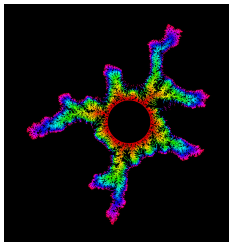


Scaling Limits of Laplacian Random Growth Models

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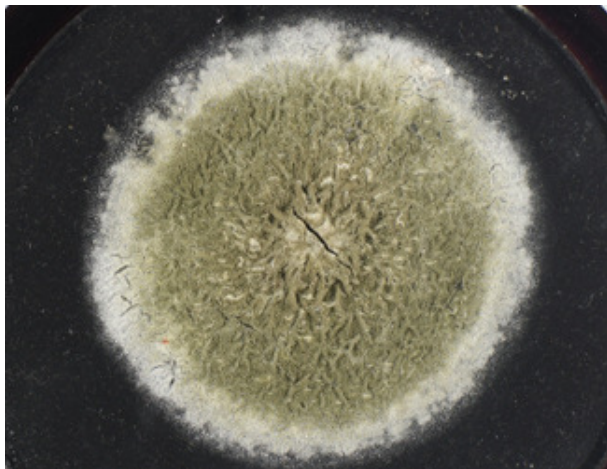
Biological growth



Photo by James Wearn



Biological growth



Gift by Sir Alexander Fleming to Edinburgh University Library, Scotland



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Mineral deposition



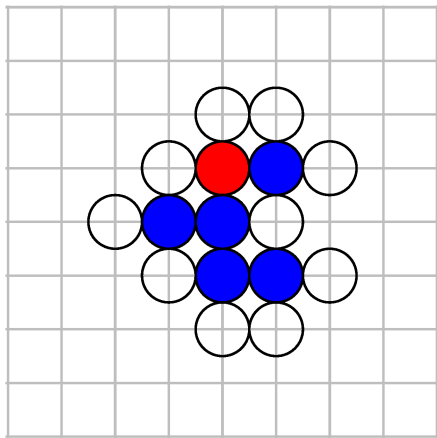
Photo by Kevin R Johnson



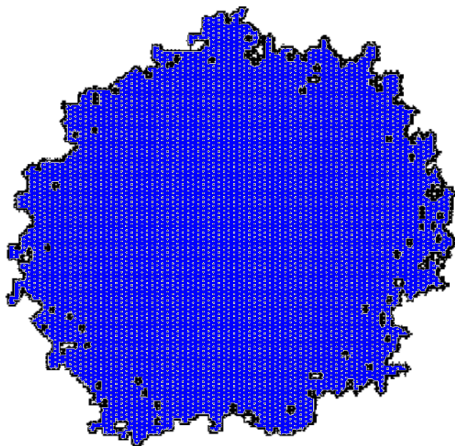
Mineral deposition



Lattice models for random growth



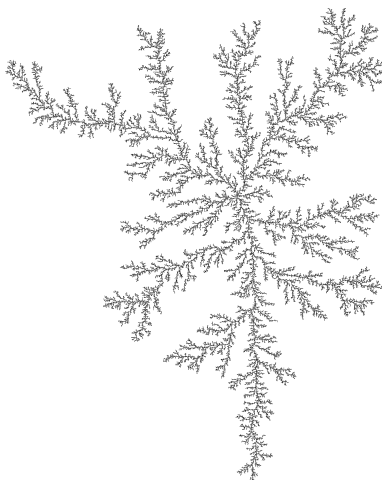
Eden model for biological growth (1,500 particles)



Simulation by H.J. Herrmann



DLA cluster for mineral deposition (2,000 particles)



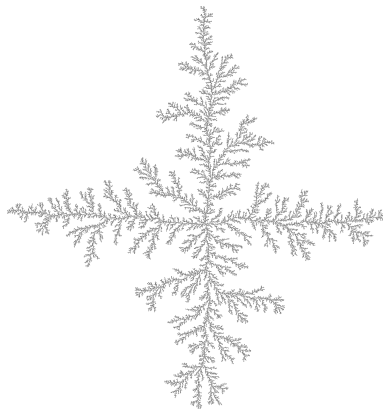
Other lattice models for random growth

- Dielectric breakdown models (DBM)
- Internal diffusion-limited aggregation (IDLA)
- First passage percolation (FPP)
- Interface models: ballistic deposition, corner growth model, etc.

What do we know about DLA?

- **Not much!**
- H. Kesten: At time t DLA is contained in a ball of radius $t^{2/3}$.
- No proof DLA does not converge to a ball.
- Main open problems:
 - Existence of universal limit.
 - Growth rate of the cluster.
 - Structure of the limiting set (e.g. fractal dimension).
 - Number of arms.

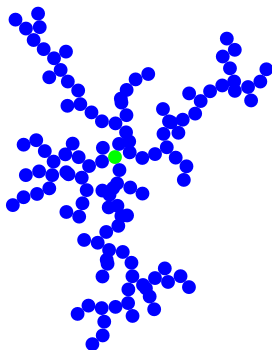
DLA cluster of size 4,096



Simulation by Vincent Beffara



Off-lattice DLA



Ball shaped particles perform BM (from infinity) until they attach to the aggregate.

Harmonic measure

- The attachment point is distributed according to harmonic measure on the cluster boundary (from infinity).
- By conformal invariance of BM, harmonic measure is conformally invariant.
- An algorithm for sampling a boundary point of a set A :
Let D_0 denote the exterior unit disk in the complex plane \mathbb{C} and let $\Phi : D_0 \rightarrow A^c$ be conformal. Choose a point $y \in \partial D_0$ uniformly. Then take $\Phi(y)$.

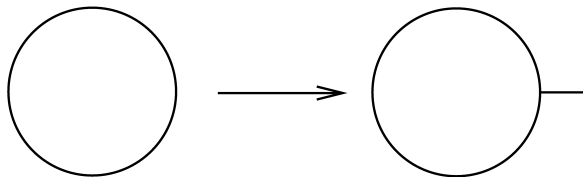
Conformal mapping representation of a slit-shaped particle

Let P denote the slit $[1, 1 + \delta]$ in the complex plane.

There exists a unique conformal mapping $F : D_0 \rightarrow D_0 \setminus P$ that fixes ∞ in the sense that

$$F(z) = e^c z + O(1) \quad \text{as } |z| \rightarrow \infty,$$

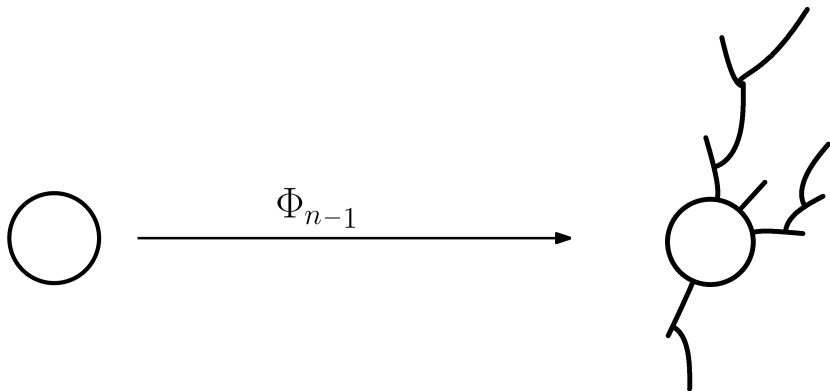
for some $c > 0$, the (log of the) capacity, which satisfies $e^c = 1 + \frac{\delta^2}{4(1+\delta)}$.



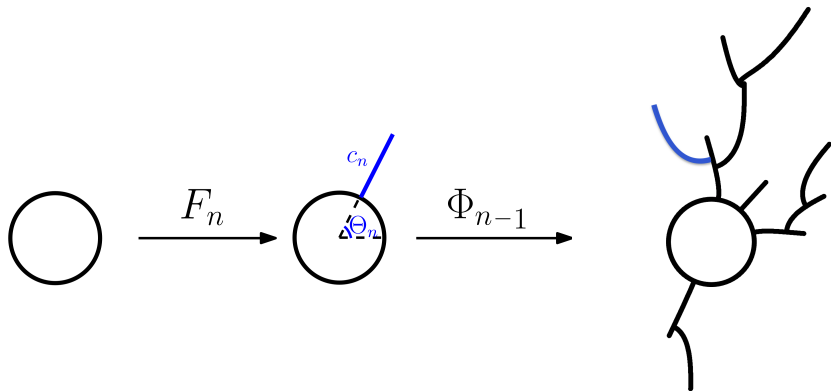
Conformal mapping representation of a cluster

- Suppose P_1, P_2, \dots is a sequence of particles, where P_n has capacity c_n (or length δ_n) and attachment angle Θ_n , $n = 1, 2, \dots$. Let F_n be the particle map corresponding to P_n .
 - Set $\Phi_0(z) = z$.
 - Recursively define $\Phi_n(z) = \Phi_{n-1} \circ F_n(z)$, for $n = 1, 2, \dots$.
- This generates a sequence of conformal maps $\Phi_n : D_0 \rightarrow K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.

Cluster formed by iteratively composing mappings



Cluster formed by iteratively composing mappings



$$\Phi_n = \Phi_{n-1} \circ F_n = F_1 \circ F_2 \circ \cdots \circ F_n$$

Loewner chain representation

Define the driving measure $\mu_t = \delta_{e^{i\xi_t}}$, where

$$\xi_t = \sum_{k=1}^N \Theta_k 1_{(C_{k-1}, C_k]}(t),$$

with $C_k = \sum_{j=1}^k c_j$, for angles $\{\Theta_k\}$ and capacities $\{c_k\}$ as above.

Consider the solution to the Loewner equation

$$\partial_t \Psi_t(z) = z \Psi_t'(z) \int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} d\mu_t(e^{i\theta}),$$

with initial condition $\Psi_0(z) = z$.

Then

$$\Phi_n = \Psi_{C_n}, \quad n = 0, 1, 2, \dots$$

Parameter choices for physical models

- By varying the sequences $\{\Theta_n\}$ and $\{c_n\}$, it is possible to describe a wide class of growth models.
- For biological growth (Eden model)

$$\mathbb{P}(\Theta_n \in (a, b)) \propto \int_a^b |\Phi'_{n-1}(e^{i\theta})| d\theta$$

and

$$c_n \approx c |\Phi'_{n-1}(e^{i\Theta_n})|^{-2}$$

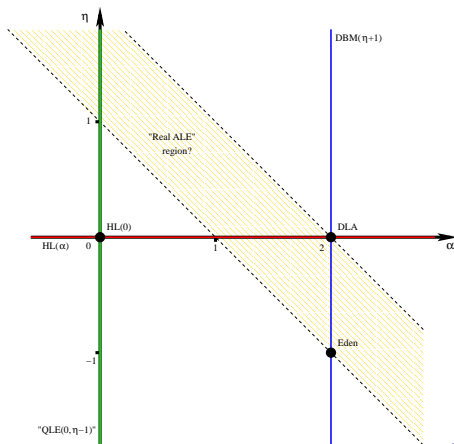
- For DLA, c_n is as above and

$$\mathbb{P}(\Theta_n \in (a, b)) = \mathbb{P}(\Phi_{n-1}^{-1}(B_\tau) \in (a, b)) \propto (b - a)$$

where B_t is Brownian motion started from ∞ and τ is the hitting time of the cluster K_{n-1} .

Aggregate Loewner Evolution, $ALE(\alpha, \eta, \sigma)$

- Θ_n distributed $\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$; $c_n = c |\Phi'_{n-1}(e^{\sigma+i\Theta_n})|^{-\alpha}$.



Regularization for ALE

- Even after the arrival of a single slit particle, the map $\theta \mapsto |\Phi'_n(e^{i\theta})|$ is badly behaved and takes the values 0 and ∞ .
- For some values of η ,

$$\int_{-\pi}^{\pi} |\Phi'_{n-1}(e^{i\theta})|^{-\eta} d\theta = \infty,$$

so regularization is necessary to even define the measure.

- A solution is to let Θ_n have distribution

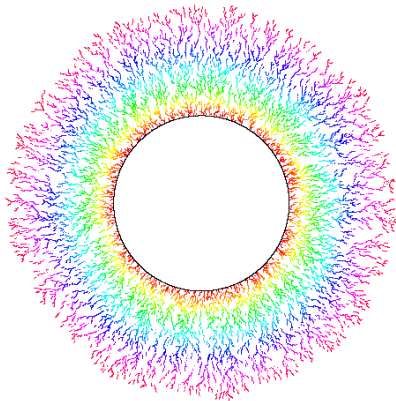
$$\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$$

for $\sigma > 0$ and take the limit $\sigma \rightarrow 0$.

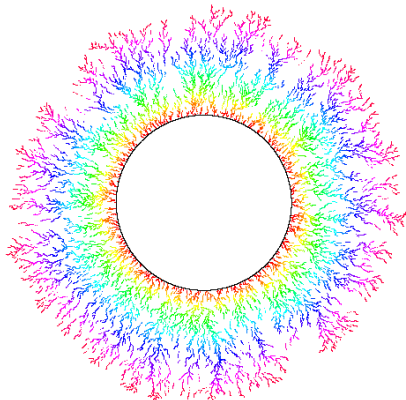
- Models are very sensitive to the rate at which $\sigma \rightarrow 0$. Can be argued that $\sigma \sim c^{1/2}$ is natural from a physical point of view.



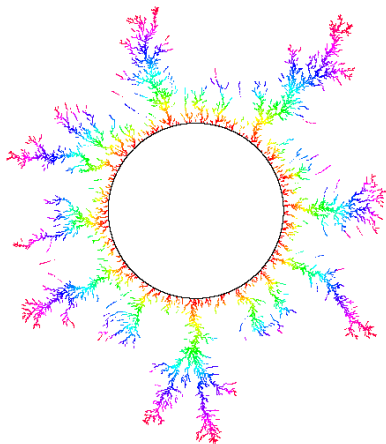
ALE(0,0) cluster with 8,000 particles for $c = 10^{-4}$



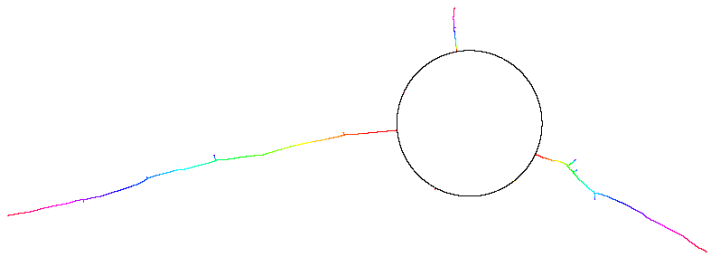
ALE(0,1,0.02) cluster with 8,000 particles for $c = 10^{-4}$



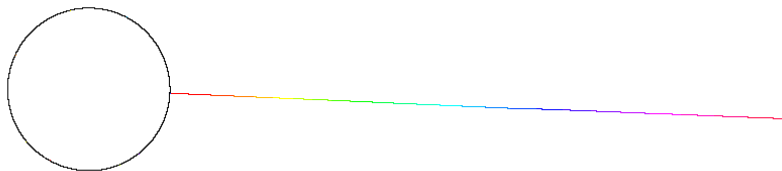
ALE(0,2,0.02) cluster with 8,000 particles for $c = 10^{-4}$



ALE(0,2, 10^{-8}) cluster with 10,000 particles for $c = 10^{-4}$



ALE(0,4, 10^{-8}) cluster with 10,000 particles for $c = 10^{-4}$



Previous results for HL(0)

Much of the previous work relates to HL(0) as particle maps are i.i.d. so the model is mathematically the most tractable.

- Norris and Turner (2012):
 - small-particle scaling limit of HL(0) is a growing disk:
 $\Phi_n(z) \approx e^{cn}z$
 - branching structure is related to the Brownian web
 - expected size of the n^{th} particle is roughly $\delta \exp cn$, so HL(0) is “unphysical”.
- Silvestri (2017): fluctuations converge to a log-correlated Fractional Gaussian Field.

Previous results for $HL(\alpha)$ for $\alpha \neq 0$

All results for $HL(\alpha)$ with $\alpha \neq 0$ require regularization.

- Rohde and Zinsmeister (2005): estimates on the dimension of scaling limits for a regularized version of $HL(\alpha)$ under capacity rescaling.
- Sola, Turner, Viklund (2015): small-particle scaling limit of a sufficiently regularized $HL(\alpha)$ is a growing disk for all α .
- Liddle and Turner (2020): fluctuations for *very* regularized $HL(\alpha)$ under capacity rescaling.

Disk-regime results for $ALE(\alpha, \eta, \sigma)$

Norris, Turner, Silvestri (2019 and 2021)

- **Scaling limit:** For all $\alpha + \eta \leq 1$, provided $\sigma \gg c^{1/2}$ as $c \rightarrow 0$ ($\sigma \gg c^{1/5}$ when $\alpha + \eta = 1$),

$$\Phi_n(z) \approx (1 + \alpha cn)^{1/\alpha} z.$$

- **Fluctuations:** Set

$\mathcal{F}_n^{(c)}(z) = c^{-1/2} ((1 + \alpha cn)^{-1/\alpha} \Phi_n(z) - z)$. Then

$\mathcal{F}_{\lfloor t/c \rfloor}^{(c)}(z) \rightarrow \mathcal{F}_t(z)$ where

$$\dot{\mathcal{F}}_t(z) = \frac{1}{1 + \alpha t} \left((1 - (\alpha + \eta)) z \mathcal{F}'_t(z) - \mathcal{F}_t(z) + \sqrt{2} \xi_t(z) \right).$$

Here $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.



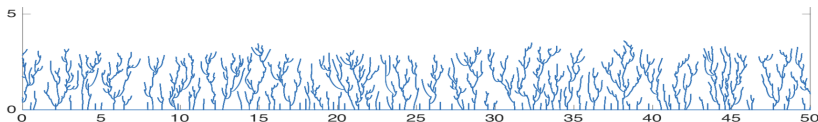
Singular-regime results for $\text{ALE}(\alpha, \eta, \sigma)$

- Sola, Turner, Viklund (2019): scaling limit of $\text{ALE}(\alpha, \eta, \sigma)$ is a growing slit if $\alpha \geq 0$ and $\eta > 1$ when using slit particles, provided $\sigma \rightarrow 0$ sufficiently fast as $c \rightarrow 0$.
- Higgs (2021): scaling limit of $\text{ALE}(0, \eta, \sigma)$ converges to a SLE_4 for $\eta < -2$ when using slit particles, provided σ is very small. Other SLE_κ 's with $\kappa > 4$ can be obtained by using different particle shapes.



Other model variants

- Sola, Turner, Viklund (2012): Anisotropic HL(0)
- Berestycki and Silvestri (2021): Constrained HL(0)
- Berger, Turner, Procaccia (2021): Stationary HL(0)



Open questions / conjectures

- Phase transitions
 - From disks to non-disks
 - From absolutely continuous support to singular support
- Universality
 - Of scaling limits
 - Of fluctuations
- Connections
 - Between model variants
 - With lattice models
 - With SLE
 - With GMC

References

- [1] M.B.Hastings and L.S.Levitov, *Laplacian growth as one-dimensional turbulence*, Physica D 116 (1998).
- [2] F.Johansson Viklund, A.Sola, A.Turner, *Small particle limits in a regularized Laplacian random growth model*, CMP, 334 (2015).
- [3] J.Norris, V.Silvestri, A.Turner, *Scaling limits for planar aggregation with subcritical fluctuations*, arXiv:1902.01376.
- [4] J.Norris, A.Turner, *Hastings-Levitov aggregation in the small-particle limit*, CMP, 316 (2012).
- [5] S.Rohde, M.Zinsmeister *Some remarks on Laplacian growth*, Topology and its Applications, 152 (2005).
- [6] A.Sola, A.Turner, F.Viklund, *One-dimensional scaling limits in a planar Laplacian random growth model*, CMP, 371 (2019).
- [7] V.Silvestri, *Fluctuation results for Hastings-Levitov planar growth*. PTRF, 167 (2017).