Loewner Dynamics for Real Rational Functions and the Multiple SLE(0) Process

Tom Alberts
Joint with Nam-Gyu Kang (KIAS) and Nikolai Makarov (Caltech)
arXiv:2011.05714

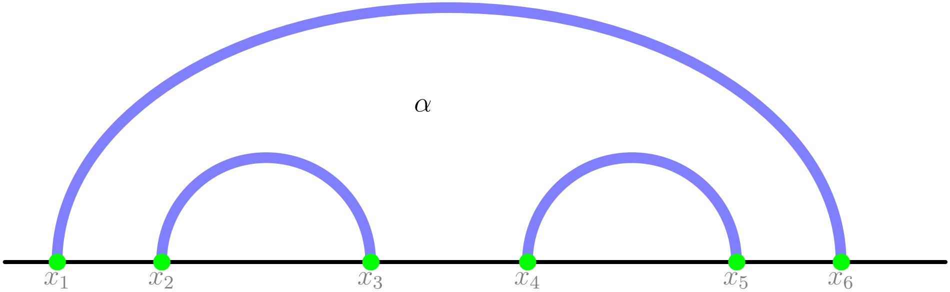
THE UNIVERSITY OF UTAH®

DEPARTMENT OF MATHEMATICS
College of Science

Multiple SLE(0)

Fix points $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$, $x_i \in \mathbb{R}$, and a non-crossing link pattern α

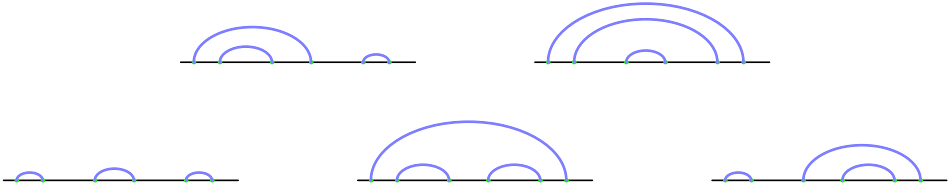
Multiple SLE$(0; \boldsymbol{x}; \alpha)$ is a special ensemble of n smooth curves in $\mathbb{H}$ that connect the points in $\boldsymbol{x}$ according to pattern α, introduced by Peltola-Wang



Multiple SLE(0)

Fix points $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$, $x_i \in \mathbb{R}$, and a non-crossing link pattern α

Multiple SLE$(0; \boldsymbol{x}; \alpha)$ is a special ensemble of n smooth curves in $\mathbb{H}$ that connect the points in $\boldsymbol{x}$ according to pattern α, introduced by Peltola-Wang



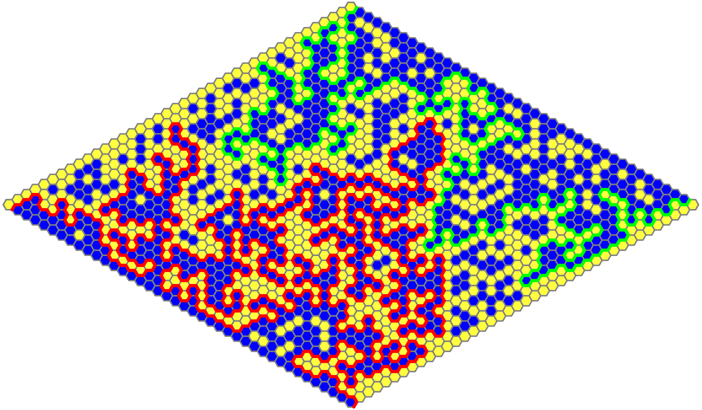
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Multiple SLE(κ)

Motivation

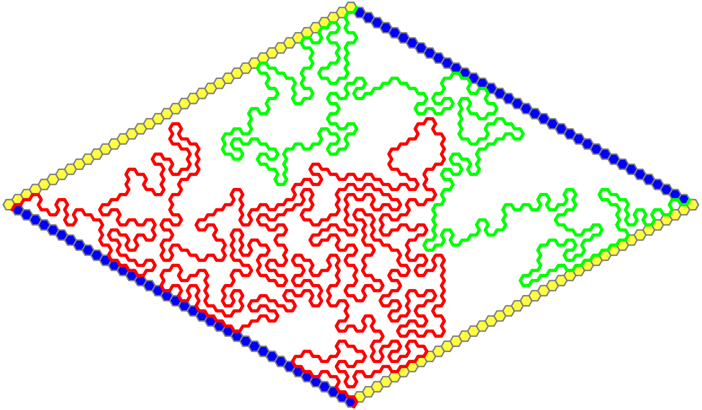
Multiple SLE(κ) as $\kappa \to 0$

Scaling limit of interfaces with *alternating boundary conditions*



Multiple SLE(κ) as $\kappa \to 0$

Scaling limit of interfaces with *alternating boundary conditions*



Multiple SLE(κ) as $\kappa \to 0$

- **Kozdron-Lawler:** Reweighting of independent SLEs by a Radon-Nikodym derivative of the form

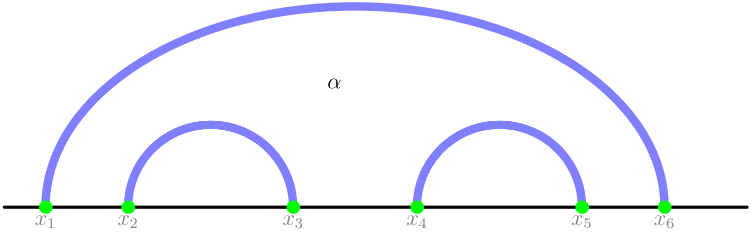
$$\exp\left\{\frac{c(\kappa)}{2} m_{\mathbb{H},\alpha}(\gamma_1, \ldots, \gamma_n)\right\} \mathbf{1}\left\{\gamma_i \cap \gamma_j = \emptyset \text{ for all } i \neq j\right\}$$

  where $m_{\mathbb{H},\alpha}$ involves **Brownian loop measure**
- $c(\kappa) \sim -12/\kappa$ as $\kappa \to 0$, so well set up for large deviations
- Extract **multichordal Loewner energy** $\mathcal{H}_{(\boldsymbol{x};\alpha)}$ as rate function
- See Eveliina's talk of last Friday

Properties of Multiple SLE(0)

Peltola-Wang prove the following properties of multiple SLE$(0; \boldsymbol{x}; \alpha)$:

- it is the deterministic limit of multiple SLE$(\kappa; \boldsymbol{x}; \alpha)$ as $\kappa \to 0$,
- is the minimizer of the multichordal Loewner energy,
- has the geodesic multichord property,
- can be generated by a Loewner flow involving $\mathcal{U} = \mathcal{U}(\boldsymbol{x}; \alpha)$,
- is the real locus of a real rational function.



Connection to Real Rational Functions

Peltola-Wang establish the connection to real rational functions by

- showing the functional $\mathcal{H}_{(\boldsymbol{x};\alpha)}$ has a unique minimizer
- the structure of the functional implies that the minimizing curves have the geodesic multichord property
- Schwarz reflection + the geodesic multichord property implies the curves are the real locus of a real rational function

Properties of Multiple SLE(0)

Peltola-Wang prove the following properties of multiple SLE$(0; \boldsymbol{x}; \alpha)$:

- it is the deterministic limit of multiple SLE$(\kappa; \boldsymbol{x}; \alpha)$ as $\kappa \to 0$,
- is the minimizer of the multichordal Loewner energy,
- has the geodesic multichord property,
- can be generated by a Loewner flow involving $\mathcal{U} = \mathcal{U}(\boldsymbol{x}; \alpha)$,
- is the real locus of a real rational function.

Properties of Multiple SLE(0)

Peltola-Wang prove the following properties of multiple SLE$(0; \boldsymbol{x}; \alpha)$:

- it is the deterministic limit of multiple SLE$(\kappa; \boldsymbol{x}; \alpha)$ as $\kappa \to 0$,
- is the minimizer of the multichordal Loewner energy,
- has the geodesic multichord property,
- can be generated by a Loewner flow involving $\mathcal{U} = \mathcal{U}(\boldsymbol{x}; \alpha)$,
- is the real locus of a real rational function.

This talk explains how to generate real locus of a real rational function via Loewner flow, without external inputs

SLE, Brownian loop measure, **conformal field theory** are lurking in many of the ideas, but not in the presentation or the proofs

Properties of Multiple SLE(0)

Peltola-Wang prove the following properties of multiple SLE$(0; \boldsymbol{x}; \alpha)$:

- it is the deterministic limit of multiple SLE$(\kappa; \boldsymbol{x}; \alpha)$ as $\kappa \to 0$,
- is the minimizer of the multichordal Loewner energy
- has the geodesic multichord property,
- can be generated by a Loewner flow involving $\mathcal{U} = \mathcal{U}(\boldsymbol{x}; \alpha)$,
- is the real locus of a real rational function.

Properties of Multiple SLE(0)

Peltola-Wang prove the following properties of multiple SLE$(0; \boldsymbol{x}; \alpha)$:

- it is the deterministic limit of multiple SLE$(\kappa; \boldsymbol{x}; \alpha)$ as $\kappa \to 0$,
- is the minimizer of the multichordal Loewner energy
- has the geodesic multichord property,
- can be generated by a Loewner flow involving $\mathcal{U} = \mathcal{U}(\boldsymbol{x}; \alpha)$,
- is the real locus of a real rational function.

Alberts-Kang-Makarov:

- new description of Loewner flow for real locus of a real rational function
- alternative formula for the functions $\mathcal{U} = \mathcal{U}(x; \alpha)$
- direct proof that real loci satisfy geodesic multichord property
- Loewner flow is an instance of **Calogero-Moser** integrable system

Real Rational Functions and their Real Loci

Real Rational Functions

- Focus on real rational functions of degree $n+1$
- Ratios of the form

$$\frac{P(z)}{Q(z)}$$

where $P, Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are polynomials with real coefficients and

$$\max\{\deg P, \deg Q\} = n+1$$

- **Real Locus:**

$$\Gamma(R) := \left\{ z \in \widehat{\mathbb{C}} : R(z) \in \widehat{\mathbb{R}} \right\}$$

Real Rational Functions

Basic properties of $\Gamma(R)$:

- $\widehat{\mathbb{R}} \subset \Gamma(R)$
- $\Gamma(R)$ is symmetric under conjugation
- $\Gamma(R)$ is path connected
- $\Gamma(R)$ is the union of disjoint arcs
- Number of arcs is related to the degree
- Arcs meet at branch points/critical points

Critical points of R: $\{z \in \widehat{\mathbb{C}} : R'(z) = 0\}$

Real Rational Functions

- In this talk we specify the critical points and assume the existence of an R with those critical points
- Restrict to the case of $2n$ real critical points $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$, so

$$R'(z) = 0 \iff z \in \{x_1, \ldots, x_{2n}\}$$

- **Notation:** $\mathrm{CRR}_{n+1}(\boldsymbol{x})$ is the set of degree $n+1$ real rational functions with $2n$ critical points at $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$
- Degree $n+1$ plus $2n$ critical points means each critical point of index 2

$$R(z) = R(x_i) + C(z - x_i)^2 + \ldots$$

so real locus is locally a + shape near each x_i

Real Rational Functions

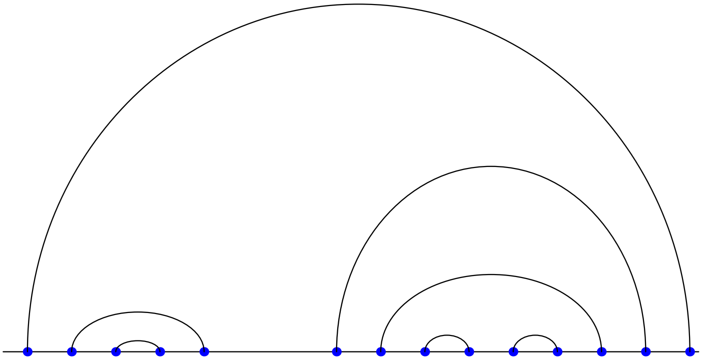
Structure of Real Locus: *If* there exists $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ the real locus is

- the real line $\mathbb{R}$,
- n non-crossing curves in $\mathbb{H}$ connecting points in $\boldsymbol{x}$
- the complex conjugates of those curves

Real Rational Functions

Structure of Real Locus: *If* there exists $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ the real locus is

- the real line $\mathbb{R}$,
- n non-crossing curves in $\mathbb{H}$ connecting points in $\boldsymbol{x}$
- the complex conjugates of those curves



Enumerative Algebraic Geometry

- Given $2n$ distinct real points x does there exist *any* $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$?
- If so how many?

Enumerative Algebraic Geometry

- Given $2n$ distinct real points $\boldsymbol{x}$ does there exist *any* $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$?
- If so how many?
- Note once you have one $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ you have infinitely many since

$$\phi \circ R = \frac{aP + bQ}{cP + dQ} \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$$

  where $\phi(z) = (az + b)/(cz + d)$ is a Möbius transform of $\mathbb{H}$ to itself
- Can easily compute that $\Gamma(\phi \circ R) = \Gamma(R)$, so enumeration is done up to equivalence under *post-composition* by PSL$(2, \mathbb{R})$

Enumerative Algebraic Geometry

- Given $2n$ distinct real points x does there exist *any* $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$?
- If so how many?

Enumerative Algebraic Geometry

- Given $2n$ distinct real points x does there exist *any* $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$?
- If so how many?
- **Goldberg (1991):** For each fixed x the number of equivalence classes is at most
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$
- **Eremenko-Gabrielov (2002) / Mukhin-Tarasov-Varchenko (2009) / Peltola-Wang (2021):** For each fixed x there are exactly C_n distinct equivalence classes

Loewner Dynamics for Real Rational Functions

Loewner Dynamics for Real Rational Functions

THE UNIVERSITY OF UTAH

- AKM shows how to grow $\Gamma(R)$ via Loewner flow
- Heavy use of the **poles** of the rational function
- Our results hold *conditionally* on existence of $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$
- Our results do *not* use SLE or Brownian loop measure, but the dynamics of our Loewner flow are best described using SLE terminology

SLE$(0, x, \boldsymbol{\rho})$ Loewner Flows

Definition (SLE$(0, x, \boldsymbol{\rho})$ Loewner Flows)

Let $x \in \mathbb{R}$ and $\boldsymbol{\rho}$ be a finite atomic measure on $\widehat{\mathbb{C}}$ that is symmetric under conjugation. Assume $\boldsymbol{\rho}(x) = 0$. SLE$(0, x, \boldsymbol{\rho})$ is the Loewner flow

$$\partial_t g_t(z) = \frac{2}{g_t(z) - x_t}, \quad g_0(z) = z,$$

where the driving function evolves as

$$\dot{x}_t = \int_{\mathbb{C}} \frac{d\boldsymbol{\rho}(w)}{x_t - g_t(w)}, \quad x_0 = x.$$

Note it matches the standard definition of SLE$(\kappa, x, \boldsymbol{\rho})$

$$dx_t = \sqrt{\kappa}\, dB_t + \int_{\mathbb{C}} \frac{d\boldsymbol{\rho}(w)}{x_t - g_t(w)}\, dt, \quad x_0 = x$$

SLE$(0, x, \boldsymbol{\rho})$ Loewner Flows

Can superimpose flows coming from different x and $\boldsymbol{\rho}$

Definition (Superposition of Flows)
Let $\nu_i : [0, \infty) \to [0, \infty)$, $i = 1, \ldots, N$ be measurable. The $\boldsymbol{\nu}$-superposition of the SLE$(0, x_j, \boldsymbol{\rho}_j)$ processes is the superposition of the corresponding flows:

$$\partial_t g_t(z) = \sum_{j=1}^{N} \frac{2\nu_j(t)}{g_t(z) - x_j(t)}, \quad g_0(z) = z,$$

where the driving functions $x_1(t), \ldots, x_N(t)$ evolve as

$$\dot{x_j} = \nu_j(t) \int_{\mathbb{C}} \frac{d\boldsymbol{\rho}_j(w)}{x_j - g_t(w)} + \sum_{k \neq j} \frac{2\nu_k(t)}{x_j - x_k}, \quad x_j(0) = x_j.$$

SLE$(0, x, \boldsymbol{\rho})$ Loewner Flows

Examples of superpositions

- Grow only the curve anchored at x_j:

$$\nu_j \equiv 1, \ \nu_k \equiv 0 \text{ for } k \neq j$$

- Grow all curves simultaneously:

$$\nu_j \equiv 1 \text{ for all } j$$

- "Adaptive" growth: vary ν_j with t depending on previous growth

Loewner Dynamics for Real Rational Functions

Theorem (Alberts-Kang-Makarov 2020)
Let $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$ be distinct real points. Assume $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_{n+1}\} \subset \widehat{\mathbb{C}}$ is closed under conjugation and solves the **stationary relation**

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \boldsymbol{\zeta}.$$

Then there exists an $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ with pole set $\boldsymbol{\zeta}$. Moreover, for

$$\boldsymbol{\rho}_j = \sum_{k \neq j} 2\delta_{x_k} - \sum_{l=1}^{n+1} 4\delta_{\zeta_l}, \quad j = 1, \ldots, 2n,$$

the curves generated by *any* $\boldsymbol{\nu}$-superposition of the SLE$(0, x_j, \boldsymbol{\rho}_j)$ Loewner flows are a subset of $\Gamma(R)$.

Loewner Dynamics for Real Rational Functions

- Theorem says the curves generated by *any* ν-superposition are a subset of $\Gamma(R)$, which is a form of **commutation**
- Dynamics remain well-defined as long as each SLE$(0, x_j, \boldsymbol{\rho}_j)$ process is well-defined or until two driving functions $x_j(t)$ collide
- Dynamics may be extendable past the collision times, and in our situation there is a natural way of doing this

Loewner Dynamics for Real Rational Functions

Two main ingredients: stationary relation + associated ρ_j

Loewner Dynamics for Real Rational Functions

Two main ingredients: stationary relation + associated $\boldsymbol{\rho}_j$

- Stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \boldsymbol{\zeta},$$

and $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_{n+1}\}$ are poles of a $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$

Loewner Dynamics for Real Rational Functions

THE UNIVERSITY OF UTAH

Two main ingredients: stationary relation + associated ρ_j

- Stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \boldsymbol{\zeta},$$

  and $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_{n+1}\}$ are poles of a $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$
- Any $R = P/Q \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ must have $n+1$ poles
- Complex poles appear in conjugate pairs
- All poles are on the real locus $\Gamma(R)$
- May be a pole at infinity
- Poles are *not* preserved under post-composition of R

Real Rational Functions

Appearance of stationary relation comes from a basic complex analysis result

Theorem

Let $\boldsymbol{x} = \{x_1, \ldots, x_{2n}\}$ be distinct real points, and $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_{n+1}\} \subset \widehat{\mathbb{C}}$ be closed under conjugation and distinct from $\boldsymbol{x}$. There exists an $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ with pole set ζ iff ζ solves the stationary relation.

Proof is based on partial fraction expansion of R'. Stationary relation is equivalent to R' having no residues at the poles ζ_k.

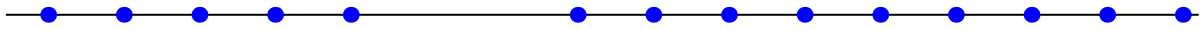
Of note: Generating solutions to the stationary relation is another way of proving existence of $\mathbb{R} \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$

Loewner Dynamics for Real Rational Functions

- Inserting poles into $\boldsymbol{\rho}_j = \sum_{k \neq j} 2\delta_{x_k} - \sum_l 4\zeta_l$ gives

$$\dot{x}_j = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \boldsymbol{\zeta}} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$

- Can show that the value of U_j is invariant under post-compositions of R



Loewner Dynamics for Real Rational Functions

- Inserting poles into $\boldsymbol{\rho}_j = \sum_{k \neq j} 2\delta_{x_k} - \sum_l 4\zeta_l$ gives

$$\dot{x}_j = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \boldsymbol{\zeta}} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$

- Can show that the value of U_j is invariant under post-compositions of R

Loewner Dynamics for Real Rational Functions

- Inserting poles into $\boldsymbol{\rho}_j = \sum_{k\neq j} 2\delta_{x_k} - \sum_l 4\zeta_l$ gives

$$\dot{x}_j = U_j(R) := \sum_{k\neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \boldsymbol{\zeta}} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$

- Can show that the value of U_j is invariant under post-compositions of R
- Peltola-Wang has a theorem for Loewner flow but less explicit formula for U_j

Loewner Dynamics for Real Rational Functions

- Inserting poles into $\boldsymbol{\rho}_j = \sum_{k \neq j} 2\delta_{x_k} - \sum_l 4\zeta_l$ gives

$$\dot{x}_j = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \boldsymbol{\zeta}} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$

- Can show that the value of U_j is invariant under post-compositions of R
- Peltola-Wang has a theorem for Loewner flow but less explicit formula for U_j
- In Peltola-Wang $U_j(\boldsymbol{x}) = \partial_{x_j} \mathcal{U}_\alpha(\boldsymbol{x})$, where $\mathcal{U}_\alpha(\boldsymbol{x})$ is the minimal value of the **multichordal Loewner potential**
- Minimal value involves the Brownian loop measure of $\Gamma(R)$ and so is difficult to evaluate explicitly

Loewner Dynamics for Real Rational Functions

Theorem (Alberts-Kang-Makarov 2020)
Functions U_j satisfy $U_j = \partial_{x_j} \log \mathcal{Z}$ where

$$\mathcal{Z}(\boldsymbol{x}) = \prod_{1\leq j<k\leq 2n} (x_j - x_k)^2 \prod_{1\leq l<m\leq n} (\zeta_l(\boldsymbol{x}) - \zeta_m(\boldsymbol{x}))^8 \prod_{k=1}^{2n} \prod_{l=1}^{n} (x_k - \zeta_l(\boldsymbol{x}))^{-4}$$

Conclusion is that $\mathcal{Z} = Ce^{\mathcal{U}}$, but there is no direct proof (as of yet)

Solutions to Null Vector Equations

Using the explicit formula for U_j we are able to show that they solve a system of quadratic equations we call the **null vector equations**

Theorem (Alberts-Kang-Makarov 2020)
Let $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ and

$$U_j(\boldsymbol{x}) = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \boldsymbol{\zeta}} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$

The functions U_j solve the system of quadratic equations

$$\frac{1}{2} U_j^2 + \sum_{k \neq j} \frac{2}{x_k - x_j} U_k - \sum_{k \neq j} \frac{6}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

Bernard-Bauer-Kytölla (2005): Classical limit of the **BPZ equations**

Solutions to Null Vector Equations

Using the explicit formula for U_j we are able to show that they solve a system of quadratic equations we call the **null vector equations**

Theorem (Alberts-Kang-Makarov 2020)
Let $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ and

$$U_j(\boldsymbol{x}) = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \boldsymbol{\zeta}} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$

The functions U_j solve the system of quadratic equations

$$\frac{1}{2} U_j^2 + \sum_{k \neq j} \frac{2}{x_k - x_j} U_k - \sum_{k \neq j} \frac{6}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

A-Kang-Makarov (2022+): Commutation + conformal invariance $\implies$ NVE

Solutions to Null Vector Equations

Proof (Null Vector Equations):

Solutions to Null Vector Equations

Proof (Null Vector Equations):

- R' has a rational primitive implies the **stationary relation**

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \boldsymbol{\zeta}$$

Solutions to Null Vector Equations

Proof (Null Vector Equations):

- R' has a rational primitive implies the **stationary relation**

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \boldsymbol{\zeta}$$

- Based on partial fraction expansion of R':

$$R = \frac{P}{Q} \implies R'(z) = \frac{P'Q - PQ'}{Q^2}(z) = \frac{\prod_{i=1}^{2n}(z - x_i)}{\prod_k (z - \zeta_k)^2}$$

Solutions to Null Vector Equations

Proof (Null Vector Equations):

- R' has a rational primitive implies the **stationary relation**

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \boldsymbol{\zeta}$$

- Based on partial fraction expansion of R':

$$R = \frac{P}{Q} \implies R'(z) = \frac{P'Q - PQ'}{Q^2}(z) = \frac{\prod_{i=1}^{2n}(z - x_i)}{\prod_k (z - \zeta_k)^2}$$

- algebra

Loewner Flow for Real Rational Functions

Summary:

- Expressing Loewner vector fields in terms of poles is beneficial
- Simple formulas for Loewner flow vector fields and solutions to null vector equations
- Our statements and proofs are ***not*** probabilistic, but idea of looking at the poles is motivated by Gaussian free field based Conformal Field Theory

THE UNIVERSITY OF UTAH

Multiple SLE(0) and Calogero-Moser Dynamics

Classical Calogero-Moser Dynamics

One-dimensional many-body problem that is **integrable** and **solvable**

$$\ddot{x}_j = \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

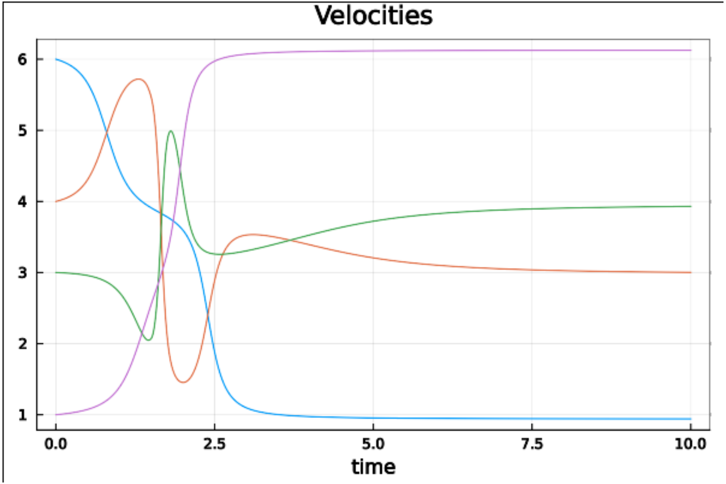
Hamiltonian system with Hamiltonian given by

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2} \sum_j p_j^2 + \sum_{j<k} \frac{1}{(x_j - x_k)^2}$$

Leads to the standard equations of motion

$$\dot{x}_j = \frac{\partial \mathcal{H}}{\partial p_j} = p_j, \quad \dot{p}_j = \ddot{x}_j = \frac{2}{(x_j - x_k)^3}$$

Classical Calogero-Moser Dynamics



Calogero-Moser Dynamics for SLE(0)

Theorem (Alberts-Kang-Makarov 2020)
Let $\boldsymbol{x} = \{x_1, \dots, x_{2n}\}$ be distinct real points and $\boldsymbol{\zeta} = \{\zeta_1, \dots, \zeta_{n+1}\} \subset \widehat{\mathbb{C}}$ be closed under conjugation and solve the **stationary relation**. Under the $1/4$-superposition of the SLE$(0, x_j, \boldsymbol{\rho}_j)$ processes

$$\ddot{x}_j = -\sum_{k \neq j} \frac{2}{(x_j - x_k)^3}.$$

Calogero-Moser Dynamics for SLE(0)

$1/4$-superposition of SLE$(0, x_j, \boldsymbol{\rho}_j)$ gives the coupled system

$$\dot{x}_j = \sum_{k \neq j} \frac{1}{x_j - x_k} - \sum_k \frac{1}{x_j - \zeta_k},$$
$$\dot{\zeta}_k = -\sum_{l \neq k} \frac{1}{\zeta_k - \zeta_l} + \sum_j \frac{1}{\zeta_k - x_j}.$$

Differentiating and stationary relation leads to Calogero-Moser

Calogero-Moser Dynamics for SLE(0)

$1/4$-superposition of SLE$(0, x_j, \boldsymbol{\rho}_j)$ gives the coupled system

$$\dot{x}_j = \sum_{k \neq j} \frac{1}{x_j - x_k} - \sum_k \frac{1}{x_j - \zeta_k},$$

$$\dot{\zeta}_k = -\sum_{l \neq k} \frac{1}{\zeta_k - \zeta_l} + \sum_j \frac{1}{\zeta_k - x_j}.$$

THE UNIVERSITY OF UTAH

Calogero-Moser Dynamics for SLE(0)

$1/4$-superposition of SLE$(0, x_j, \boldsymbol{\rho}_j)$ gives the coupled system

$$\dot{x}_j = \sum_{k \neq j} \frac{1}{x_j - x_k} - \sum_k \frac{1}{x_j - \zeta_k},$$
$$\dot{\zeta}_k = -\sum_{l \neq k} \frac{1}{\zeta_k - \zeta_l} + \sum_j \frac{1}{\zeta_k - x_j}.$$

Gives two ways of describing the Loewner evolution that generates $\Gamma(R)$:

- as two coupled first order systems of ODEs, with the underlying vector field determined by the poles and critical points of R, and
- via an autonomous second order Calogero-Moser system for the critical points that has no reference to the poles, ***but*** must be started with very particular initial momenta.

Calogero-Moser Dynamics for SLE(0)

Generate curves via

$$\partial_t g_t(z) = \sum_{j=1}^{2n} \frac{1/2}{g_t(z) - x_j(t)}, \quad \ddot{x}_j = -\sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

Calogero-Moser Dynamics for SLE(0)

Generate curves via

$$\partial_t g_t(z) = \sum_{j=1}^{2n} \frac{1/2}{g_t(z) - x_j(t)}, \qquad \ddot{x}_j = -\sum_{k\neq j} \frac{2}{(x_j - x_k)^3}$$

Theorem (Alberts-Kang-Makarov 2022+)
Necessary condition for generated curves to be in the real locus of a real rational function is that $\boldsymbol{x}(0), \dot{\boldsymbol{x}}(0)$ satisfy, for $j = 1, \ldots, 2n$,

$$\dot{x}_j^2 - \sum_{k\neq j} \frac{\dot{x}_j + \dot{x}_k}{x_j - x_k} - \sum_{k\neq j} \frac{1}{(x_j - x_k)^2} + \frac{1}{2} \sum_{k\neq j} \sum_{l\neq k} \frac{1}{(x_j - x_k)(x_j - x_l)} = 0,$$

Calogero-Moser Dynamics for SLE(0)

Generate curves via

$$\partial_t g_t(z) = \sum_{j=1}^{2n} \frac{1/2}{g_t(z) - x_j(t)}, \quad \ddot{x}_j = -\sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

Theorem (Alberts-Kang-Makarov 2022+)
Necessary condition for generated curves to be in the real locus of a real rational function is that $\boldsymbol{x}(0), \dot{\boldsymbol{x}}(0)$ satisfy, for $j = 1, \ldots, 2n$,

$$\dot{x}_j^2 - \sum_{k \neq j} \frac{\dot{x}_j + \dot{x}_k}{x_j - x_k} - \sum_{k \neq j} \frac{1}{(x_j - x_k)^2} + \frac{1}{2} \sum_{k \neq j} \sum_{l \neq k} \frac{1}{(x_j - x_k)(x_j - x_l)} = 0,$$

Show this condition is preserved under Hamiltonian flow using the **Lax pair**

THE UNIVERSITY OF UTAH

Geodesic Multichord Property

Geodesic Multichord Property

Theorem (Alberts-Kang-Makarov 2022+)
If $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ then $\Gamma(R)$ satisfies the geodesic multichord property.

"Converse" to Peltola-Wang argument that geodesic multichords are the real loci of real rational functions

Geodesic Multichord Property

Proof is inductive and based on a preservation property of Loewner flows

Theorem (Alberts-Kang-Makarov 2022+)
Let $R \in \mathrm{CRR}_{n+1}(\boldsymbol{x})$ and $\zeta_1, \ldots, \zeta_{n+1}$ be the poles of R. Then

$$R \circ g_t^{-1} \in \mathrm{CRR}_{n+1}(\boldsymbol{x}(t))$$

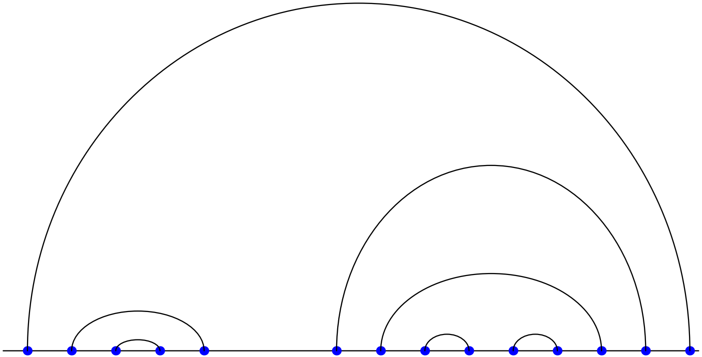
for any $\boldsymbol{\nu}$-superposition of the SLE$(0, x_j, \boldsymbol{\rho}_j)$ flows

This is a consequence of our integral of motion result

Geodesic Multichord: Pictorial Proof

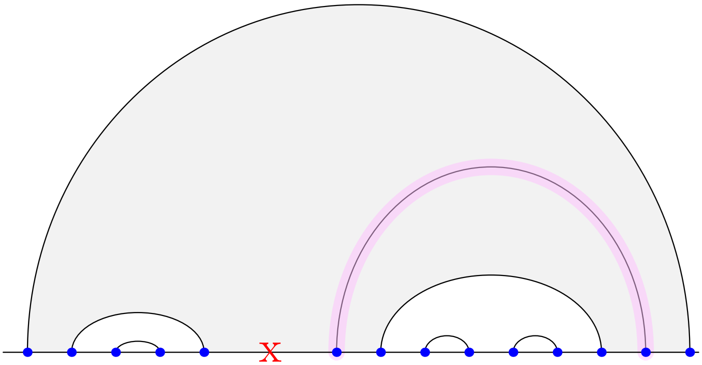
14 points chosen as critical points

Geodesic Multichord: Pictorial Proof



Real locus of any $R \in \mathrm{CRR}_8(\boldsymbol{x})$ consists of 7 curves connecting the 14 points in some non-crossing way

Geodesic Multichord: Pictorial Proof



Want to show the pink highlighted curve is a hyperbolic geodesic in the shaded region. Use a Möbius inversion to expose pink curve to ∞.

Geodesic Multichord: Pictorial Proof

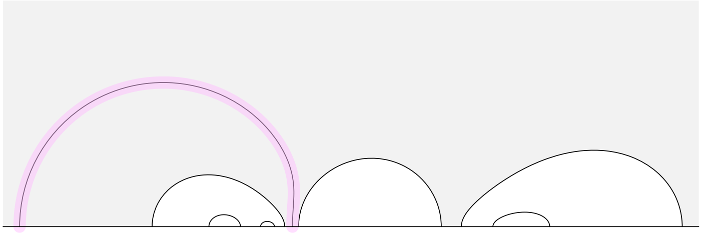
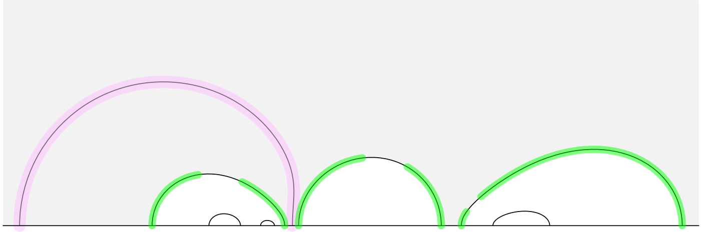


Image of curves under Möbius inversion ϕ. Is also $\Gamma(R \circ \phi^{-1})$, noting that $R \circ \phi^{-1} \in \mathrm{CRR}_8(\phi(\boldsymbol{x}))$.

Geodesic Multichord: Pictorial Proof



Use superposition of SLE$(0, x_j, \boldsymbol{\rho}_j)$ flows to grow curves that border the shaded region. Previous theorem shows that $R \circ \phi^{-1} \circ g_t^{-1}$ is also a real rational function. Limiting argument shows $R \circ \phi^{-1} \circ g_\tau^{-1}$ is also real rational function with only two critical points, where τ is the time at which all green curves simultaneously complete.

Geodesic Multichord: Pictorial Proof

$\Gamma(R \circ \phi^{-1} \circ g_\tau^{-1})$, which is also the image of the original pink curve under $g_\tau \circ \phi$. Since $R \circ \phi^{-1} \circ g_\tau^{-1}$ is real rational with only two critical points can directly compute that it is the hyperbolic geodesic in $\mathbb{H}$. Conformal invariance of hyperbolic geodesics completes the proof.

Conformal Field Theory Motivation

From CFT to Real Rational Functions

Two main ideas

- $\kappa \to 0+$ limits of the **method of screening** for solutions to BPZ equations
- Gaussian free field as a **martingale observable** for the multiple SLE$(\kappa; \boldsymbol{x}; \alpha)$ process under Loewner evolution, and its $\kappa \to 0+$ limit

Multiple SLE(κ) Driving Functions

A single arm of a multiple SLE(κ) ensemble has driving function

$$dx_j(t) = \sqrt{\kappa}\, dB_t + \kappa(\partial_{x_j} \log Z)(\boldsymbol{x}(t)), \quad x_k(t) = g_t(x_k)$$

Dubédat: Commutation $\implies Z = Z(\boldsymbol{x}; \kappa)$ solves the BPZ equations: a system of $2n$ linear partial differential equations

Two standard ways of constructing solutions $Z = Z(\boldsymbol{x})$:

- reweighting independent SLEs via **Brownian loop measure** terms (Lawler, Lawler/Kozdron)
- contour integration/the method of **screening**/Coulomb gas integrals (Dubédat, Flores/Kleban, Kytölla/Peltola)

Large deviations come from limits of $\kappa \partial_{x_j} \log Z$ as $\kappa \to 0$

Solutions via Method of Screening

Introduce $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ and define

$$Z^p(\boldsymbol{x}, \boldsymbol{\zeta}) = \prod_{i \neq j} (x_i - x_j)^{2/\kappa} \prod_{i \neq j} (\zeta_i - \zeta_j)^{8/\kappa} \prod_{i,j} (x_i - \zeta_j)^{-4/\kappa}$$

Solutions via Method of Screening

Introduce $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ and define

$$Z^p(\boldsymbol{x}, \boldsymbol{\zeta}) = \prod_{i \neq j} (x_i - x_j)^{2/\kappa} \prod_{i \neq j} (\zeta_i - \zeta_j)^{8/\kappa} \prod_{i,j} (x_i - \zeta_j)^{-4/\kappa}$$

For appropriate contours $\mathcal{C}_1, \ldots, \mathcal{C}_n$

$$\boldsymbol{x} \mapsto \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_n} Z^p(\boldsymbol{x}, \boldsymbol{\zeta}) \, d\zeta_1 \ldots d\zeta_n =: Z(\boldsymbol{x}; \kappa)$$

solves the BPZ equations.

Solutions via Method of Screening

Introduce $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ and define

$$Z^p(\boldsymbol{x}, \boldsymbol{\zeta}) = \prod_{i \neq j} (x_i - x_j)^{2/\kappa} \prod_{i \neq j} (\zeta_i - \zeta_j)^{8/\kappa} \prod_{i,j} (x_i - \zeta_j)^{-4/\kappa}$$

For appropriate contours $\mathcal{C}_1, \ldots, \mathcal{C}_n$

$$\boldsymbol{x} \mapsto \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_n} Z^p(\boldsymbol{x}, \boldsymbol{\zeta}) \, d\zeta_1 \ldots d\zeta_n =: Z(\boldsymbol{x}; \kappa)$$

solves the BPZ equations. Ideal for **steepest descent**. As $\kappa \to 0$ integrals concentrate on poles of the rational function

$$\kappa \log Z(\boldsymbol{x}; \kappa) \xrightarrow{\kappa \to 0} \sum_{i \neq j} \log(x_i - x_j)^2 + \sum_{i \neq j} \log(\zeta_i(\boldsymbol{x}) - \zeta_j(\boldsymbol{x}))^8 + \sum_{i,j} \log(x_i - \zeta_j(\boldsymbol{x}))^{-4}$$

and $U_j = \partial_{x_j}$(RHS)

Method of Screening and Integrals of Motion

Given $\boldsymbol{x}, \boldsymbol{\zeta}$ and $z \in \mathbb{H}$

$\Phi(z; \boldsymbol{x}, \boldsymbol{\zeta}; \kappa) = \text{harmonic extension of boundary conditions determined by } \boldsymbol{x}, \boldsymbol{\zeta}, \kappa$

Then it can be shown that

$$\oint_{\mathcal{C}_1} \cdots \oint_{\mathcal{C}_n} \Phi(z; \boldsymbol{x}, \boldsymbol{\zeta}; \kappa)\, d\boldsymbol{\zeta} \bigg\| g_t^{-1}$$

is a martingale for the multiple SLE$(\kappa; \boldsymbol{x})$ process

Apply stationary phase to get an integral of motion for multiple SLE$(0; \boldsymbol{x}; \alpha)$

Method of Screening and Integrals of Motion

Theorem (Alberts-Kang-Makarov 2020)

For any $\boldsymbol{\nu}$-superposition of SLE$(0, x_j, \boldsymbol{\rho}_j)$ processes the quantities

$$g_t'(z)\frac{\prod_{j=1}^{2n}(g_t(z) - x_j(t))}{\prod_{k=1}^{n+1}(g_t(z) - g_t(\zeta_k(\boldsymbol{x})))^2}$$

are integrals of motion, for each $z \in \mathbb{H}$.

This does *not* require ζ to satisfy the stationary relation

When stationary relation is satisfied, integral of motion is key in the proof that generated hull is in $\Gamma(R)$

Integral of motion also leads to **geodesic multichord property** for $\Gamma(R)$

Open Questions

Critical Points of Higher Multiplicities

Throughout we've assumed $R = P/Q$ is real rational with critical points $\boldsymbol{x} = (x_1, \ldots, x_{2n})$, and critical points are order 1, i.e.

$$R'(z) \sim C(z - x_i), \quad z \to x_i$$

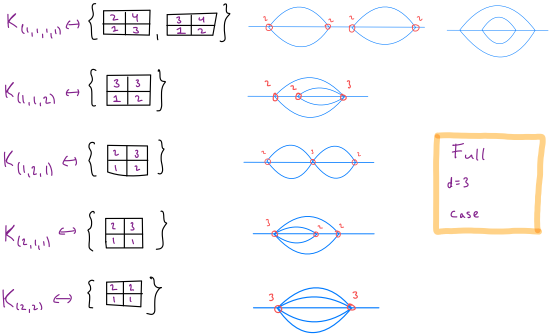
What about higher multiplicities?

Critical Points of Higher Multiplicities

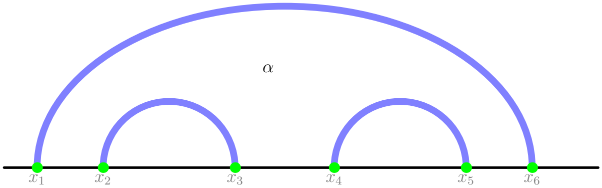
Throughout we've assumed $R = P/Q$ is real rational with critical points $\boldsymbol{x} = (x_1, \ldots, x_{2n})$, and critical points are order 1, i.e.

$$R'(z) \sim C(z - x_i), \quad z \to x_i$$

What about higher multiplicities?



Natural Time Parameterizations



$$\partial_t g_t(z) = \sum_{j=1}^{2n} \frac{2\nu_j(t)}{g_t(z) - x_j(t)}, \quad g_0(z) = z, \quad x_j(0) = x_j$$

where $\boldsymbol{x}(t) = (x_1(t), \ldots, x_{2n}(t))$ evolve according to

$$\dot{x}_j(t) = U_j(R \circ g_t^{-1})\nu_j(t) + \sum_{k \neq j} \frac{2\nu_k(t)}{x_j(t) - x_k(t)}, \quad j = 1, \ldots, 2n.$$

Enumeration: Solutions to Stationary Equation

Fix $\boldsymbol{x} = (x_1, \ldots, x_{2n})$. Up to permutation of coordinates, how many solutions $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ are there to the stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \qquad k = 1, \ldots, n?$$

Goldberg + Eremenko-Gabrielov/Peltola-Wang implies there should be exactly C_n, i.e. solutions are enumerated by link patterns

Is there a way to generate all solutions from one particular solution?

Enumeration: Solutions to Null Vector Equations

Fix $\boldsymbol{x} = (x_1, \ldots, x_{2n})$. Can one directly enumerate the number of (real) solutions $(U_1, \ldots, U_{2n})$ to the null vector equations

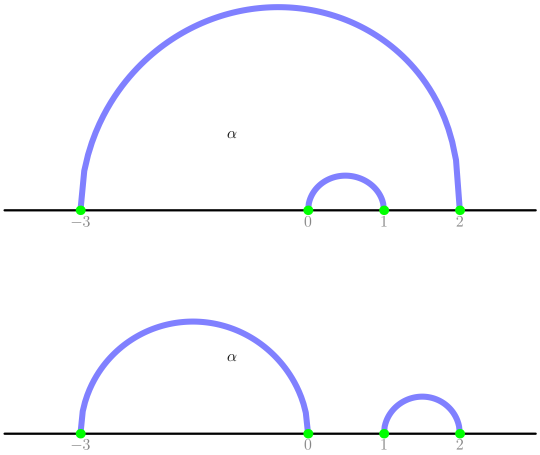
$$\frac{1}{2}U_j^2 + \sum_{k \neq j} \frac{2}{x_k - x_j} U_k - \sum_{k \neq j} \frac{6}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

and conformal Ward identities

$$\sum_{j=1}^{2n} U_j = 0, \quad \sum_{j=1}^{2n} x_j U_j = -6n, \quad \sum_{j=1}^{2n} x_j^2 U_j = -6 \sum_{j=1}^{2n} x_j.$$

Actual Pictures

Multiple SLE(0)



Multiple SLE(0)