Multiple SLE_κ from a loop measure perspective

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Joint work with Gregory F. Lawler

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Topic

Goal: combine dynamical and configurational interpretations of *SLE* to understand multiple radial *SLE* (two kinds)

Today's talk:

N-Sided Radial Schramm-Loewner Evolution (with Gregory F. Lawler). *Probab. Theory Relat. Fields* **181**, pages 451-488 (2021).

Many authors:

- Dubédat '[07]
- Kozdron, Lawler ['07]
- Kytölä and Peltola ['16].
- Jahangoshahi, Lawler ['18]
- Peltola, Wu ['19]
- Zhan ['18, '19]
- Beffara, Peltola, Wu ['21]

Notes:

- Loop measure will be a main tool
- Discrete models are motivation
- This talk: $\kappa \leq 4$

Main Result

Chordal case:

Known that multiple *SLE* is absolutely continuous w.r.t. n independent *SLEs* (boundary perturbation property)

Obstacle for radial: want to do an analogous construction, but (Radon-Nikodym derivative = ∞) $\times$ (partition function = 0) = ??

Short answer:

- Tilt independent n-radial *SLE* $\rightarrow$ **locally independent** *SLE*
- Tilt again and *take a limit* $\rightarrow$ **global n-radial** *SLE*

Main tools: Brownian loop measure, analysis of radial Bessel process

Main Result:

Radial Bessel / Dyson BM naturally appears as the driving function!
(Different drift for locally independent vs. global)

Contents

I. Background

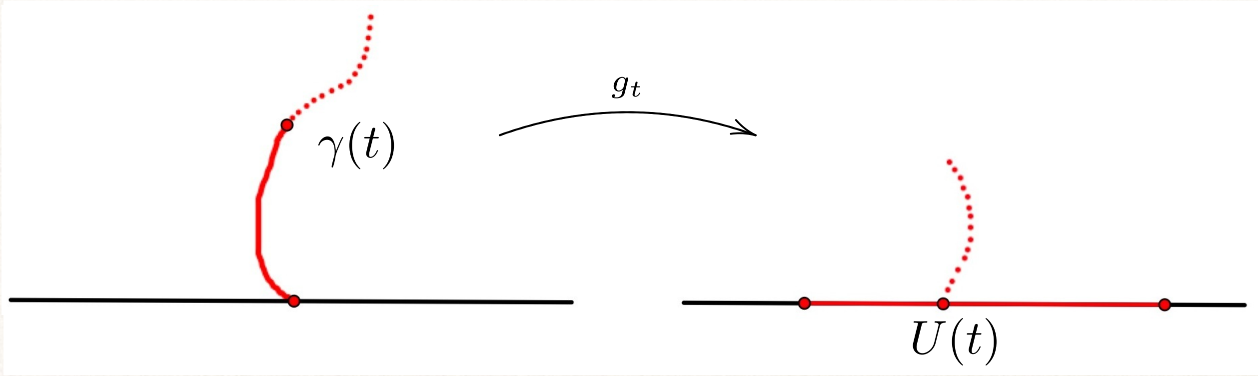
- Loewner Equation & *SLE*
  - Interpretations: dynamical vs configurational
- Loop measures
  - Loop-erased random walk
  - Restriction property and RW loop measure
- Radial restriction property & Brownian loop measure

II. *n*-Radial SLE

- Which loops?
- Locally Independent *n*-radial *SLE*, connection to radial Bessel
- (Global) *n*-radial *SLE*, connection to radial Bessel

Chordal Loewner Equation

$\gamma : (0,T] \to \mathbb{H}$ simple curve, $\gamma(0) \in \mathbb{R}$.



$b(t) = hcap\ \gamma_t$,

$U(t)$ called the driving function

Composition property: $g_{s,t} \circ g_s(z) = g_t(z)$.

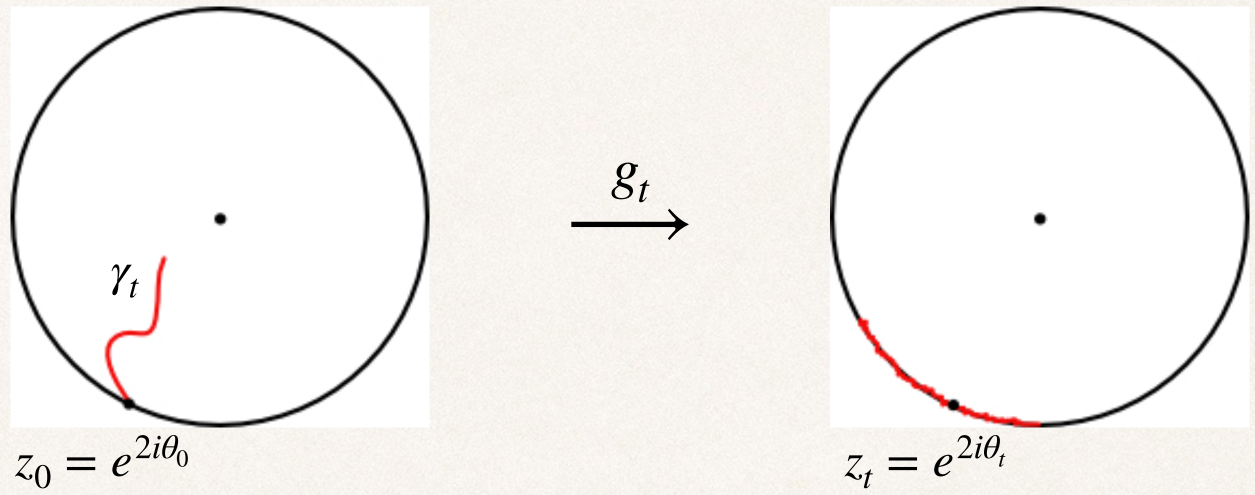
Loewner (1920s): g_t satisfies

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U(t)}, \quad g_0(z) = z.$$

Radial Loewner Equation (one curve)

Radial Loewner equation: (from a boundary point to 0)

- $\gamma : (0,T] \to \mathbb{D}$ a simple curve starting on unit circle.



- Conformal mappings $g_t : \mathbb{D} \setminus \gamma_t \to \mathbb{D}$ satisfy

$$\dot{g}_t(w) = 2a\, g_t(w)\, \frac{z_t + g_t(w)}{z_t - g_t(w)} .$$

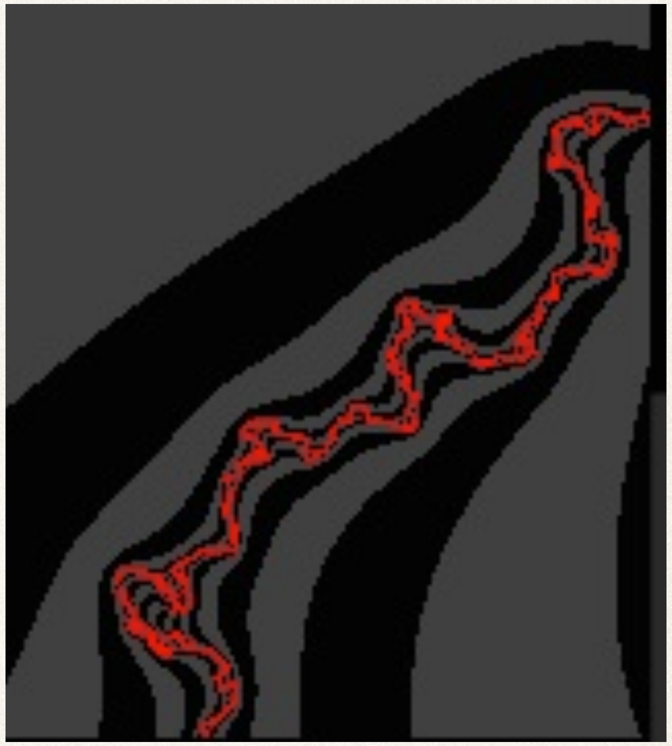
- Parameterized so that $g_t'(0) = e^{2at}$.

Schramm-Loewner Evolution

Dynamical interpretation

- Chordal:

$$\dot{g}_t(z) = \frac{2a}{g_t(z) - B_t}, \qquad g_0(z) = z\,.$$



- Radial:

$$\dot{g}_t(w) = 2a\, g_t(w)\, \frac{e^{2iB_t} + g_t(w)}{e^{2iB_t} - g_t(w)}, \quad g_0(w) = w\,.$$

Results usually stated in terms of κ, where $a = 2/\kappa$.

Schramm-Loewner Evolution

Configurational interpretation

- **Conformal invariance/covariance**
  - Idea: $f(SLE \text{ in } D) = SLE \text{ in } f(D)$
  - Measures with total mass:

$$f \circ \underbrace{\mu_D(z,0)}_{\substack{SLE \text{ measure} \\ \text{on curves from } z \text{ to } 0}} = |f'(z)|^b \, |f'(0)|^{\tilde{b}} \, \mu_{f(D)}(f(z),0)\,.$$

  - Partition functions (i.e. total mass)

$$\Psi_D(z,0) = |f'(z)|^b \, |f'(0)|^{\tilde{b}} \, \Psi_{f(D)}(f(z),0)\,.$$

Non-probability measures: keep more information as we change the domain

$$b = \frac{6-\kappa}{2\kappa}$$

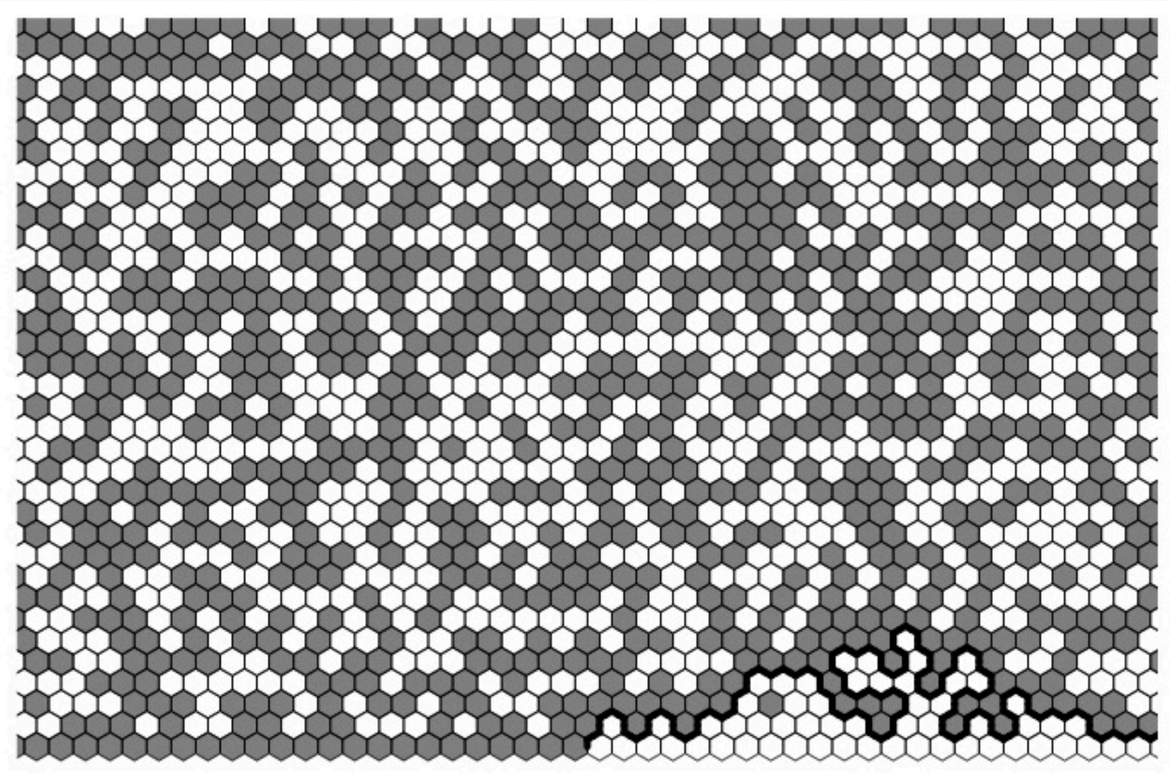
$$\tilde{b} = \frac{b(1-a)}{2a}$$

$$a = 2/\kappa$$

- **Domain Markov property**
  - Idea: curve views its own past as part of the boundary

Schramm-Loewner Evolution

- Universal scaling limit of many discrete proceses, including:
  - Loop-erased random walk
  - Critical Ising model
  - Uniform spanning tree
  - Critical percolation

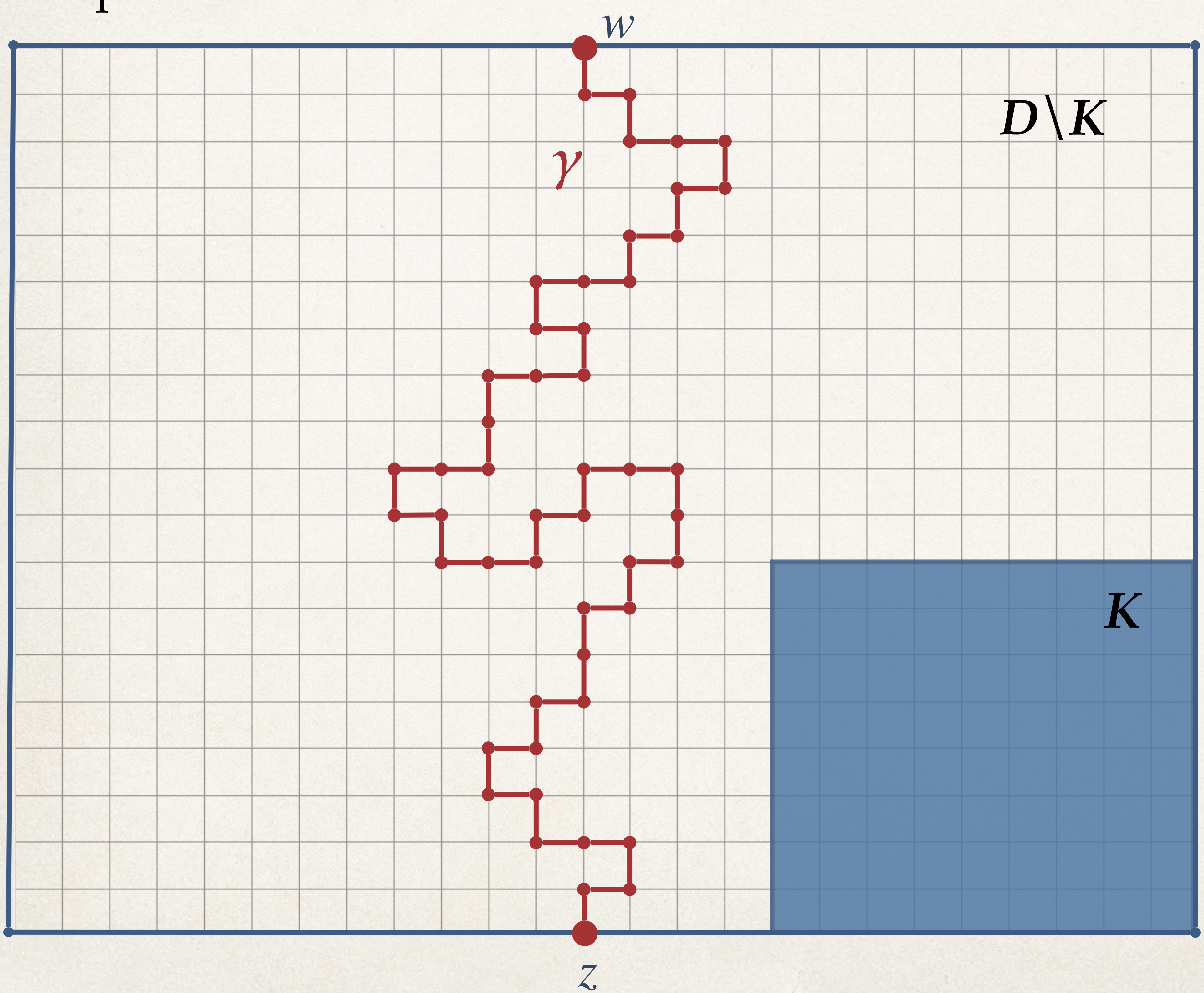


Critical percolation: Oded Schramm '99

- Today: use discrete models to build intuition
- We'll look at loop-erased random walk to understand loop measures

Loop measure and LERW

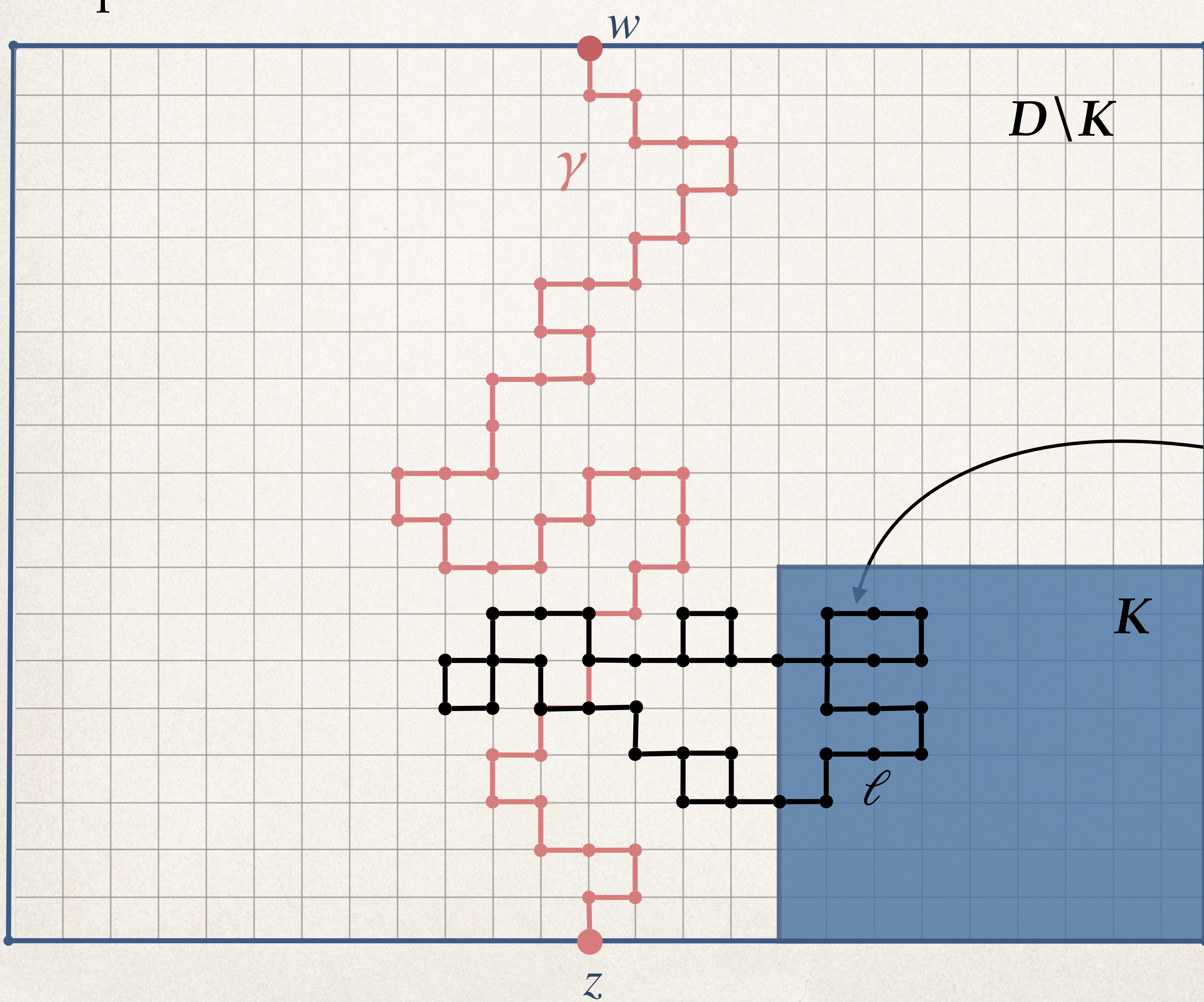
Loop-erased random walk:



How does the measure of γ in D compare to its measure in $D \setminus K$?

Loop measure and LERW

Loop-erased random walk:



How does the measure of γ in D compare to its measure in $D\setminus K$?

LERW path carries measure of SRW loops that intersect γ and K.

Need to reweight the measure by $m\{\ell \in D : \ell \cap K \neq \emptyset\}$.

Random Walk Loop Measure in $\mathbb{Z}^2$

- (Unrooted) **loop measure**:

$K(\ell)$ = # of representatives of ℓ

$$m(\ell) = \frac{K(\ell)}{|\ell| \cdot 4^{|\ell|}}.$$

Length of loop

Prob. of a simple random walk making the loop

- Why this def? **Limit = Brownian loop measure**. (Want *SLE* results.) [Lawler-Werner-Trujillo Ferreras]

- Brownian loop measure $m_{\mathbb{C}}$ on unrooted loops given by

$$\text{(duration 1 Brownian bridge)} \times \text{(Area meas.)} \times \left(\frac{1}{2\pi t^2}\, dt\right)$$

Base loop

Basepoint (i.e. root)

time duration × root location

Restriction

Measure on paths in D	Measure on paths in $D \setminus K$
• LERW	• weight by $m\{\ell \in D : \ell \cap K \neq \emptyset\}$ ↑ Random walk loop measure
• SLE_κ	• weight by $m\{\ell \in D : \ell \cap K \neq \emptyset\}$ ↑ Brownian loop measure

- Want to study: *SLE* paths weighted by Brownian loop measure
  - Need stochastic calculus to make sense of this

Girsanov Theorem (stoch. calc.)

Girsanov Theorem: (giving drift to B_t via change of measure)

- B_t Brownian motion under probability measure $\mathbb{P}$.
- $\boldsymbol{M_t}$ a non-negative martingale wrt $\mathbb{P}$, $\boldsymbol{M_0} = 1$,

$$dM_t = A_t M_t \, dB_t .$$

- Let $\dfrac{d\tilde{\mathbb{P}}_t}{d\mathbb{P}} = M_t .$
- Then B_t satisfies

$$dB_t = A_t \, dt + dW_t,$$

for W_t Brownian motion wrt $\tilde{\mathbb{P}}$.

Idea: gives a way to condition on measure 0 events.

Application to SLE:

- SLE_κ has driving function $\sqrt{\kappa} B_t$.
- new measure → new driving function.

Restriction property for radial SLE_κ

- γ radial SLE_κ from 1 to 0 in $\mathbb{D}$. $\gamma_t = \gamma[0,t]$.
- $U = \mathbb{D}\backslash K \subset \mathbb{D}$ simply connected
- Let $D_t = \mathbb{D}\backslash\gamma_t$, $U_t = U\backslash\gamma_t$,

Initial segment

$$\Psi_t = \frac{\Psi_{U_t}(\gamma(t),0)}{\Psi_{D_t}(\gamma(t),0)}.$$

Total mass of paths in U_t

Total mass of paths in D_t

- ***SLE_κ* in *U*** is SLE_κ in $\mathbb{D}$ "weighted locally" by Ψ_t
  - Find a local martingale $M_t = A_t\Psi_t$, where A_t is differentiable.
  - Use Girsanov theorem.
  - Can calculate: $A_t = 1\{\gamma_t \subset U\}\exp\left\{\frac{\mathbf{c}}{2} m_D(\gamma_t, K)\right\}$.

Proof: Jahangoshahi & Lawler '18, earlier folklore?

Multiple Radial SLE_κ

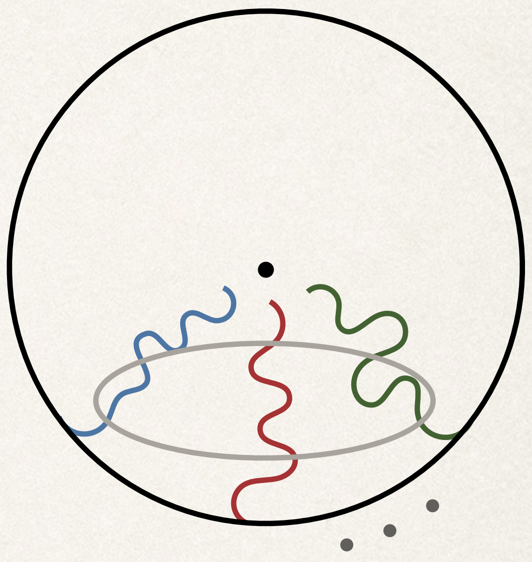
- **Radial Loewner equation**: (from a boundary point to 0)

$$\dot{g}_t(z) = 2a\, g_t(z) \sum_{j=1}^{n} \frac{z_t^j + g_t(z)}{z_t^j - g_t(z)}$$

- *SLE* = a measure on paths (with partition function).
- *SLE* also = **a measure on parametrized curves with killing**.
    - Curves are growing: keep time parameter t
    - Process up to a stopping time t < T.
    - Paths killed, so not a probability measure:
    - Total mass at time t = partition function $\Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)$.
- **Can use both "configurational" and "dynamical" information!**

Multiple Radial SLE_κ

<u>Q:</u> Use restriction to define multiple radial *SLE*?

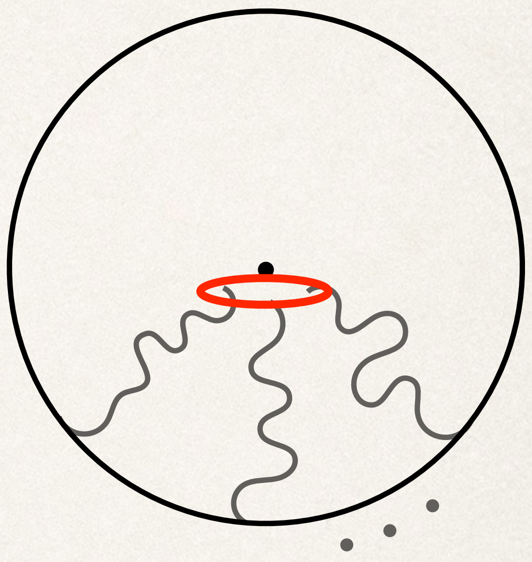


- n curves in the disk from unit circle to origin.
- Grow all curves at the "same rate."
  - Measure on parametrized curves, not just paths.
  - Weight by loops that hit multiple curves
- Procedure works for chordal case

  [Jahangoshahi & Lawler '18]

Multiple Radial SLE_κ

<u>Q:</u> **Use restriction to define multiple radial *SLE*?**



- n curves in the disk from unit circle to origin.
- Grow all curves at the "same rate."
  - Measure on parametrized curves, not just paths.
  - Weight by loops that hit multiple curves
- Procedure works for chordal case

  [Jahangoshahi & Lawler '18]
- But in radial case, this measure is infinite! (tiny red loops)
- We'll need to define it as a limit process.

SLE_κ Results [H-Lawler, '21]

Setup

- n points on the unit circle:

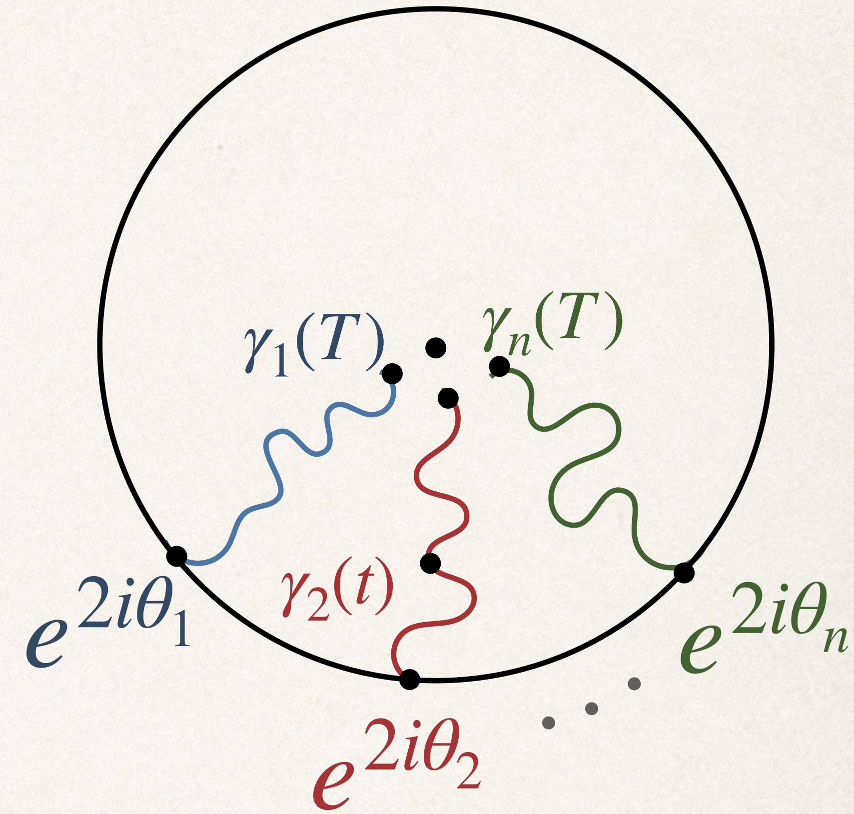
$$z_t^j = \exp\{2i\theta_t^j\}, \quad j = 1,\ldots,n\,.$$

- Define h by:

$$g_t(e^{2i\zeta}) = e^{2ih_t(\zeta)}\,.$$

Then

$$\dot{h}_t(\zeta) = a\sum_{j=1}^{n}\cot(h_t(\zeta) - \theta_t^j)\,.$$



- Killing: $0 \le t \le T$ stop process when any $\gamma_t^j \cap \gamma_t^k \neq \emptyset\,.$

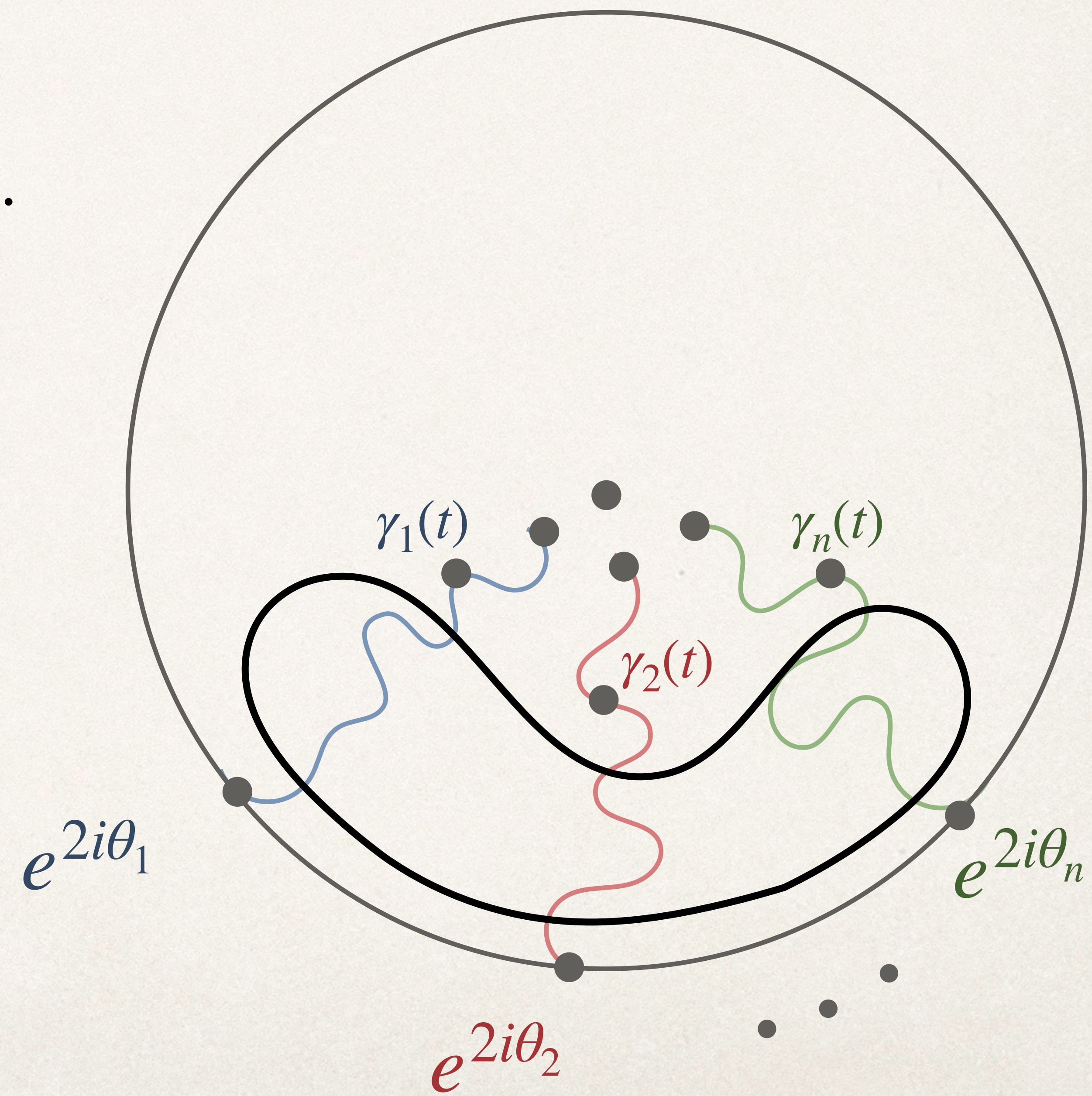
Loop notation setup

- Denote the n-tuple of curves $\boldsymbol{\gamma}_t = (\gamma_t^1, \ldots, \gamma_t^n)$
- Let $\hat{L}_t^j = \hat{L}_t^j(\boldsymbol{\gamma})$ be the set of loops that hit γ_t^j after hitting another γ_t^k.
- Define

$$\hat{\mathscr{L}}_t = \hat{I}_t \exp\left\{ \frac{c}{2} \sum_{j=1}^{n} m_{\mathbb{D}}(\hat{L}_t^j) \right\}.$$

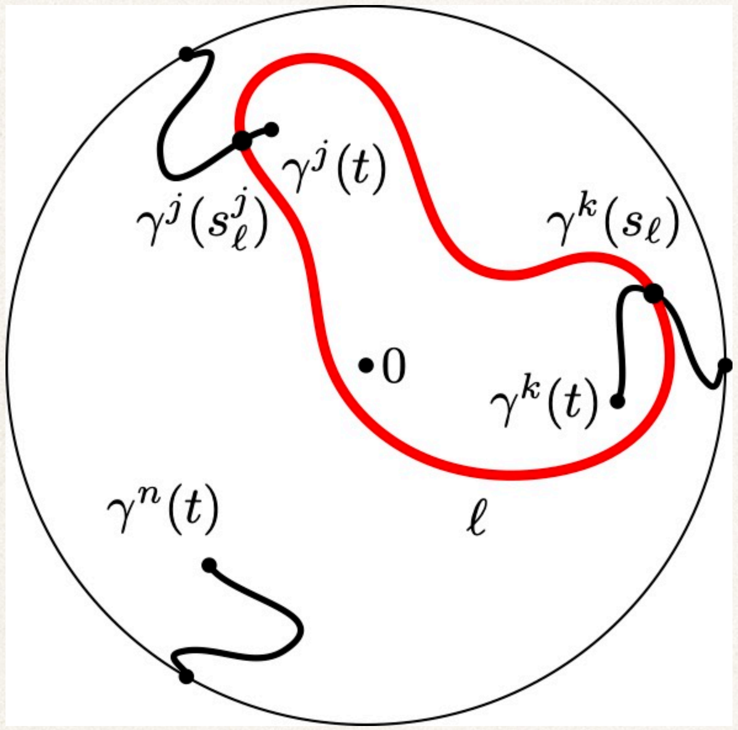
Indicator that curves don't intersect

Brownian loop measure



Which loops?

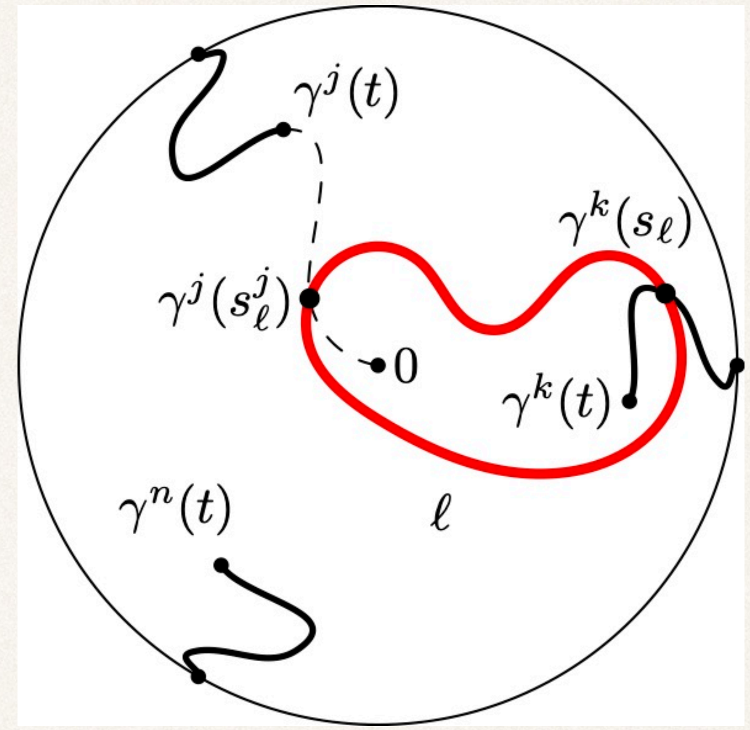
Locally independent *SLE*



Loops used for $\hat{\mathscr{L}}_t$

- At time t, each curve γ^j_t sees the **past** of every other curve
- t-measurable loops

Truncation whose limit is global *n*-radial *SLE*



Loops used for $\mathscr{L}_t$

- **Future** loops also included
- Truncate by assuming one hit is before time t.

Locally Independent SLE_κ

"Theorem" (Locally independent SLE_κ) [H-Lawler '21]:
If each of the n SLE curves grows as if in the domain $D \setminus \gamma_t$, then the driving functions satisfy

$$d\theta_t^j = a \sum_{k:k\neq j} \cot(\theta_t^j - \theta_t^k)\, dt + dW_t^j,$$

$a = 2/\kappa$

for $W_t^1, \ldots, W_t^n$ independent Brownian motions.

Radial Bessel equation with parameter $a = 2/\kappa$ generates locally independent SLE_κ.

Locally Independent SLE_κ

Theorem (Locally independent SLE_κ) [H-Lawler '21]:

If $\gamma_t \sim$ independent n-path SLE_κ, then

Loop term

$$M_t = \hat{\mathscr{L}}_t \, \Psi_t \, \exp\left\{ -2a\tilde{b}n(n-1)t + ab\int_0^t \phi(\theta_s)ds \right\}$$

is a local martingale for $0 \le t \le T$.

$$\phi(\theta_t) = \sum_{j=1}^{n} \sum_{k:k\neq j} \csc^2(\theta^j - \theta^k)$$

If $\mathbb{P}_*$ is defined by

$$\frac{d\mathbb{P}_*}{d\mathbb{P}} = M_t,$$

Ratio of part. functions

then

$$d\theta_t^j = a \sum_{k:k\neq j} \cot(\theta_t^j - \theta_t^k)\, dt + dW_t^j,$$

for $W_t^1, \ldots, W_t^n$ independent Brownian motions under $\mathbb{P}_*$.

$$a = 2/\kappa$$

$$b = \frac{6-\kappa}{2\kappa}$$

$$\tilde{b} = \frac{b(1-a)}{2a}$$

Radial Bessel equation with parameter $a = 2/\kappa$ generates locally independent SLE_κ.

Idea of Proof

Can express $M_t = \prod_{j=1}^{n} M_t^j$

- M_t^j is just the "j-version" of every term in M_t:

$$M_t^j = I_t^j \Psi_t^j \exp\left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\hat{L}_t^j) \right\} \exp\left\{ ab \int_0^t \sum_{k \neq j} \csc^2(\theta_s^j - \theta_s^k) ds \right\}$$

- After tilting by M_t^j, the curve γ^j at time t is locally growing as *SLE* in $D_t = \mathbb{D} \backslash \boldsymbol{\gamma}_t$
  - Reason for the name locally independent *SLE*

- Computation verifies local mart; Girsanov finishes the proof.

Locally Independent SLE_κ

Locally independent SLE_κ: In the measure $\mathbb{P}_*$, the paths $\gamma_t^1, \ldots, \gamma_t^n$ locally grow like independent *SLE* in slit domain.

- Driving functions satisfy

$$d\theta_t^j = a \sum_{k:k \neq j} \cot(\theta_t^j - \theta_t^k)\, dt + dW_t^j\,.$$

- Drift strong enough to guarantee non-intersection of driving functions, but *paths* could collide.
  - $\hat{\mathscr{L}}_t$ doesn't see any loops that hit in the future.
  - Idea: γ_t^j only "planning" to avoid past of other curves

- Need to tilt again!

Global n-radial SLE_κ

Theorem (Global n-radial SLE_κ) [H-Lawler '21]:
Let t be fixed, and let $\gamma_t \sim$ independent n-path SLE_κ.
For $T > t$, let $\mu_T = \mu_{T,t}$ denote the measure on γ_t whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is

$$\frac{\mathscr{L}_T}{\mathbb{E}^{\theta_0}\left[\mathscr{L}_T\right]}.$$

Then as $T \to \infty$, μ_T converges in variation distance to a prob measure under which the driving functions satisfy

$$d\theta_t^j = 2a \sum_{k:k \neq j} \cot(\theta_t^j - \theta_t^k)\, dt + d\hat{W}_t^j,$$

for $\hat{W}_t^1, \ldots, \hat{W}_t^n$ independent Brownian motions.

Radial Bessel equation with parameter $2a = 4/\kappa$ generates n-radial SLE_κ.

Idea of Proof

<u>**Radial Bessel calculations:**</u>

- Define martingale

$$M_{t,\alpha} = \prod_{1 \le j < k \le n} |\sin(\theta^k - \theta^j)|^\alpha \exp\left\{\frac{\alpha^2 n(n^2-1)}{6} t\right\} \exp\left\{\frac{\alpha - \alpha^2}{2} \int_0^t \psi(\boldsymbol{\theta}_s) ds\right\}$$

- Tilting by $M_{t,\alpha}$ gives a prob measure $\mathbb{P}_\alpha$ with

$$d\theta_t^j = \alpha \sum_{k:k \neq j} \cot(\theta_t^j - \theta_t^k)\, dt + dW_t^j\,.$$

Locally indep *SLE*
if $\alpha = a$

- Define

$$N_{t,\alpha,2\alpha} = \prod_{1 \le j < k \le n} |\sin(\theta^k - \theta^j)|^\alpha \exp\left\{\frac{\alpha^2 n(n^2-1)}{2} t\right\} \exp\left\{-\frac{\alpha(3\alpha - 1)}{2} \int_0^t \psi(\boldsymbol{\theta}_s) ds\right\}$$

- This is a $\mathbb{P}_\alpha$-martingale.
- Tilting by $M_{t,\alpha}$ and then by $N_{t,\alpha,2\alpha}$ gives $\mathbb{P}_{2\alpha}$.

Idea of Proof

Truncations

- On the other hand, the truncations we need are obtained by tilting by $\tilde{N}_{t,T} = \mathbb{E}^{\boldsymbol{\theta}_0}\left[\mathscr{L}_T \mid \boldsymbol{\gamma}_t\right]$.

- Compute that
$$\tilde{N}_{t,T} = \hat{\mathscr{L}}_t \Psi_t \mathbb{E}^{\boldsymbol{\theta}_t}[\mathscr{L}_{T-t}].$$

- Prove an exponential rate of convergence:
$$\mathbb{E}^{\boldsymbol{\theta}_0}[\mathscr{L}_T] = e^{-2an\beta t}\mathscr{F}_a(\boldsymbol{\theta})[1 + O(e^{-uT})]\ .$$

- Compare with the previous calculation. $\square$

Open Questions

- Understanding $4 < \kappa < 8$ (loop interp does not apply, but Bessel results do)

- Take the number of curves to infinity? Use random matrix theory.

- How to unify loop-measure approach with PDE/pure partition function approach to multiple SLE?

Thank you!