Characterising the Gaussian free field

Connections Workshop, MSRI, 20th January 2022

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Gaussian free field

Definition

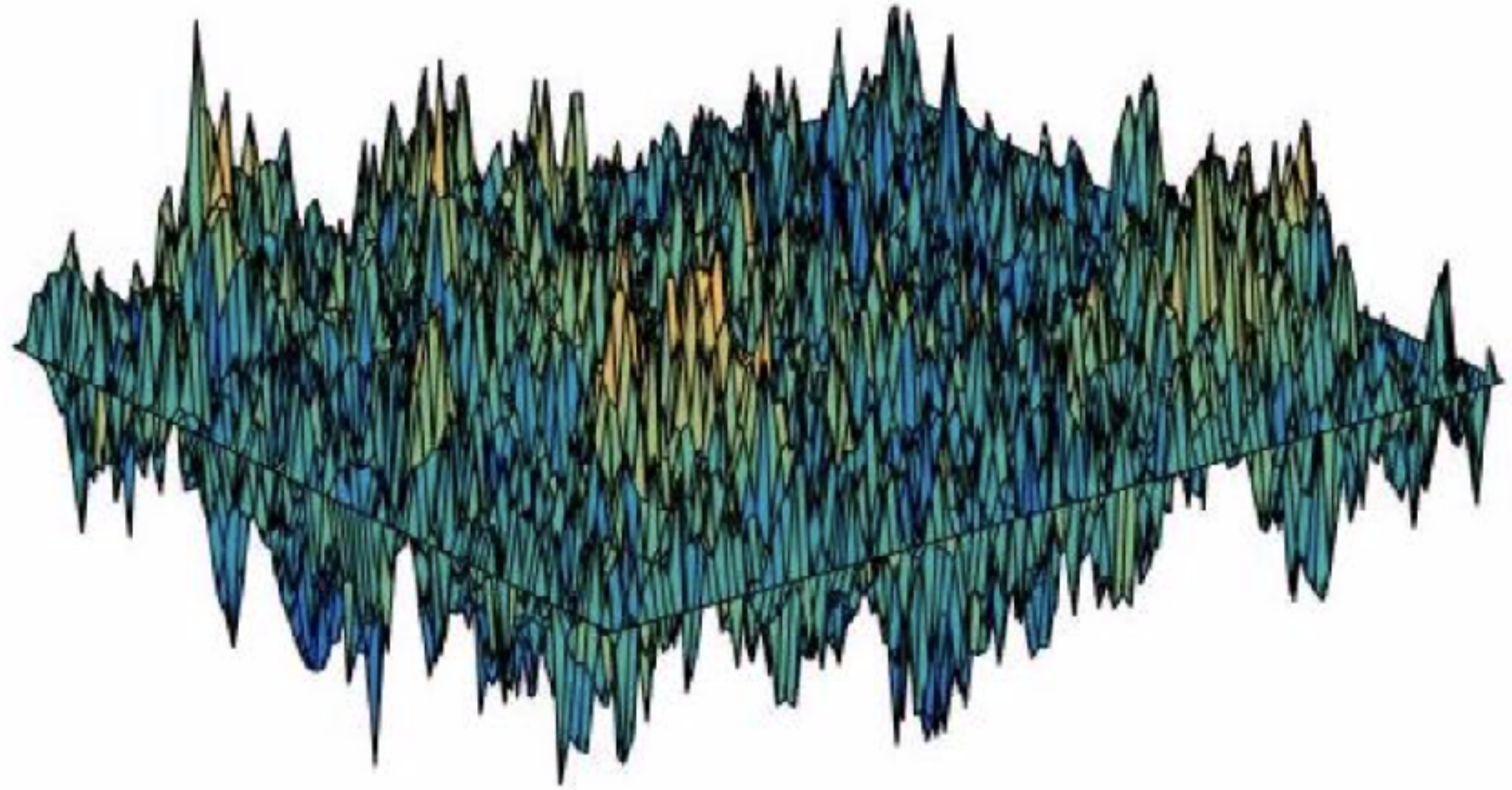
(With 0 boundary conditions, in the unit ball $\mathbb{B} \subset \mathbb{R}^d$, $d \geq 1$)

Random Schwarz distribution h such that $(h, f)_{f \in C_c^\infty(\mathbb{B})}$ is a **centred, Gaussian process** with

$$\mathbb{E}((h,f)(h,g)) = \iint_{\mathbb{B}^2} f(x) G^{\mathbb{B}}(x,y) g(y)\, dxdy$$

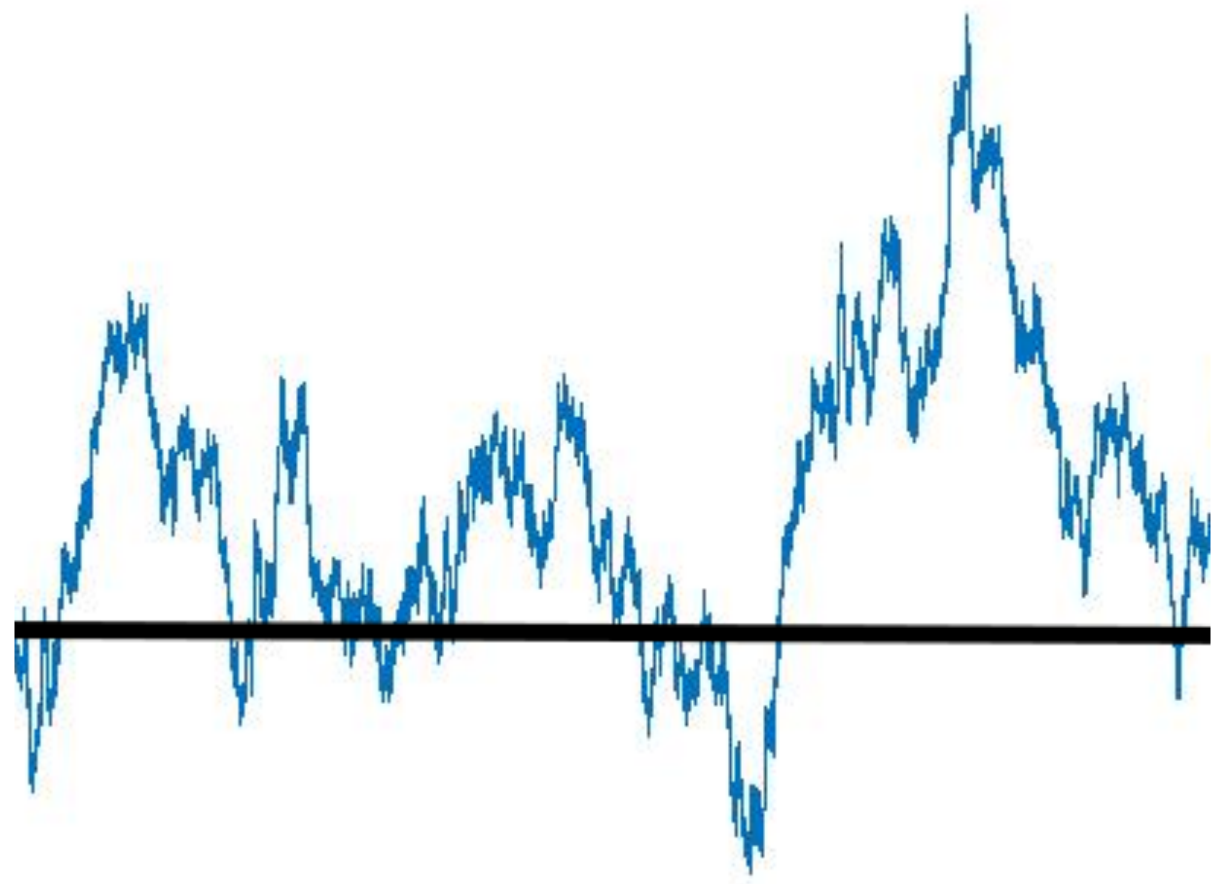
for all $f, g \in C_c^\infty(\mathbb{B})$

$G^{\mathbb{B}}$ is the Greens function for the Laplacian with zero boundary conditions in $\mathbb{B}$



Example: $d = 1$

Brownian bridge



- $G^{\mathbb{B}}(s, t) = s(1 - t)$ for $0 \leq s < t \leq 1$
- $\Rightarrow$ standard Brownian bridge on $[0,1]$
- This Schwarz distribution is actually a **well-defined function** (not true for $d \geq 2$)
- **Universal scaling limit** of random walks with zero boundary conditions
- **Lots of characterisations** (at least for Brownian motion)

Example: $d = 2$

Planar GFF

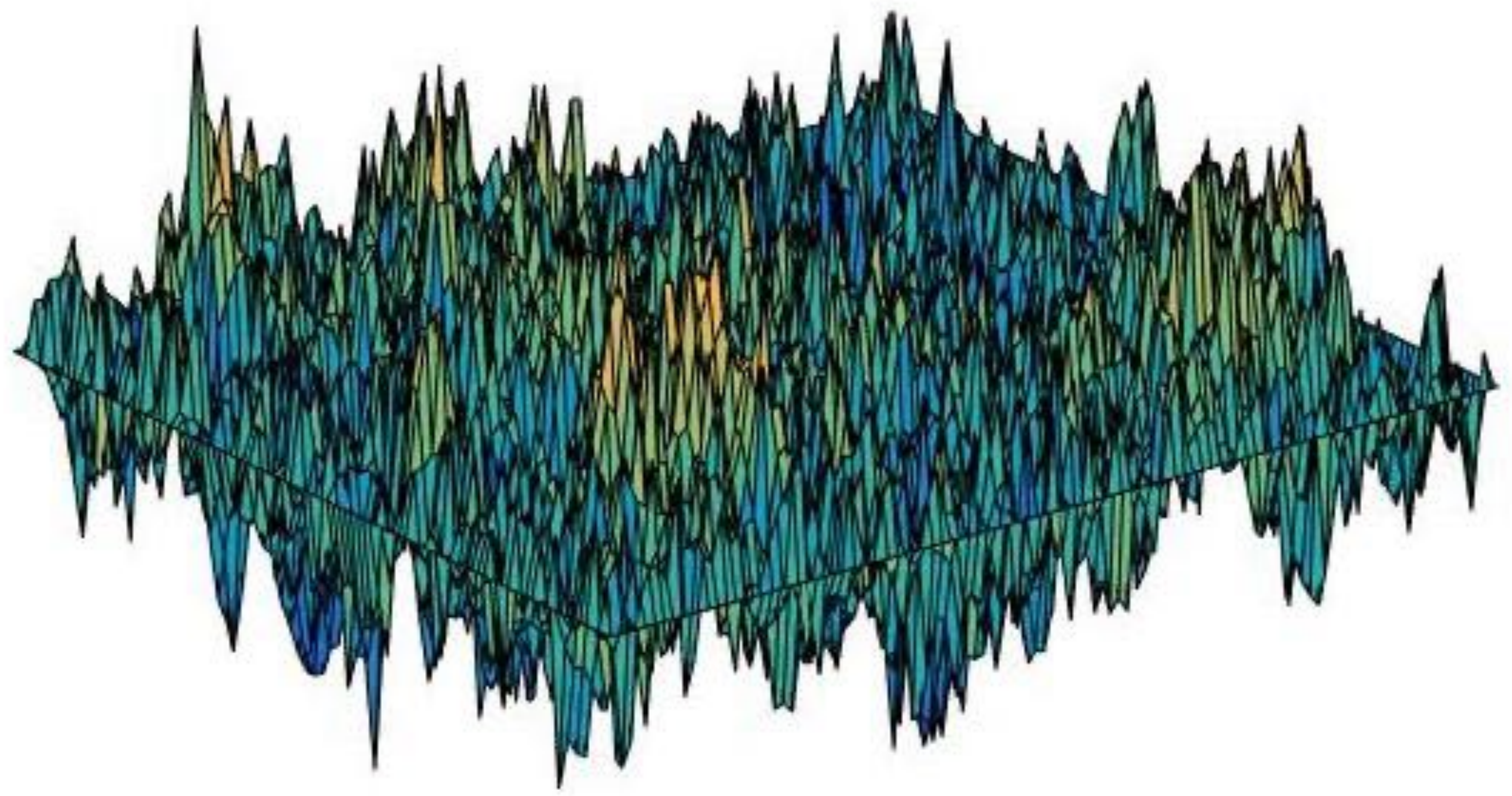
$$G^{\mathbb{B}}(0,z) = -\frac{1}{2\pi}\log|z| \text{ for } z \in \mathbb{B}$$

- If $D \subset \mathbb{C}$ is simply connected and $\varphi : D \to \mathbb{B}$ is conformal then

$$G^{D}(x,y) = G^{\mathbb{B}}(\varphi(x),\varphi(y)) \ \forall x, y \in D$$

- Can define GFF in any simply connected D; **conformally invariant**

- Conjectured/proven to arise as a **universal scaling limit**



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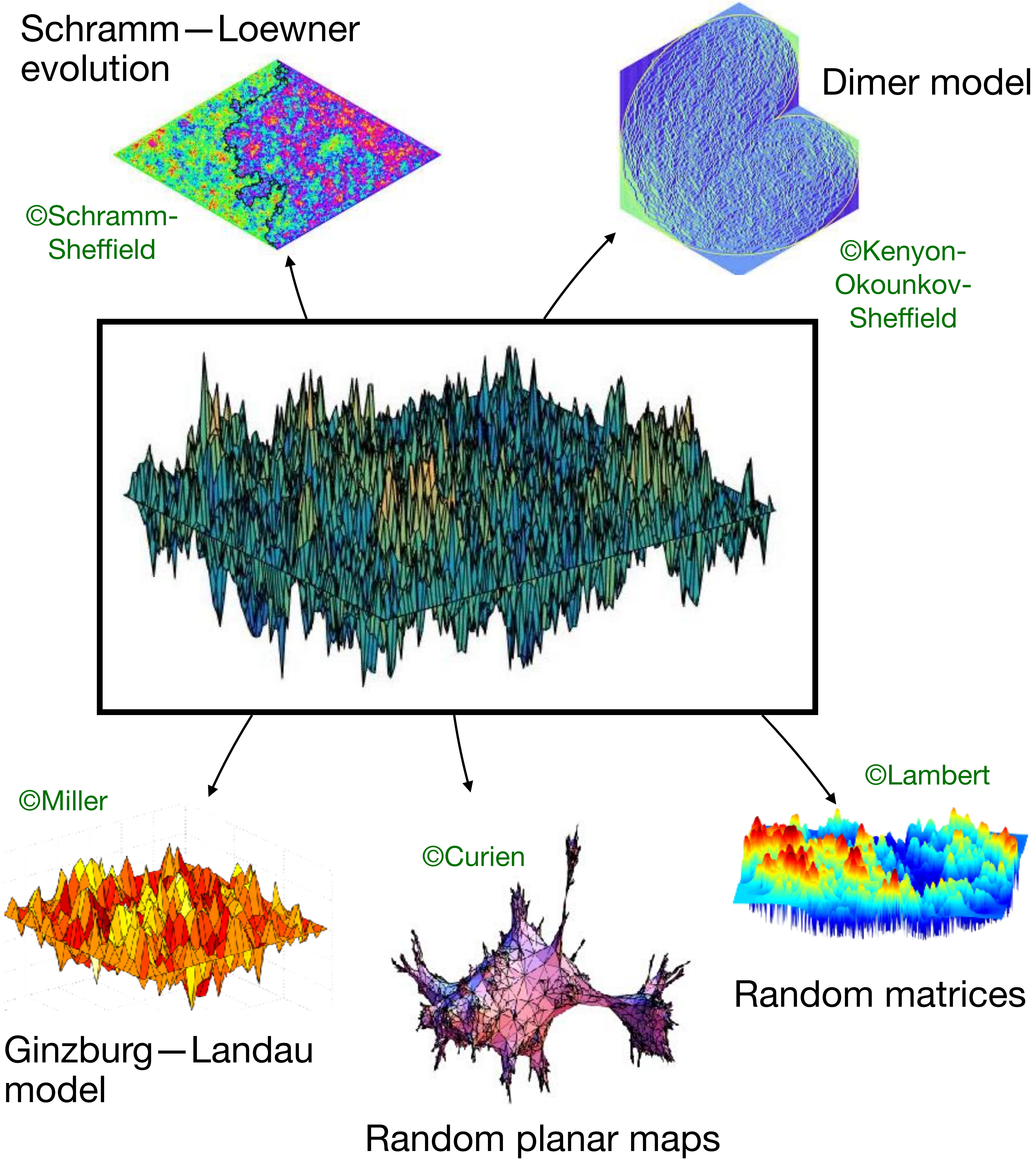
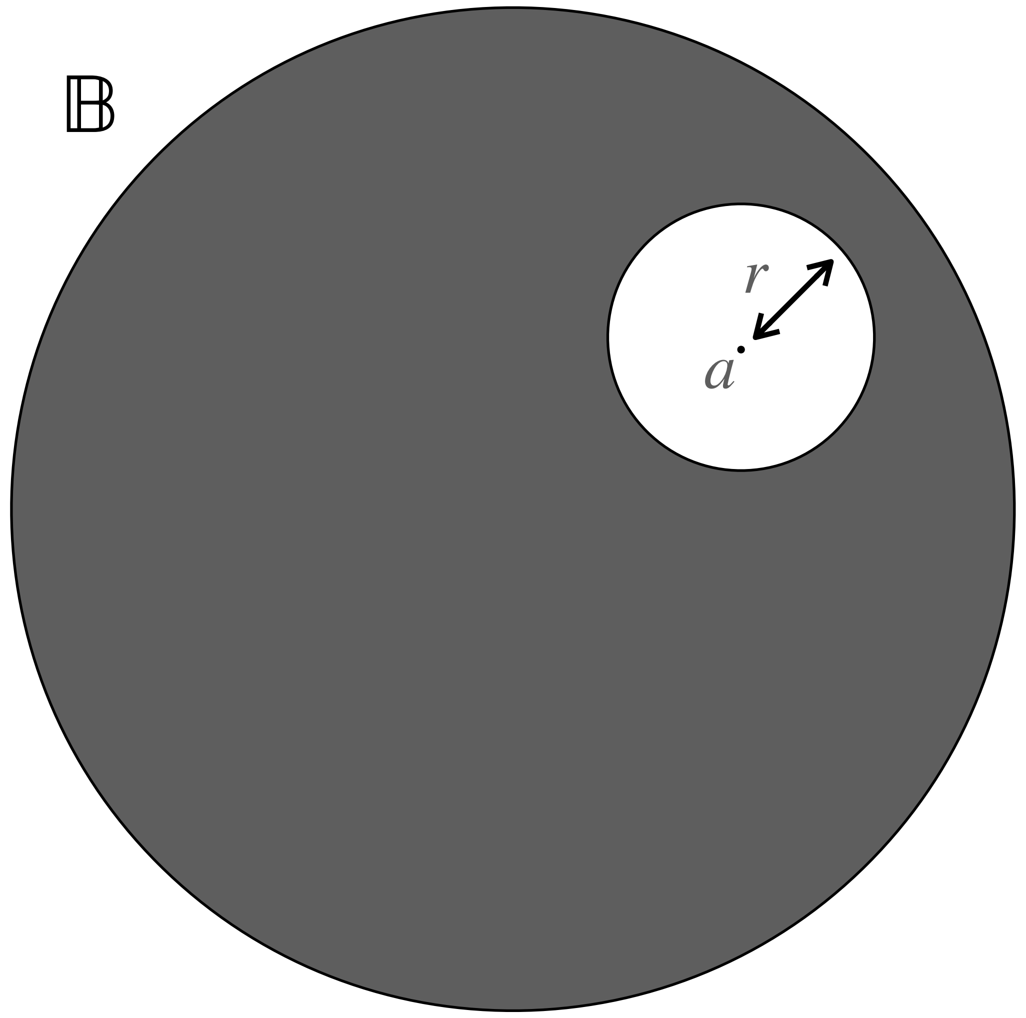


The domain Markov property



$$h \;=\; h^{a+r\mathbb{B}} \;+\; \varphi^{a+r\mathbb{B}}$$

Scaled zero boundary field + independent harmonic function

- $\varphi^{a+r\mathbb{B}}$ is a random Schwarz distribution in $\mathbb{B}$ which a.s. corresponds to a **harmonic function** when restricted to $a + r\mathbb{B}$

- The process $(h^{a+r\mathbb{B}}, r^{-d}f(r\cdot + a))_{f\in C_c^\infty(\mathbb{B})}$ is equal in law to $r^{1-\frac{d}{2}}(h,f)_{f\in C_c^\infty(\mathbb{B})}$

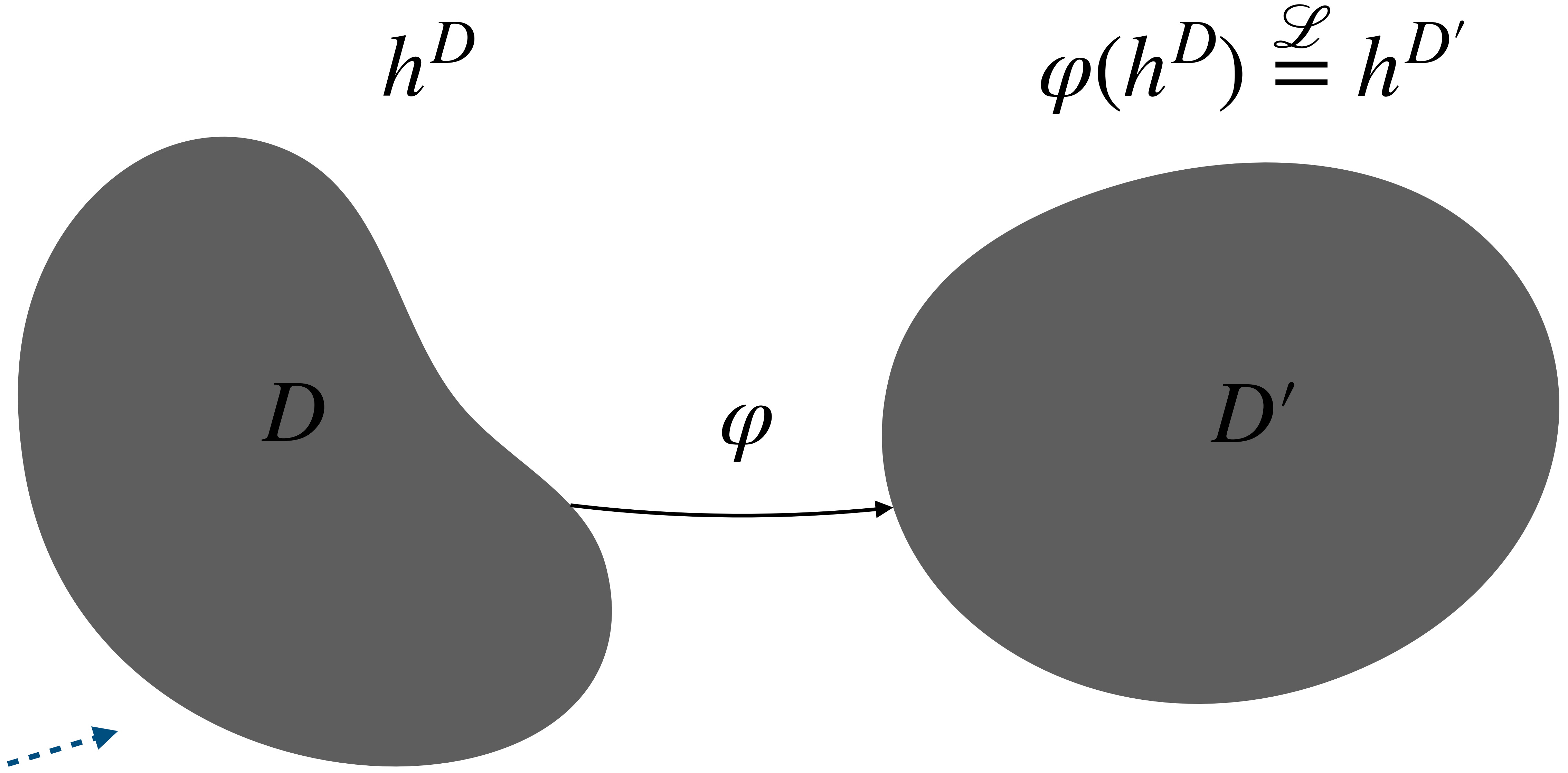
- $h^{a+r\mathbb{B}}$ and $\varphi^{a+r\mathbb{B}}$ are **independent**

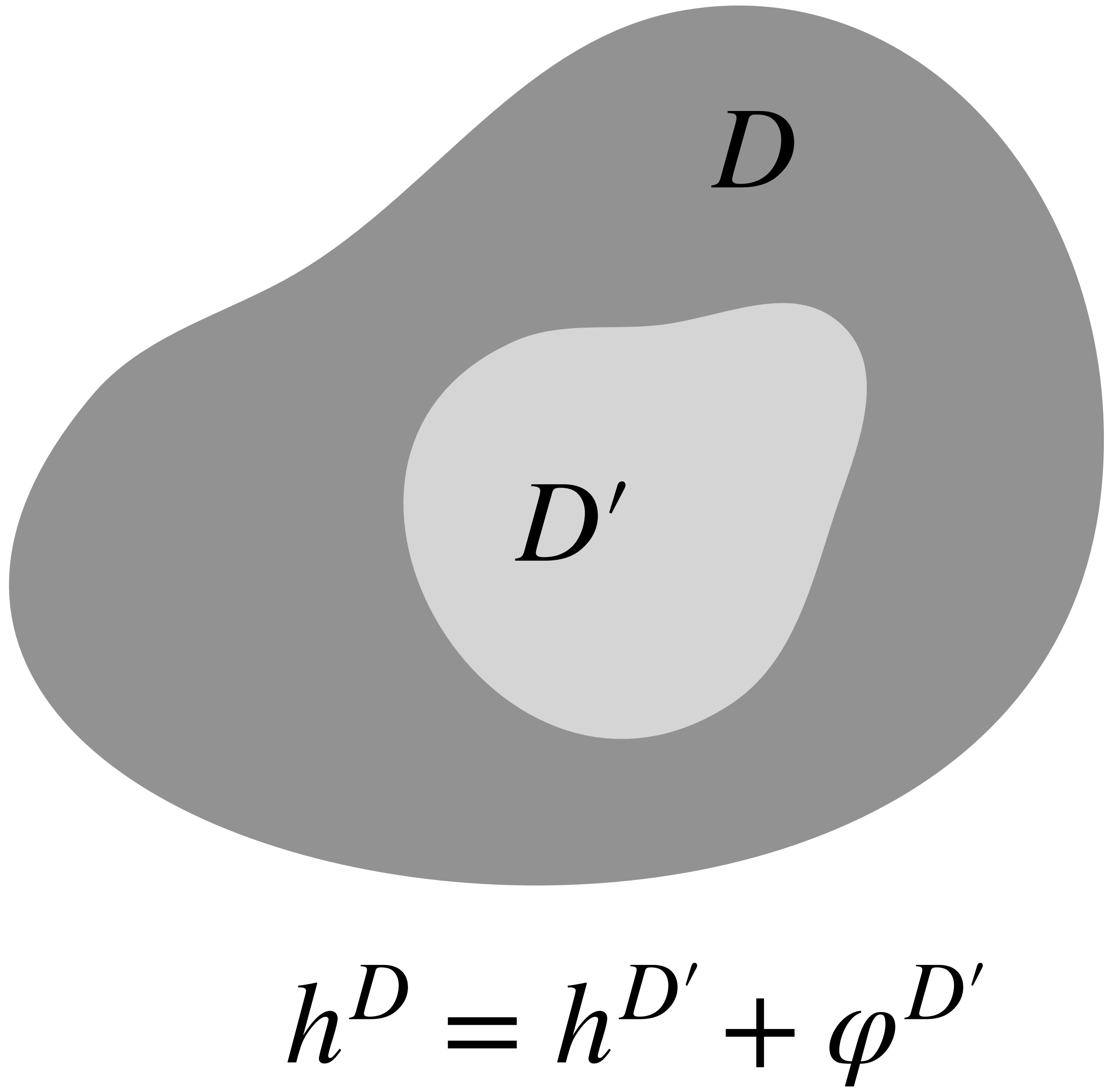
2d Case

Berestycki-P-Ray

Suppose $(h^D, f)_{f \in C_c^\infty(D)}$ is a centred linear stochastic process, defined for each simply connected $D \subset \mathbb{C}$ satisfying:

- **conformal invariance**
- the conformal **Markov property**
- $(1+\varepsilon)$-**moments** for some $\varepsilon > 0$
- **zero boundary conditions** and stochastic **continuity**





Main Result

$d \geq 2$, $\mathbb{B} \subset \mathbb{R}^d$ unit ball

Aru-P.

Suppose that h a centred random Schwarz distribution on $\mathbb{B}$ satisfying

- **domain Markov property** for balls
- **fourth moments** $(\mathbb{E}((h,f)^4) < \infty \ \forall f \in C_c^\infty(\mathbb{B}))$
- **zero boundary conditions** ($(h,f_n) \to 0$ in L^2 for $(f_n)_{n\geq 0}$ smooth & positive with support $\to \partial\mathbb{B}$ and $\sup_n\left(\sup_{r>1}\sup_{x,y\in\partial r\mathbb{B}} |f_n(x)/f_n(y)| + \|f_n\|_{L^1(\mathbb{B})}\right) < \infty$)).

Then h is a multiple of a GFF on $\mathbb{B}$ with zero boundary conditions

Comments

And questions

- Could this be used to identify **scaling limits?**
- **Rotational invariance** is true but not needed!
- Can probably **weaken some assumptions,** e.g. exact copy in DMP, moments
- **Harmonicity** in the Markov property is **key**
- Are there other interesting fields characterised by a different Markov property? (e.g., stable bridges/fields, CLE nesting field?)
- What about **GFFs on other manifolds?**

Idea for the proof

Outline

Two steps

- Covariance is the Greens’ function (simpler step)
- Gaussianity (more challenging)

Covariance is the Greens' function

Idea of proof

Key ingredient: Suppose that for $y \in \mathbb{B}$, $k_y(x)$ is a **harmonic** function defined in $\mathbb{B}\backslash\{y\}$, such that $k_y(x) - bs(|x-y|)$ is bounded in a neighbourhood of y for some $b > 0$ and such that $(k_y, f_n)_{L^2} \to 0$ for any sequence of functions f_n as in our zero boundary condition. Then $k_y(x) = bG^{\mathbb{B}}(x, y)$ for all $x \neq y\,;\, x, y \in \mathbb{B}$.

Harmonicity + scaling + boundary conditions ⇒ Greens' function

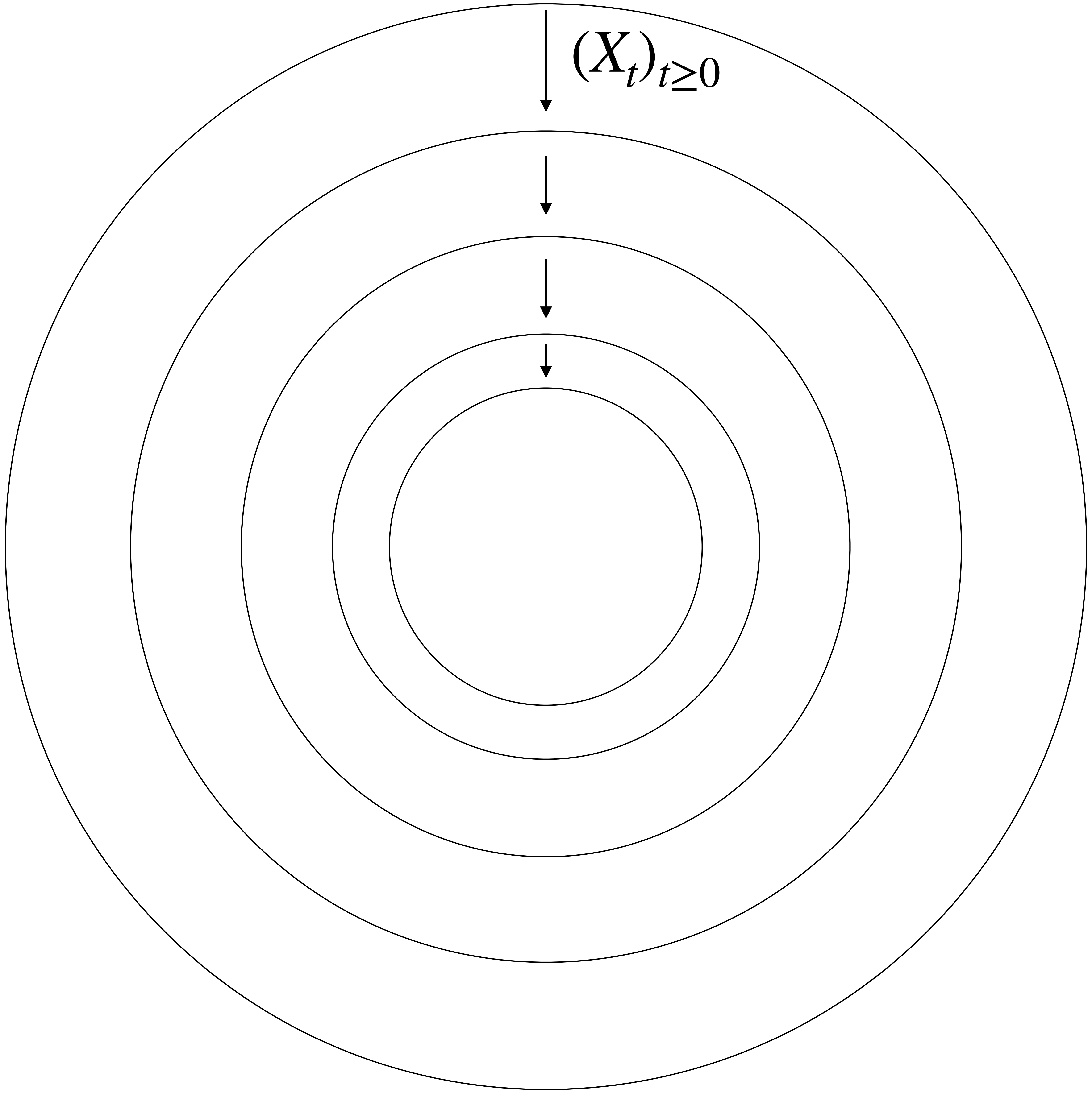
This condition can be checked quite easily using the assumptions (esp. DMP)

Gaussianity

Warm up

$$X_t := \oint_{|x|=e^{-t}} \text{''}h(x)\,dx\text{''}$$

- In 2d, $(X_t)_{t\geq 0}$ is **centered** and has **stationary** and **independent increments** by the domain Markov property
- Using the 4th moment assumption, and Kolmogorov's criterion, also has a **continuous modification**
- **$\Rightarrow X_t$ is Brownian motion $\Rightarrow$ jointly Gaussian**
- In $d \geq 3$ everything is the same except the stationarity. Still get Gaussianity!



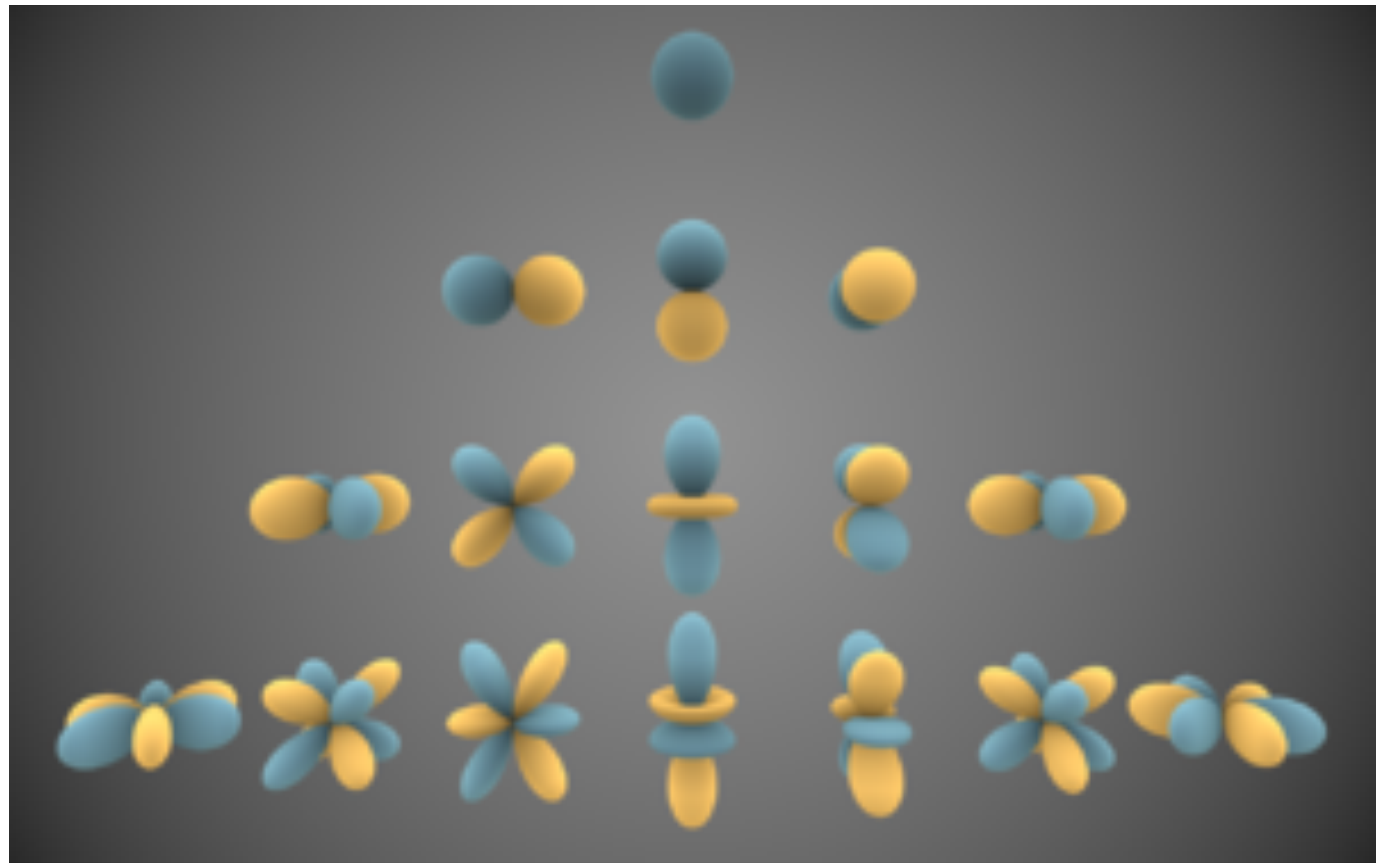
X_t is the "average value" of h on the spherical shell of radius e^{-t}

Gaussianity

Spherical harmonics

Example In 2d,
$\psi_{n,1} = \sin(n\theta), \psi_{n,2} = \cos(n\theta)$ for
$n \geq 1$

Let $(\psi_{n,j})_{n\geq 0, 1\leq j\leq M_n}$ be an orthonormal basis of **spherical harmonics** for $L^2(\partial\mathbb{B})$. In particular, $x \mapsto |x|^n \psi_{n,j}(x/|x|)$ is harmonic in $\mathbb{B}$



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- The same argument as for the spherical average case then gives that

$$X_r^{n,j} := r^{-n} \int_{|x|=r} \text{''} h(x)\, \psi_{n,j}\left(\frac{x}{|x|}\right) dx \text{''}$$

  for $r \in (0,1]$ is a Gaussian process

- Using Markov property again ⇒ "spherical harmonic averages", as a process indexed by the radius and the choice of harmonic $\psi_{n,j}$, is **Gaussian**

Gaussianity

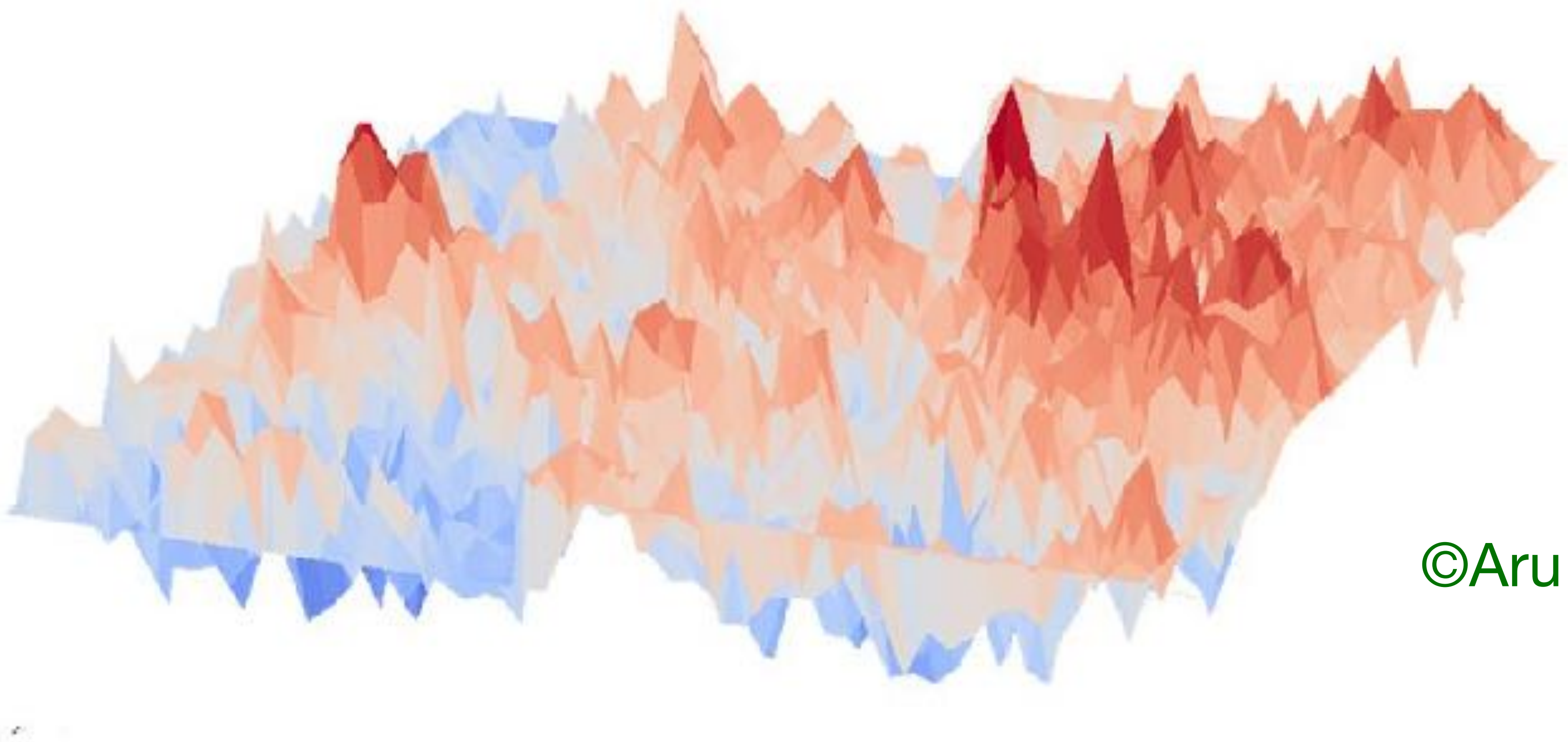
Conclusion

- There exist radial functions $(f_{n,i})_{i,n\geq 0}$ such that

$$x \mapsto f_{n,i}(|x|)\,\psi_{n,j}\left(\frac{x}{|x|}\right)$$

form an orthonormal basis of $L^2(\mathbb{B})$

- Previous slide $\Rightarrow$ h tested against these functions is jointly Gaussian $\Rightarrow$ Result!



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Thanks!