 & \frac{-\imath\chi_j}{N} e^{2\imath\theta(k,x,t)} \\ 0 & 0 \end{bmatrix} \tag{1}$$

where $\theta(x,k,t) = 4tk^3 + xk$, and $\chi_j \in \mathbb{R}$;

3. $M(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{k}\right)$ as $k \to \infty$.

The N-soliton potential $u(x,t)$ is determined from M via

$$u(x,t) = 2\imath \lim_{k\to\infty} k(M(k;x,t))_{21} = 2\imath \lim_{k\to\infty} k(M(k;x,t)_{12}.$$

$$u(x,t) = \frac{i}{2}\frac{\partial}{\partial x}\log\det(\mathbb{I}+M) - \frac{i}{2}\frac{\partial}{\partial x}\log\det(\mathbb{I}+M^\dagger),$$

$$M_{j\ell} = \sqrt{\chi_j\chi_\ell}\frac{e^{-2i(\theta(\kappa_j,x,t)+\theta(\kappa_\ell,x,t))}}{i(\kappa_j+\kappa_\ell)}.$$