On large deviations of SLEs, real rational functions, and zeta-regularized determinants of Laplacians

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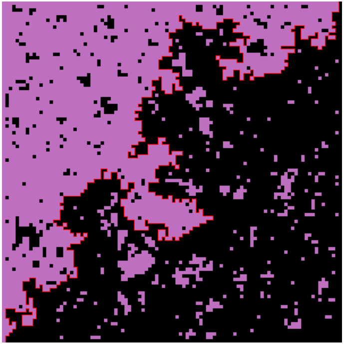
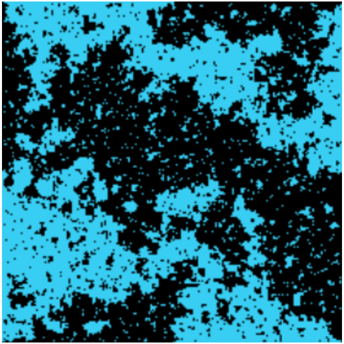
Based mainly on [arXiv:2006.08574] with **Yilin Wang**.

Rough outline

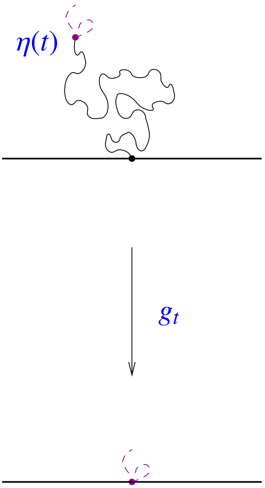
1. *Schramm-Loewner evolution* (SLE_κ) and multiple chordal SLE_κ
2. Large deviations and *Loewner energy*
3. Loewner energy in terms of
   (zeta-regularized) determinants of Laplace-Beltrami operators
4. Classification of minimizers?
   - real rational functions with prescribed critical points
   - Shapiro-Shapiro conjecture
5. Interpretation of minima?
   - semiclassical conformal blocks in conformal field theory
   - Calogero-Moser systems
6. Further questions

What is SLE$_\kappa$?





Schramm-Loewner evolution



Thm. [Loewner '23]

Any **simple chordal curve** η
(more generally, a locally growing family of hulls)
can be encoded in
a **Loewner evolution** of conformal maps
$g_t : \mathbb{H} \setminus \eta[0,t] \to \mathbb{H}$ which solve the ODE

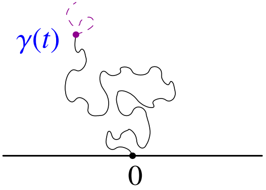
$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}, \qquad g_0(z) = z,$$

where W is a (continuous) real-valued function.

(Here, we have chosen the capacity parameterization.)

$W_t = g_t(\eta(t))$ on $\mathbb{R}$

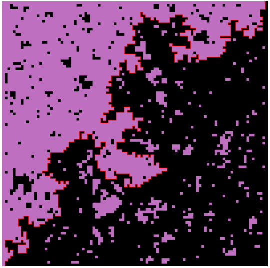
↑ **Loewner driving function** $W\colon [0,\infty) \to \mathbb{R}$



Thm. [Schramm '00]

$\exists!$ one-parameter family $(\mathrm{SLE}_\kappa)_{\kappa \geq 0}$ of probability measures on chordal curves with **conformal invariance** and **domain Markov property**

$g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$

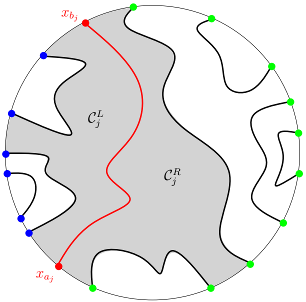


$W_t = g_t(\gamma(t)) = \sqrt{\kappa} B_t$ on $\mathbb{R}$

↳ Loewner driving process: **Brownian motion of "speed"** $\kappa \geq 0$

Multiple (chordal) SLE$_\kappa$ (let's assume $\kappa < 8/3$)

- family of **random chordal curves** $(\gamma_1^\kappa, \ldots, \gamma_N^\kappa)$ in $(D; x_1, \ldots, x_{2N})$
- connectivities encoded in **planar pairings** α of curve endpoints $\{\{x_{a_j}, x_{b_j}\}\}_{j=1,\ldots,N}$
- **re-sampling** symmetry ($\rightsquigarrow$ Markov chain)



Conditionally on $N-1$ of the curves, the remaining one is the chordal SLE$_\kappa$ in the random domain where it can live.

Cardy '03; Bauer, Bernard & Kytölä '05;
Dubédat '06–'07; Kozdron & Lawler '07; Lawler '09;
Kytölä & P. '16; Miller & Sheffield '16; P. & Wu '19;
Miller, Sheffield & Werner '20; Beffara, P. & Wu '21

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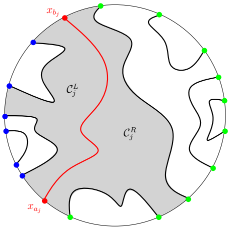


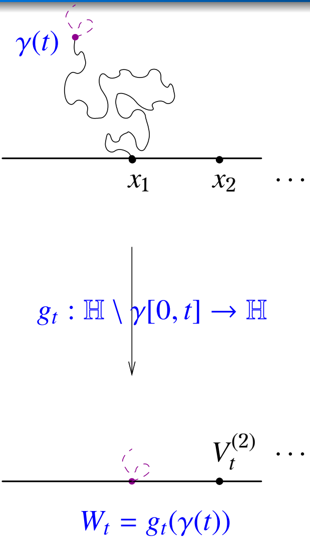
Thm.

For any fixed connectivity α of $2N$ points, **there exists a unique N-SLE$_\kappa$ probability measure $\mathbb{Q}_\alpha^\#$.**

$$\frac{\mathrm{d}\mathbb{Q}_\alpha}{\bigotimes_{1\le i\le N} \mathrm{d}\mathbb{P}_{\mathrm{SLE}}^{(i)}} := \exp\left(\frac{(3\kappa-8)(6-\kappa)}{2\kappa}\, m^{\mathrm{loop}}(D;\gamma_1^\kappa,\ldots,\gamma_N^\kappa)\right), \qquad \mathbb{Q}_\alpha^\# = \frac{\mathbb{Q}_\alpha}{|\mathbb{Q}_\alpha|}$$

- m^{loop} is a combinatorial expression involving Brownian loop measure
- alternatively, describe interaction of curves by "(pure) partition function"

$$\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) := |\mathbb{Q}_\alpha|(D; x_1, \ldots, x_{2N}) \prod_{j=1}^{N} P_D(x_{a_j}, x_{b_j})^{\frac{6-\kappa}{2\kappa}}$$



- re-sampling symmetry (*Dubédat's commutation relations*): can grow one curve at a time
- driving process of one curve γ: image of its tip:

$$W_t := \lim_{z \to \gamma(t)} g_t(z)$$

- interaction of curves encoded in **partition function $\mathcal{Z}$** of the Loewner chain:

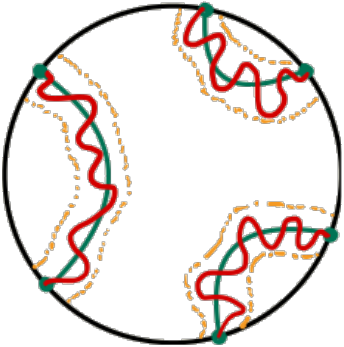
$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}(W_t, V_t^{(2)}, V_t^{(3)}, \ldots)\,\mathrm{d}t$$

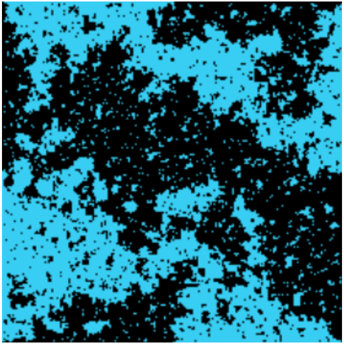
$$\mathrm{d}V_t^{(i)} = \frac{2\,\mathrm{d}t}{V_t^{(i)} - W_t},$$

$$V_0^{(i)} = x_i, \quad \text{for } i \neq 1, \qquad W_0 = x_1.$$

E.g. $\mathcal{Z}_\alpha(\mathbb{H}; x_1, \ldots, x_{2N}) := |\mathbb{Q}_\alpha|(\mathbb{H}; x_1, \ldots, x_{2N}) \prod_{j=1}^{N} |x_{b_j} - x_{a_j}|^{\frac{\kappa-6}{\kappa}}$

Large deviations of SLE_κ as $\kappa \to 0+$





(Chordal) Loewner energy

Dubédat '05; Friz & Shekhar '17; Wang '19; Bishop '19, ...

- Let's consider given smooth curve η. Idea:

$$\text{“ } \mathbb{P}[\mathrm{SLE}_\kappa \text{ curve stay close to } \eta] \overset{\kappa \to 0+}{\approx} \exp\Big(- \frac{I(\eta)}{\kappa} \Big) \text{ ”}$$

- SLE driven by standard *Brownian motion* B
- decay rate: **Loewner energy** of the curve η

  defined as the *Dirichlet energy* of its driver W:

$$I(\eta) := \frac{1}{2} \int_0^\infty \Big(\frac{\mathrm{d}}{\mathrm{d}t} W_t \Big)^2 \mathrm{d}t \quad \in \quad [0, +\infty]$$

Thm. Large Deviation Principle for BM [Schilder '66]

Fix $T > 0$. The random path $\sqrt{\kappa} B_{[0,T]}$ satisfies LDP in $C^0[0,T]$

(sup norm, with good rate function $I_T(W) := \frac{1}{2} \int_0^T \big(\frac{\mathrm{d}}{\mathrm{d}t} W_t \big)^2 \mathrm{d}t$)

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}[\sqrt{\kappa} B_{[0,T]} \in C] \leq - \inf_{W \in C} I_T(W) \qquad \text{for any closed set } C$$

$$\liminf_{\kappa \to 0+} \kappa \log \mathbb{P}[\sqrt{\kappa} B_{[0,T]} \in O] \geq - \inf_{W \in O} I_T(W) \qquad \text{for any open set } O$$

Large deviations of SLE_κ as $\kappa \to 0+$

- SLE_κ curves $\bar{\gamma}^\kappa := (\gamma_1^\kappa, \ldots, \gamma_N^\kappa)$ fluctuate near "optimal" curves when $\kappa > 0$ is small. **Idea:** *minimal energy* $\implies$ "*optimal*"
- for given smooth curves $\bar{\eta} := (\eta_1, \ldots, \eta_N)$, expect:

$$\text{“ } \mathbb{P}[\mathrm{SLE}_\kappa \text{ curves stay close to } \bar{\eta}] \overset{\kappa \to 0+}{\approx} \exp\Big(- \frac{I(\bar{\eta})}{\kappa} \Big) \text{ ”}$$

- $I(\bar{\eta}) \geq 0$ **Loewner energy** of the curves $\bar{\eta}$

Thm. [Wang '19; P. & Wang '21]

The family of laws $(\mathbb{P}^\kappa)_{\kappa>0}$ of SLE_κ curves $\bar{\gamma}^\kappa$ satisfies LDP:

(for Hausdorff distance, with good rate function I)

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^\kappa[\bar{\gamma}^\kappa \in C] \leq - \inf_{\bar{\eta} \in C} I(\bar{\eta}) \qquad \text{for any closed set } C$$

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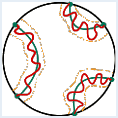

Proof idea: Schilder thm for BM, Varadhan's lemma + *careful analysis* □

Large deviations of SLE$_\kappa$ as $\kappa \to 0+$

" $\mathbb{P}[\text{SLE}_\kappa \text{ curves } \bar{\gamma}^\kappa \text{ stay close to } \bar{\eta}] \overset{\kappa \to 0+}{\approx} \exp\left(-\frac{I(\bar{\eta})}{\kappa}\right)$ "

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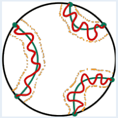

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- SLE$_\kappa$ is driven by one-dimensional Brownian motion:
  Schilder's thm gives a LDP for $\sqrt{\kappa}B_t$ for *finite times* $t \in [0, T]$

Large deviations of SLE$_\kappa$ as $\kappa \to 0+$

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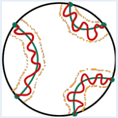

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- transfer this to SLE$_\kappa$ curves for *finite times* $t \in [0,T]$

  (But: encounter topological trouble: no direct contraction principle!)

Large deviations of SLE$_\kappa$ as $\kappa \to 0+$

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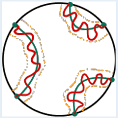

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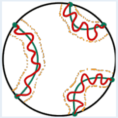

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  (But: encounter topological trouble: no direct contraction principle!)
- extend to $T \to \infty$ via (non-trivial) SLE$_\kappa$ estimates
- multiple curves using Varadhan's lemma and RN-derivative wrt. independent SLEs via pure partition functions $\mathcal{Z}_\alpha$ □

Multiple SLE$_\kappa$ by weighting independent SLEs

- family of **random chordal curves** $(\gamma_1^\kappa, \ldots, \gamma_N^\kappa)$ in $(D; x_1, \ldots, x_{2N})$
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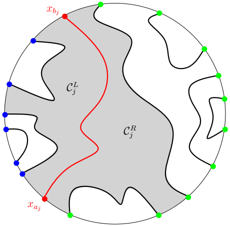

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Intrinsic object: Loewner potential

- **Multi-chord Loewner energy** of curves $\bar{\eta} := (\eta_1, \ldots, \eta_N)$:

$$I(\bar{\eta}) := 12 \left(\mathcal{H}(\bar{\eta}) - \inf_{\bar{\gamma}} \mathcal{H}(\bar{\gamma}) \right)$$

- **Loewner potential** $\mathcal{H}(\bar{\eta})$ of curves $\bar{\eta} := (\eta_1, \ldots, \eta_N)$:

$$\mathcal{H}(\bar{\eta}) := \frac{1}{12} \sum_{j=1}^{N} I(\eta_j) + m^{\text{loop}}(\bar{\eta}) - \frac{1}{4} \sum_{j=1}^{N} \log P(x_{a_j}, x_{b_j})$$

- multiple SLE_κ probability measure $\mathbb{Q}_\alpha^\# = \frac{\mathbb{Q}_\alpha}{|\mathbb{Q}_\alpha|}$:

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- interaction of curves: "(pure) partition function"

$$\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) := |\mathbb{Q}_\alpha|(D; x_1, \ldots, x_{2N}) \prod_{j=1}^{N} P_D(x_{a_j}, x_{b_j})^{\frac{6-\kappa}{2\kappa}}$$

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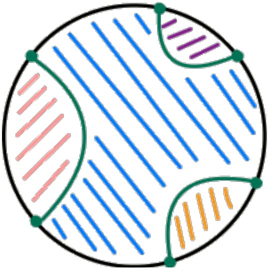
- $I(\eta) := \frac{1}{2}\int_0^\infty (\frac{\mathrm{d}}{\mathrm{d}t} W_t)^2 \mathrm{d}t$
  one-curve Loewner energy
- "interaction": $m^{\mathrm{loop}}(\bar{\eta})$
  Brownian loop measure term
- $P(x_{a_j}, x_{b_j})$ boundary Poisson kernel
- x_{a_j}, x_{b_j} endpoints of curve η_j



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- for given smooth curves $\bar{\eta} := (\eta_1, \ldots, \eta_N)$, expect:

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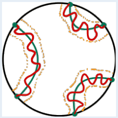
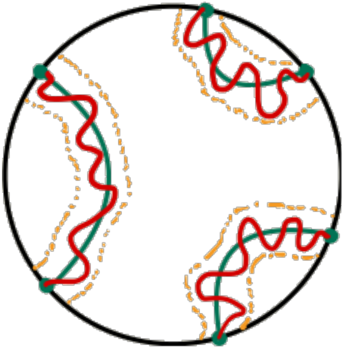
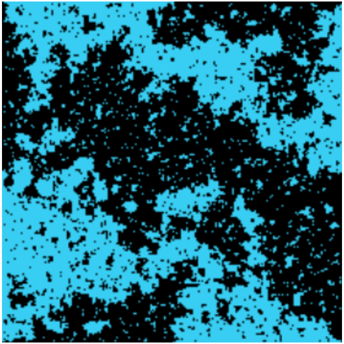


Proof idea: Schilder thm for BM, Varadhan's lemma + *careful analysis* □

Loewner potential





IN ANOTHER FORM

Loewner potential – more intuitive formula

As $\mathcal{H}(\bar{\eta})$ is a bit complicated, let's write it differently:

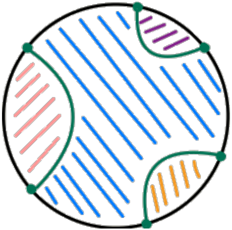
Thm. [Wang '19; P. & Wang '21]

For any smooth $\bar{\eta}$ in bounded smooth domain $(D; x_1, \ldots, x_{2N})$,

$$\mathcal{H}_D(\bar{\eta}) = \log \det{}_\zeta \Delta_D - \sum_{\text{c.c. } C} \log \det{}_\zeta \Delta_C - \frac{N}{2} \log \pi$$

Proof idea: Both sides have the same conformal covariance; use Polyakov-Alvarez anomaly formula (for domains with corners) [Aldana, Kirsten, Rowlett '20] □

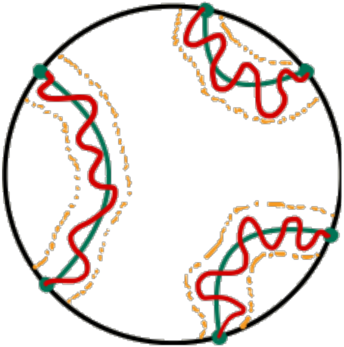
- $\log \det_\zeta \Delta$ zeta-regularized determinant of Laplacian Δ with Dirichlet b.c.
- sum over *connected components* C of $D \setminus \bigcup_i \eta_i$
- $\frac{1}{2} \log \pi \approx 0.5724$ *universal constant*

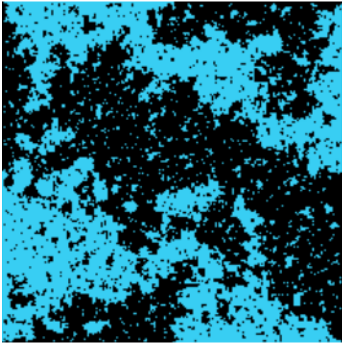


NB: Also makes sense on Riemannian surfaces (depends on metric).

Potential minimizers

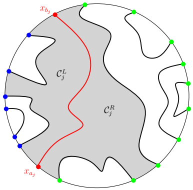
⟹ optimal curves





Solution to Shapiro conjecture (special case)

- **Proposition.** *Any minimizer* of Loewner potential $\mathcal{H}(\cdot)$ is so-called "**geodesic multichord**", i.e., $\forall\ j$, η_j is *hyperbolic geodesic* in its own component



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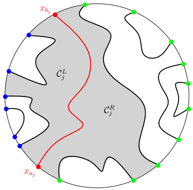

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POTENTIAL MINIMIZERS $\Longrightarrow$ OPTIMAL CURVES

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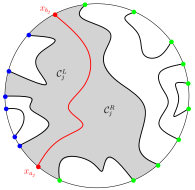

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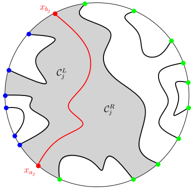
Recall: connectivities (planar pairings) of curves labeled by α

NB: $\exists$ a Catalan number $C_N = \frac{1}{N+1}\binom{2N}{N}$ of them

Potential minimizers $\Longrightarrow$ optimal curves



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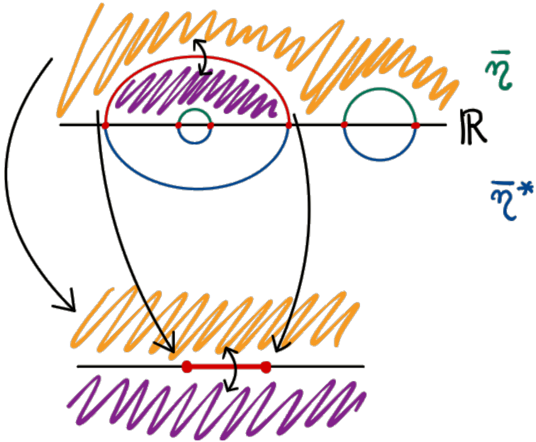
Thm. [P. & Wang '21]

1. Geodesic multichord gives rise to unique* rational function on $\mathbb{C} \cup \{\infty\}$ of degree $N+1$ with $2N$ critical points on real line.
2. There exists a unique geodesic multichord for each α.
3. In particular, there exists exactly* C_N rational functions of degree $N+1$ with given $2N$ critical points on the real line.

$\star$ (up to post-composition by Möbius map)

Proof: Explicit construction & upper bound result [Goldberg '91] □

Proposition. Let $\bar{\eta}$ be a geodesic multichord in $\mathbb{H}$. The union of $\bar{\eta}$, its complex conjugate $\bar{\eta}^*$, and the real line is the real locus of a rational function of degree $N+1$ with critical points $\{x_1, \ldots, x_{2N}\}$.



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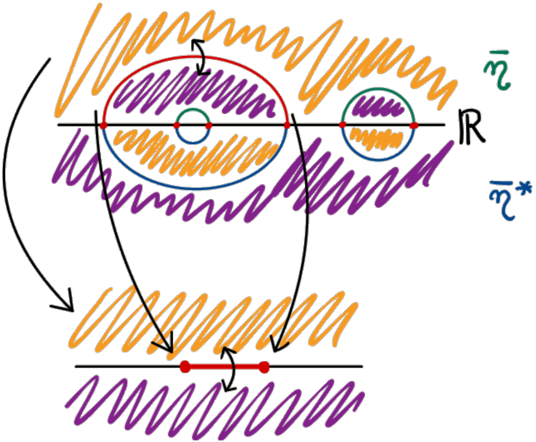

Potential minimizers $\Longrightarrow$ Shapiro conjecture

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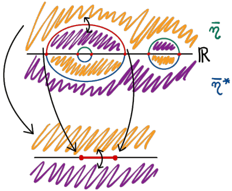
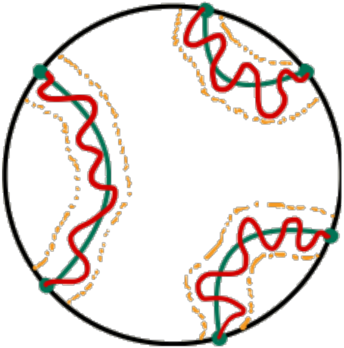
Cor. (Shapiro conjecture)

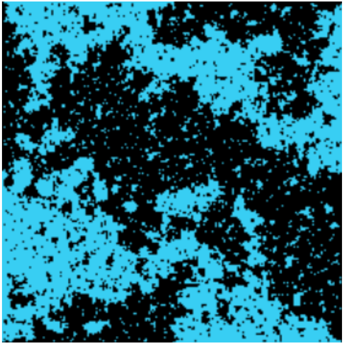
If all critical points of rational function are real, then it is **real rational function***.



- Special case of Shapiro conjecture [B. & M. Shapiro '95]
- First proven: [Eremenko & Gabrielov, Ann.Math.'00]
- General case: [Mukhin, Tarasov & Varchenko, Ann.Math.'09]

Potential minima...





... Semiclassical CFT
conformal blocks

Potential minima $\Longrightarrow$ Semiclassical conformal blocks

- Fix domain data $D = \mathbb{H}$ and $x_1 < \cdots < x_{2N}$ and connectivity α.
- Set $\mathcal{U}(x_1, \ldots, x_{2N}) := 12 \inf_{\bar{\gamma}} \mathcal{H}_{\mathbb{H}; x_1, \ldots, x_{2N}}(\bar{\gamma})$ (minimum potential)

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$$\frac{1}{2}(\partial_j \mathcal{U}(x_1, \ldots, x_{2N}))^2 - \sum_{i \neq j} \frac{2}{x_i - x_j} \partial_i \mathcal{U}(x_1, \ldots, x_{2N}) = \sum_{i \neq j} \frac{6}{(x_i - x_j)^2} \quad \forall j$$

Proof: Study $\mathcal{U}$ & use self-similarity of Loewner flow of geodesic multichords □

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- Alberts, Kang, Makarov [arXiv:2011.05714]: evol. of critical pts & poles of the rational function described by Calogero-Moser

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Thm. [Alberts, Kang, Makarov '20]

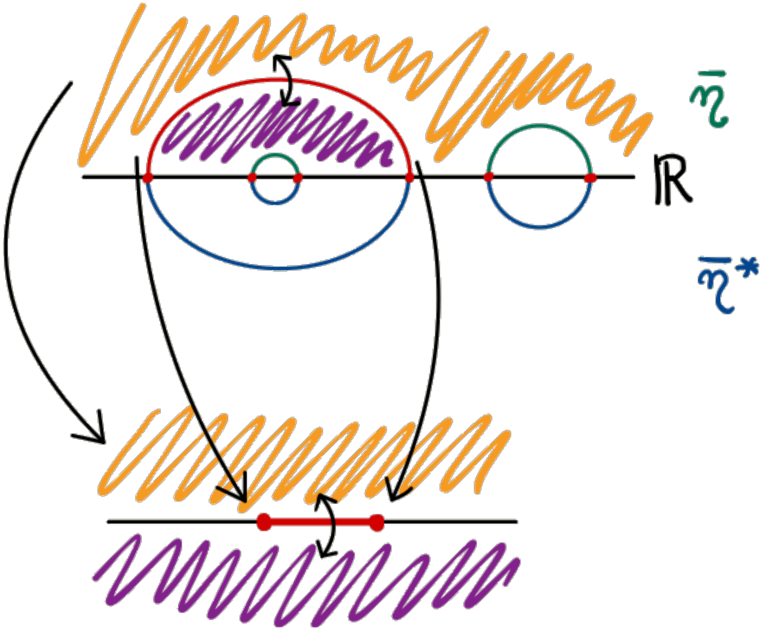
Up to a multiplicative constant (only depending on N), $\mathcal{U}$ equals:

$$-\log\left(\prod_{1 \le i < j \le 2N} (x_j - x_i)^2 \prod_{1 \le r < s \le N} (u_s - u_r)^8 \prod_{r=1}^{N} \prod_{k=1}^{2N} (u_r - x_k)^{-4} \right)$$

where $u_1, \ldots, u_N$ are the poles of the* rational function $h_{\bar{\eta}}$ associated to the unique geodesic multi-chord $\bar{\eta}$ that minimizes the Loewner potential.

Proof sketch: This "partition function" generates curves $\bar{\eta}'$ that belong to the real locus of $h_{\bar{\eta}}$. But the real locus is uniquely determined: it comprises $\bar{\eta} \cup \bar{\eta}^* \cup \mathbb{R}$. $\square$

$\star$ (hydrodynamically normalized at ∞)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- Classification of minimizers [Bonk, Eremenko '21]


Thank you!