Crossing Probabilities in 2D Critical Lattice Models

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Based on joint works with

- Vincent Beffara (CNRS)
- Jian Ding (University of Pennsylvania)
- Yong Han (Shenzhen University)
- Mingchang Liu (Tsinghua University)
- Eveliina Peltola (University of Bonn, Aalto University)
- Mateo Wirth (University of Pennsylvania)

Outline

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Ising Model

Ising Model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x\sim y} \sigma_x\sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))$$

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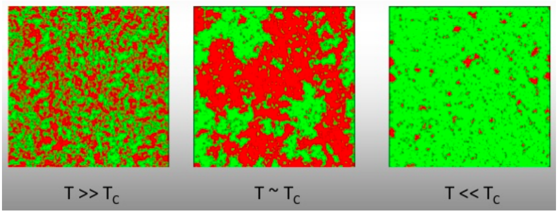

- $\beta > \beta_c$: ordered
- $\beta \approx \beta_c$: critical
- $\beta < \beta_c$: chaotic

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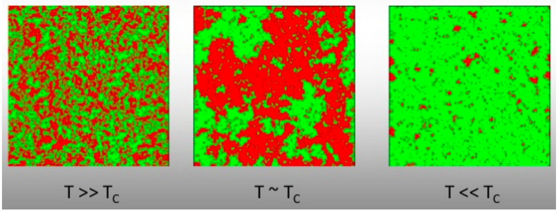

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Critical phase ?

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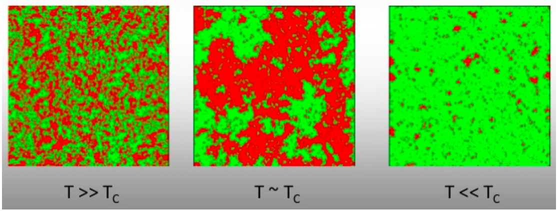

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Conformally invariant.

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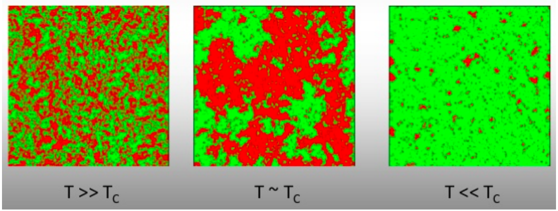

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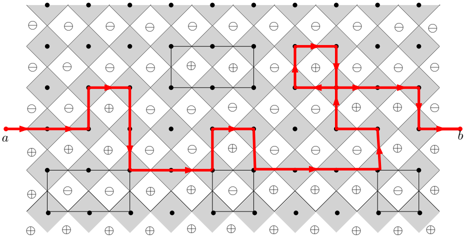
Correlation function

$\mu[\sigma_{z_1} \cdots \sigma_{z_n}] \to \phi(z_1, \ldots, z_n)$.

Schramm Loewner Evolution (SLE)

The law of interfaces is conformally invariant.

Conformal Invariance of Interfaces



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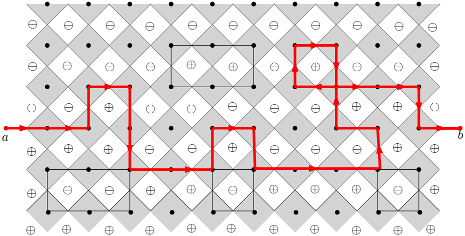

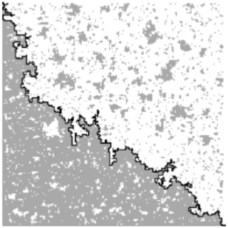

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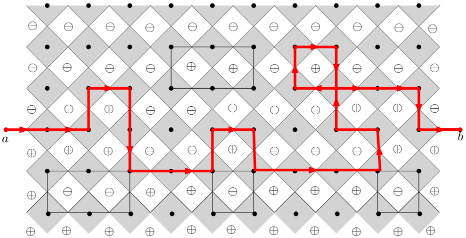
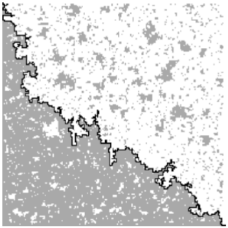
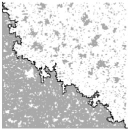
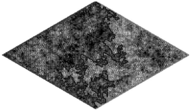
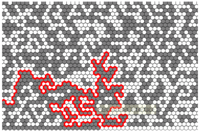
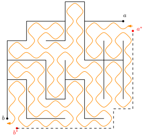




Theorem [Chelkak-Smirnov et al. Invent. 2012]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE_3.

Conformal Invariance in 2D Lattice Model

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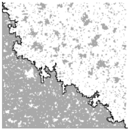
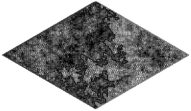
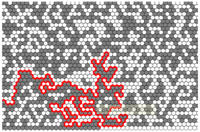
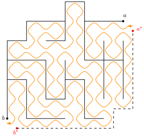

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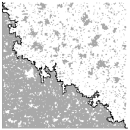
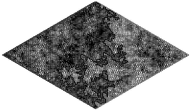
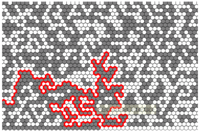
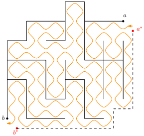

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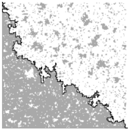
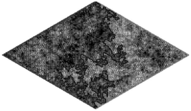
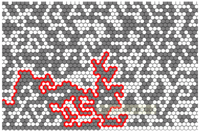
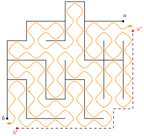

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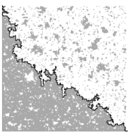
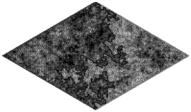
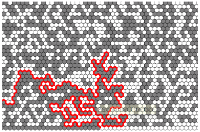
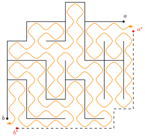

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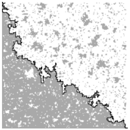
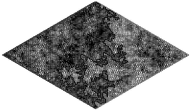
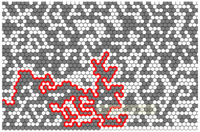
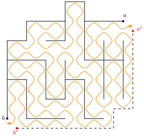

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- Uniform spanning tree (UST) SLE_8 : [Lawler-Schramm-Werner, AOP 2004]

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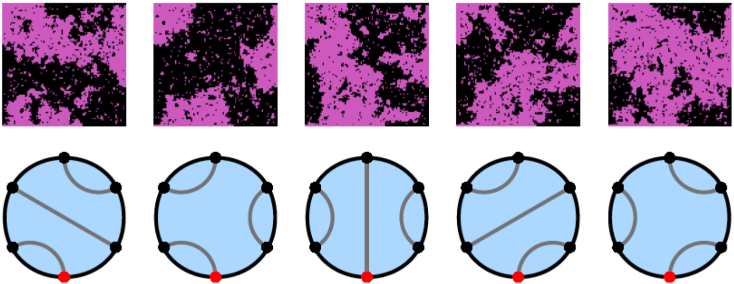
Crossing probabilities



Theorem [Peltola-W. 2018]

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta\to 0} \mathbb{P}[\mathcal{A}_\delta = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{Ising}(\Omega; x_1, \ldots, x_{2N})}, \quad \mathcal{Z}_{Ising} = \sum_{\alpha\in \mathrm{LP}_N} \mathcal{Z}_\alpha,$$

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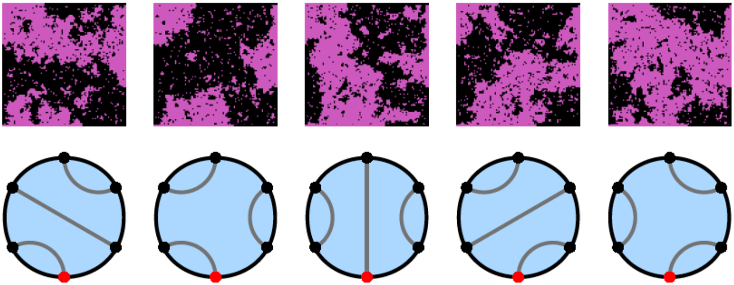

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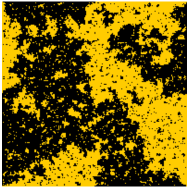
- Partially conjectured in [Bauer-Bernard-Kytölä, JSP 2005].
- Partially solved in [Izyurov, CMP 2015].

Pure Partition Functions



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$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{(6-\kappa)/\kappa}{(x_j - x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0.$

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N}\varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})).$

ASY : $\lim_{x_j, x_{j+1}\to\xi}\frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1}-x_j)^{-2h}} = \begin{cases}\mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \text{if } \{j, j+1\} \in \alpha;\\ 0, & \text{else.}\end{cases}$

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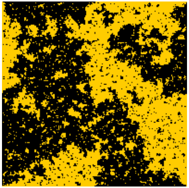

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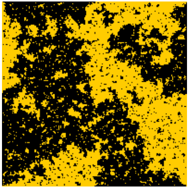
Questions

Existence ? Uniqueness ?

Pure Partition Functions : Uniqueness and Existence



Uniqueness [Flores-Kleban, CMP 2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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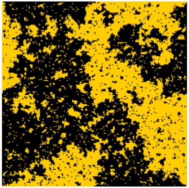

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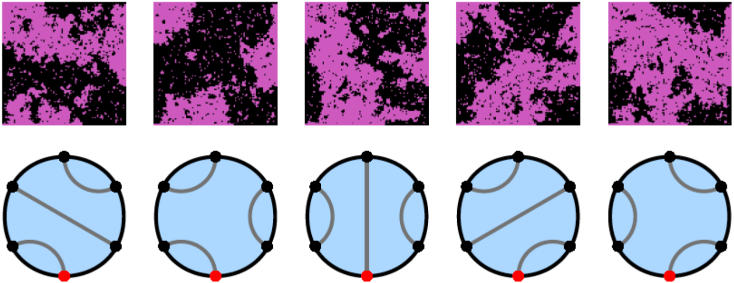
Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP 2016]
- $\kappa \in (0, 4]$ [Peltola-W. CMP 2019, Beffara-Peltola-W. AOP 2021]
- $\kappa \in (0, 6]$ [W. CMP 2020]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

Crossing Probabilities of Ising Interfaces



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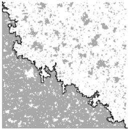
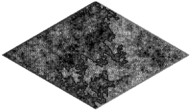
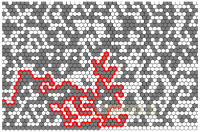
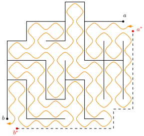
where $\{\mathcal{Z}_\alpha\}$ is the pure partition functions for multiple SLE_3.

$$\mathcal{Z}_{Ising}(\mathbb{H};x_1,\ldots,x_{2N})=Pf\left((x_j-x_i)^{-1}\right)_{i,j=1}^{2N}.$$

Connection Probabilities

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- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP 2019] ✓

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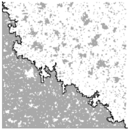
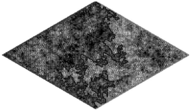
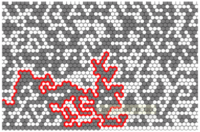
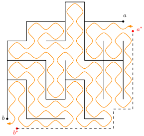

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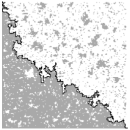
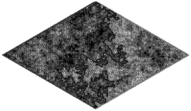
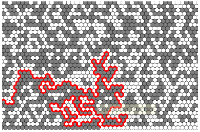
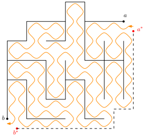
Connection Probabilities

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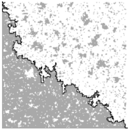
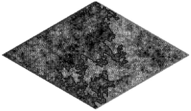
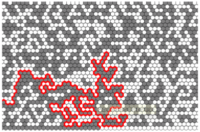
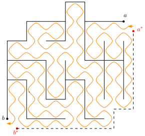

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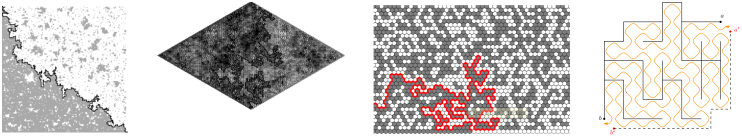
Connection Probabilities



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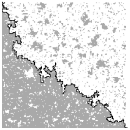
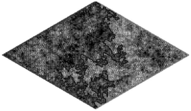
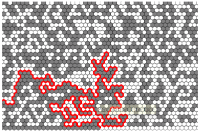
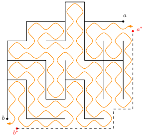
Connection Probabilities

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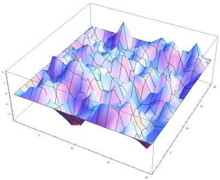
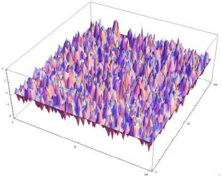
- Multiple Peano curves in UST : $\kappa = 8$.
  [Han-Liu-W. 2020], [Liu-Peltola-W. 2021], [Liu-W. 2021] ★

Table of contents

dGFF (Discrete Gaussian Free Field)





dGFF with mean zero :
a measure Γ on functions $\rho : D \to \mathbb{R}$ and $\rho = 0$ on ∂D with density

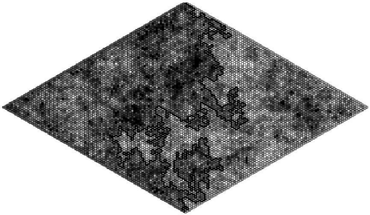
$$\frac{1}{\mathcal{Z}} \exp(-\frac{1}{2} \sum_{x\sim y} (\rho(x) - \rho(y))^2).$$

- For each vertex x, $\Gamma(x)$ Gaussian random variable
- Covariance : Green's function for SRW
- Mean value : zero.

dGFF with mean Γ_∂ :
dGFF with mean zero plus a harmonic function Γ_∂.

- For each vertex x, let $\Gamma(x)$ be a Gaussian random variable
- Covariance : Green's function for SRW
- Mean value : $\Gamma_\partial(x)$

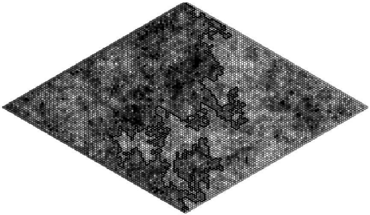
Level lines of dGFF



[Schramm-Sheffield, ACTA 2009]

- dGFF with boundary value $+\lambda$ on $\mathbb{R}_+$ and $-\lambda$ on $\mathbb{R}_-$
- γ^δ : the level line of dGFF with height zero
- γ^δ converges in distribution to SLE_4 as δ goes to zero

Level lines of dGFF



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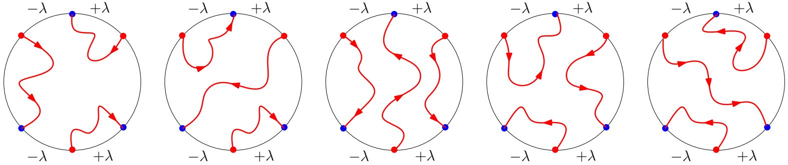
- γ^δ : the level line of dGFF with height zero
- γ^δ converges in distribution to SLE_4 as δ goes to zero
- $\rightarrow$ SLE_4 is the “level line" of GFF with height zero

Connection Probabilities



- $2N$ marked points
- N level lines
- LP_N : planar link patterns

Theorem [Peltola-W. CMP 2019]

The connection of level lines of GFF forms a planar link pattern $\mathcal{A}$:

$$\mathbb{P}[\mathcal{A}=\alpha]=\frac{\mathcal{Z}_\alpha(\Omega;x_1,\ldots,x_{2N})}{\mathcal{Z}_{\mathrm{GFF}}(\Omega;x_1,\ldots,x_{2N})},\quad \mathcal{Z}_{\mathrm{GFF}}=\sum_{\alpha\in\mathrm{LP}_N}\mathcal{Z}_\alpha,$$

where $\{\mathcal{Z}_\alpha:\alpha\in\mathrm{LP}_N\}$ is the pure partition functions for multiple SLE_4.

Discrete GFF and metric graph GFF

dGFF with mean Γ_∂ :

- For each vertex x, let $\Gamma(x)$ be a Gaussian random variable
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Discrete GFF and metric graph GFF

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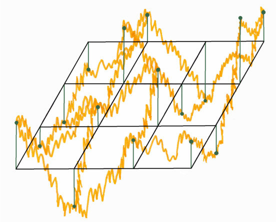
Metric graph GFF (mGFF) with mean Γ_∂ :

- Graph $\mathcal{G} = (V, E) \longrightarrow$ metric graph $\tilde{\mathcal{G}}$
- For each point $x \in \tilde{\mathcal{G}}$, let $\tilde{\Gamma}(x)$ be a Gaussian random variable
- Covariance : Green's function for BM on $\tilde{\mathcal{G}}$
- Mean value : $\Gamma_\partial(x)$.



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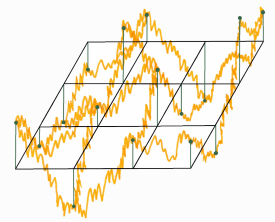

dGFF and mGFF

Given dGFF $(\Gamma(x), x \in V)$, mGFF $\tilde{\Gamma}$: $\Gamma +$ Brownian bridges.

Discrete GFF and metric graph GFF

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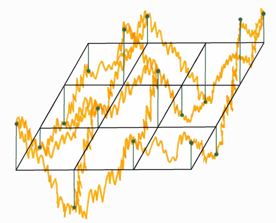

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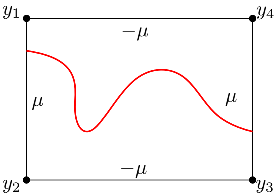
dGFF and mGFF

dGFF $\longrightarrow$ GFF, mGFF $\longrightarrow$ GFF

Crossing probability in dGFF and mGFF



dGFF and mGFF

$$\text{dGFF} \longrightarrow \text{GFF}, \qquad \text{mGFF} \longrightarrow \text{GFF}.$$

Discrete GFF

- $\exists c_d = c_d(L, \mu) \in (0, 1)$

$$\mathbb{P}\left[(y_1^\delta y_2^\delta) \overset{\Gamma^\delta \geq 0}{\longleftrightarrow} (y_3^\delta y_4^\delta)\right] \in (c_d, 1 - c_d).$$

Metric graph GFF

- $\exists c_m = c_m(L, \mu) \in (0, 1)$

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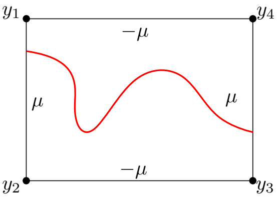

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Theorem [Ding-Wirth-W. AIHP 2022+]

$$\exists c = c(L,\mu) > 0, \quad \mathbb{P}\left[(y_1 y_2) \overset{\Gamma^\delta \geq 0}{\longleftrightarrow} (y_3 y_4)\right] - \mathbb{P}\left[(y_1 y_2) \overset{\tilde{\Gamma}^\delta \geq 0}{\longleftrightarrow} (y_3 y_4)\right] \geq c.$$

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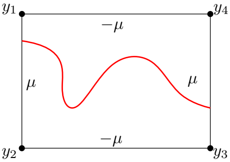

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Question : why ?

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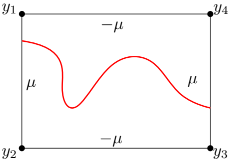

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- Crossing event is not preserved in the cvg of distributions

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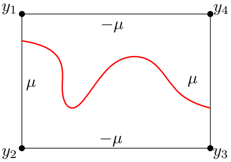

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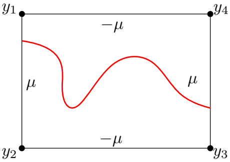
Crossing probability in dGFF and mGFF



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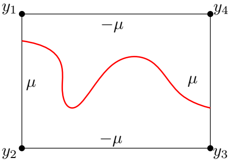

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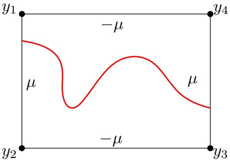
Crossing probability in dGFF and mGFF



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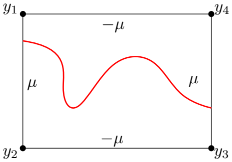
- dGFF, if $\mu = \lambda$, we have $\lim_{\delta \to 0} \mathbb{P}\left[(y_1 y_2) \overset{\Gamma^\delta \geq 0}{\longleftrightarrow} (y_3 y_4)\right] = q$.

Crossing probability in dGFF and mGFF



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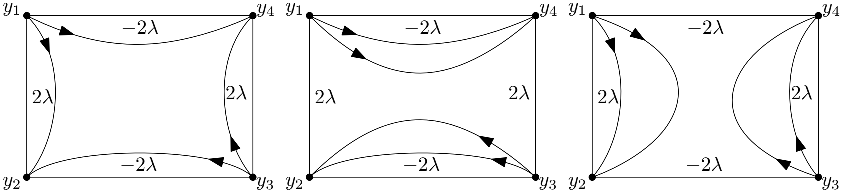
- mGFF, if $\mu = 2\lambda$, we have $\lim_{\delta \to 0} \mathbb{P}\left[(y_1 y_2) \overset{\tilde{\Gamma}^\delta \geq 0}{\longleftrightarrow} (y_3 y_4)\right] = q^4.$

Connection Probabilities in mGFF



Theorem [Liu-W. EJP 2021]

The connection probabilities in mGFF are given by

$$\lim_{\delta} \mathbb{P}[\mathcal{A}^\delta = \hat{\alpha}] = \mathcal{M}_{\omega,\tau(\hat{\alpha})} \frac{\mathcal{Z}_{\hat{\alpha}}(\Omega; y_1, \ldots, y_{2N})}{\mathcal{Z}_{\text{mGFF}}(\Omega; y_1, \ldots, y_{2N})},$$

where $\mathcal{M}_{\omega,\tau(\hat{\alpha})}$ is certain coefficient and $\mathcal{Z}_{\hat{\alpha}}$ is “fusion" of pure partition function $\mathcal{Z}_\alpha$.

Fusion of partition functions

Pure partition functions

PDE : $\left[\partial_i^2 + \sum_{j\neq i}\left(\frac{1}{x_j-x_i}\partial_j - \frac{1/4}{(x_j-x_i)^2}\right)\right]\mathcal{Z}_\alpha(x_1,\ldots,x_{2N}) = 0.$

$$\mathcal{Z}_{\hat{\alpha}}(y_1,\ldots,y_{2N}) = \lim_{\substack{x_{2j-1},x_{2j}\to y_j \\ \forall 1\le j\le 2N}} \frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{4N})}{\sqrt{\prod_{1\le j\le 2N}(x_{2j}-x_{2j-1})}}.$$

Fusion of pure partition functions

PDE : $\left[\partial_i^3 - 4\mathcal{L}^{(i)}_{-2}\partial_i + 2\mathcal{L}^{(i)}_{-3}\right]\mathcal{Z}_{\hat{\alpha}}(y_1,\ldots,y_{2N}) = 0,$

$$\mathcal{L}^{(i)}_{-2} := \sum_{j\neq i}\left(\frac{1}{(y_j-y_i)^2} - \frac{1}{y_j-y_i}\partial_j\right), \quad \mathcal{L}^{(i)}_{-3} := \sum_{j\neq i}\left(\frac{2}{(y_j-y_i)^3} - \frac{1}{(y_j-y_i)^2}\partial_j\right).$$

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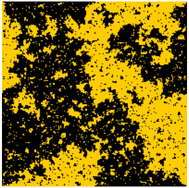
Pure Partition Functions



Pure Partition Functions

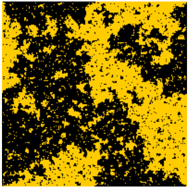
$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{(6-\kappa)/\kappa}{(x_j - x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0.$

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})).$

ASY : $\lim_{x_j, x_{j+1}\to\xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \begin{cases} \mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \text{if } \{j, j+1\} \in \alpha; \\ 0, & \text{else}. \end{cases}$

Pure Partition Functions



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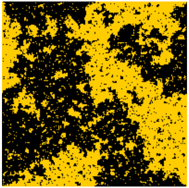
Uniqueness

Uniqueness for $\kappa \in (0, 8)$: [Flores-Kleban, CMP 2015]

Existence

- Existence for $\kappa \in (0, 8) \setminus \mathbb{Q}$: [Kytölä-Peltola, CMP 2016]
- Existence for $\kappa \in (0, 4]$: [Peltola-W. CMP 2019, Beffara-Peltola-W. AOP 2021]
- Existence for $\kappa \in (0, 6]$: [W. CMP 2020]
- Existence conjectured for $\kappa \in (6, 8)$: see e.g. [Peltola, JMP 2019]

Pure Partition Functions



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Question : What about $\kappa = 8$?

Uniform spanning tree

- $G = (V, E)$: a finite connected graph.
- A tree is a subgraph of G without loops.
- A spanning tree is a tree that covers all the vertices.
- UST : uniform spanning tree.
- UST in topological polygon and Peano curves

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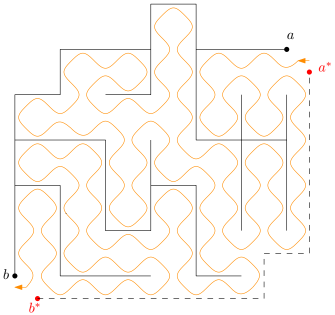
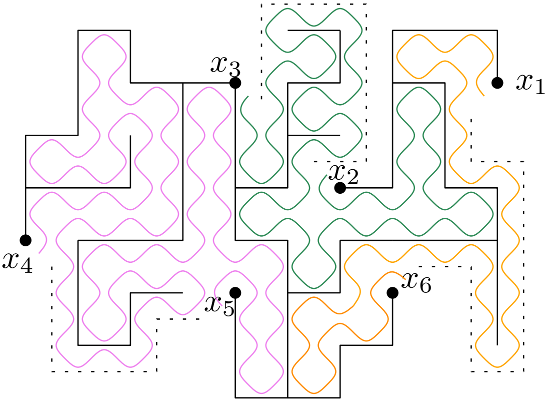


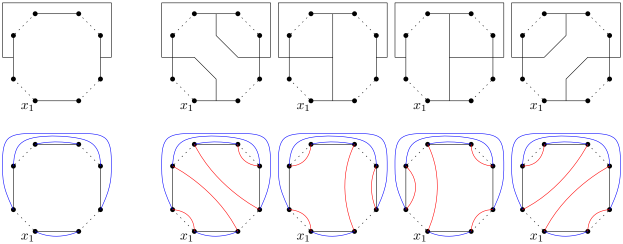
Theorem [Lawler-Schramm-Werner, AOP 2004]

The Peano curve in UST with Dobrushin boundary conditions converges weakly to SLE_8.

Uniform spanning tree



Uniform spanning tree in polygons



boundary conditions :

$$\beta = \{\{1,2\},\{3,8\},\{4,7\},\{5,6\}\}$$

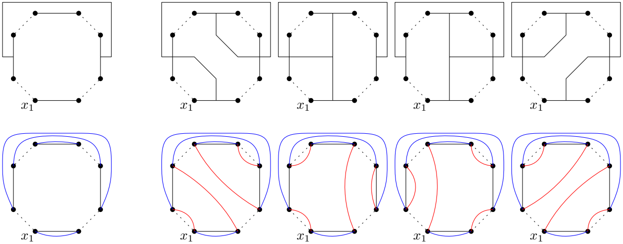
possible link patterns :

$$\begin{aligned}\alpha_1 &= \{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}\\ \alpha_2 &= \{\{1,8\},\{2,5\},\{3,4\},\{6,7\}\}\\ \alpha_3 &= \{\{1,6\},\{2,3\},\{4,5\},\{7,8\}\}\\ \alpha_4 &= \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\}\end{aligned}$$

the (renormalized) meander matrix

$$\mathcal{M}_{\alpha,\beta} = 1.$$

Uniform spanning tree in polygons



Theorem [Liu-Peltola-W. 2021]

Consider UST in polygon $(\Omega; x_1, \ldots, x_{2N})$ with boundary conditions $\beta \in \mathrm{LP}_N$. The connection of Peano curves forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{A}_\delta = \alpha] = \mathcal{M}_{\alpha,\beta} \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{F}_\beta(\Omega; x_1, \ldots, x_{2N})}$$

where $\{\mathcal{F}_\beta : \beta \in \mathrm{LP}_N\}$ is a collection of Coulomb gas integrals, $\{\mathcal{M}_{\alpha,\beta} : \alpha, \beta \in \mathrm{LP}_N\}$ is the (renormalized) meander matrix, and $\mathcal{Z}_\alpha = \sum_\gamma \mathcal{M}^{-1}_{\alpha,\gamma} \mathcal{F}_\gamma$.

Previous results : [Kenyon-Wilson, Trans. AMS 2011], [Dubédat, JSP 2006]

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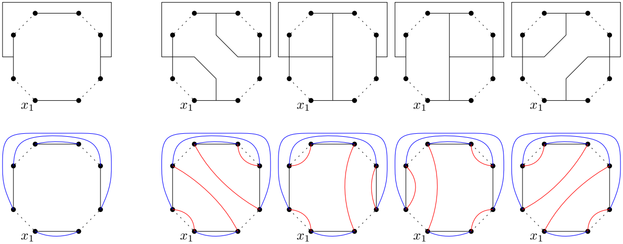

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Previous results : [Kenyon-Wilson, Trans. AMS 2011], [Dubédat, JSP 2006]

Difficulty : proper observable

Coulomb gas integrals

Boundary conditions :

$\beta = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$
with link endpoints ordered as $a_1 < a_2 < \cdots < a_N$ and $a_r < b_r$ for all $1 \le r \le N$,

Coulomb gas integrals : suppose $x_1 < \cdots < x_{2N}$,

$$\mathcal{F}_\beta(x_1, \ldots, x_{2N}) := \prod_{1 \le i < j \le 2N} (x_j - x_i)^{1/4} \int_{x_{a_1}}^{x_{b_1}} \cdots \int_{x_{a_N}}^{x_{b_N}} \prod_{1 \le r < s \le N} (u_s - u_r) \prod_{r=1}^{N} \frac{\mathrm{d}u_r}{\prod_{k=1}^{2N} (u_r - x_k)^{1/2}},$$

where the branch of the multivalued integrand is chosen to be real and positive when

$$x_{a_r} < u_r < x_{a_r+1} \qquad \text{for all } 1 \le r \le N.$$

Theorem [Liu-Peltola-W. 2021]

- $\mathcal{F}_\beta$ satisfies PDE.
- $\mathcal{F}_\beta$ satisfies COV.
- $\mathcal{F}_\beta$ is POS.
- $\mathcal{F}_\beta$ satisfies ASY.

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{F}_\beta(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{1/4}} = \pi \mathcal{F}_{\beta/\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), \qquad \text{if } \{j, j+1\} \in \beta;$$

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{F}_\beta(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{1/4} |\log(x_{j+1} - x_j)|} = \mathcal{F}_{\wp_j(\beta)/\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), \qquad \text{if } \{j, j+1\} \notin \beta.$$

Uniform spanning tree and log-CFT

Theorem [Liu-Peltola-W. 2021]

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J. Cardy. J. Phys. A. 46 :49, 31 pp, 2013 :

which is in general non-unitary. The case $Q = 1$ corresponds to bond percolation, for which the partition function $Z = 1$, so in this case we expect the scaling limit to be a logCFT, although other values of Q (-2, corresponding to uniform spanning trees, and $+2$, corresponding to the extended Ising model) also turn out to be logarithmic.

- V. Gurarie. Logarithmic operators in conformal field theory. Nucl. Phys. B,1993.
- M. Gaberdiel, H. Kausch. Indecomposable fusion products. Nucl. Phys. B,1996.
- H. Kausch. Symplectic fermions. Nucl. Phys. B, 2000.
- P.Pearce, J. Rasmussen. Solvable critical dense polymers. J. Stat. Mech.,2007

- We provide a rigorous result towards a log-CFT description of the scaling limit of the UST : $c = -2$.

Thanks !

1. Peltola-W. Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. *Comm. Math. Phys.* 366(2) : 469-536, 2019.
2. W. Hypergeometric SLE : conformal Markov characterization and applications *Comm. Math. Phys.* 374(2) : 433-484, 2020.
3. Beffara-Peltola-W. On the uniqueness of global multiple SLEs *Ann. Probab.* 49(1) : 400-434, 2021.
4. Liu-W. Scaling limits of crossing probabilities in metric graph GFF *Electron. J. Probab.* 26 : article no. 37, 1-46, 2021.
5. Ding-Wirth-W. Crossing estimates from metric graph and discrete GFF *Ann. Inst. H. Poincaré Probab. Statist.* 2022+.
6. Peltola-W. Crossing probabilities of multiple Ising interfaces arXiv:1808.09438. 2018
7. Han-Liu-W. Hypergeometric SLE with $\kappa = 8$: convergence of UST and LERW in topological rectangles. arxiv:2008.00403. 2020.
8. Liu-Peltola-W. Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in $c = -2$ logarithm CFT. arXiv:2108.04421. 2021.
9. Liu-W. Loop-erased random walk branch of uniform spanning tree in topological polygons. arXiv:2108.10500. 2021.