Conformal welding in Liouville quantum gravity

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Joint works with M. Ang and X. Sun and with M. Lehmkuehler

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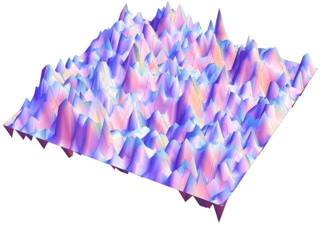
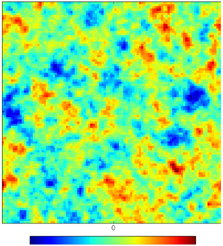
Agenda

1. Background
   - The Gaussian free field (GFF)
   - Liouville quantum gravity (LQG)
   - Schramm-Loewner evolution (SLE)
   - Conformal welding
2. Conformal welding of LQG surfaces
   - Recent and classical results
   - Applications

GFF, LQG and SLE: Three random planar objects



Gaussian Free Field
(generalized function)



Liouville quantum gravity
(2d Riemannian manifold)

Schramm-Loewner evolution
(non-crossing curve)

All three objects

- satisfy **conformal invariance** and **domain Markov property**, and
- describe the scaling limit of many discrete models (universality).

Gaussian free field (GFF)

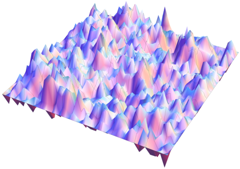
- Free boundary Gaussian free field h in $\mathbb{D}$: Gaussian random field with mean zero and covariance

$$\mathrm{Cov}(h(z), h(w)) = G(z, w), \qquad z, w \in \mathbb{D},$$

where $G : \mathbb{D} \times \mathbb{D} \to [0, \infty)$ is the Neumann Green's function

$$G(z, w) = \log |z - w|^{-1} + \log |1 - z\overline{w}|^{-1}.$$

- h **not** well defined as a function since $G(z, z) = \infty$.
- h well-defined as a **random generalized function (distribution)**.
  - $\int_{\mathbb{D}} hf \, d^2z$ is well-defined for f a smooth test function.
- See talks of Berestycki and Powell for more details.



Discrete GFF

Liouville quantum gravity (LQG)

- Let $\gamma \in (0,2)$ and let h be the Gaussian free field in $S = (0,1)^2$.
- LQG surface: $e^{\gamma h}(dx^2 + dy^2)$

$$\text{Area measure: } \mu = \text{“}e^{\gamma h} d^2 z\text{”},$$

$$\text{Boundary measure: } \nu = \text{“}e^{\gamma h/2} dz\text{”},$$

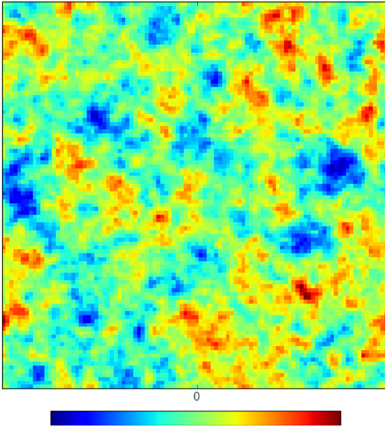
$$\text{Distance: } D = \text{“}e^{\gamma h/d_\gamma} |dz|\text{”}, \quad d_\gamma = \text{dimension} > 2.$$

- The definition of an LQG surface does not make literal sense since h is a distribution and not a function.
- μ, ν, D defined rigorously via regularized version h_ϵ of h, e.g.

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \quad U \subset [0,1]^2.$$

- References:
  - μ, ν: Hoegh-Krohn'71, Kahane'85, Duplantier-Sheffield'08, Rhodes-Vargas'13, Berestycki'15, etc.
  - D: Ding-Dubedat-Dunlap-Falconet'19, Gwynne-Miller'19
  - See also talks of Rhodes, Saksman and Sheffield

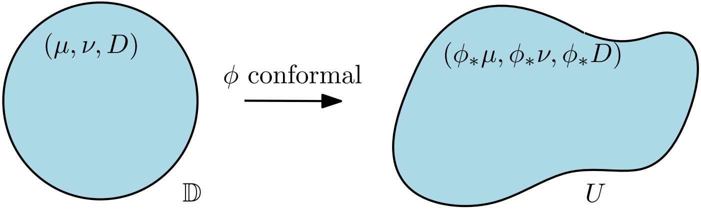
LQG area measure



Random area measure $\mu =$ "$e^{\gamma h} d^2 z$" (figure by M. Park)

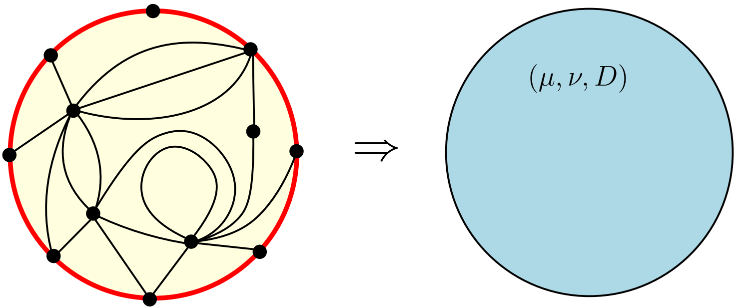
Liouville quantum gravity (LQG) surface

The tuple (μ, ν, D) describes the geometry of the γ-**LQG surface** $(\mathbb{D}, h)$.



Two different embeddings of the same γ-LQG surface

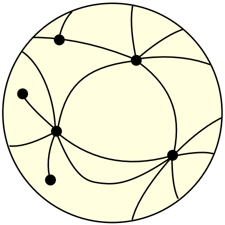
LQG as a scaling limit of random planar maps



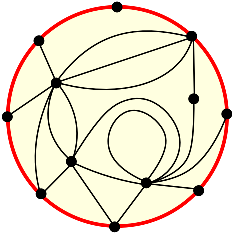
random planar map $\Rightarrow$ $\sqrt{8/3}$-LQG disk

See e.g. Le Gall'11, Miermont'11, Bettinelli-Miermont'15, Duplantier-Miller-Sheffield'14, Miller-Sheffield'16, H.-Sun'19.

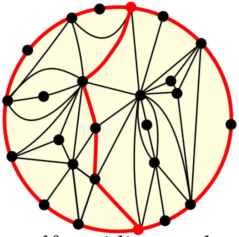
LQG as a scaling limit of random planar maps



$\Rightarrow$ $\sqrt{8/3}$-LQG sphere

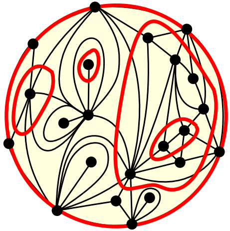


$\Rightarrow$ $\sqrt{8/3}$-LQG disk



self-avoiding path

$\Rightarrow$ $\sqrt{8/3}$-LQG disk with two $\frac{1}{\sqrt{6}}$-singularities

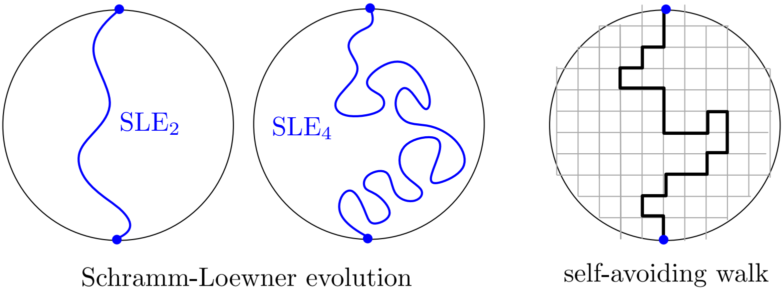


loop $O(n)$ model

$\Rightarrow$ γ-LQG disk ($\gamma = \gamma(n)$)

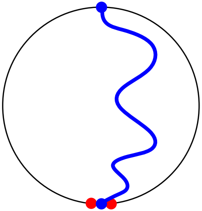
See e.g. Le Gall'11, Miermont'11, Bettinelli-Mierm.'15, Duplantier-Miller-Sheffield'14, Miller-Sheff.'16, Gwynne-Miller'16, H.-Sun'19.

Schramm-Loewner evolution (SLE)

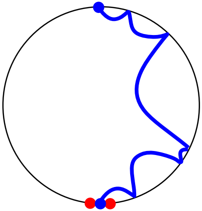


- Conformally invariant fractal curve η introduced in Schramm'99.
- Scaling limit of statistical physics models
  - Examples: self-avoiding walk (conjectured) and Ising model.
- Uniquely characterized by conf. inv. and domain Markov property.
- Parameter $\kappa > 0$.
- See talks of Healey, Wu, Peltola, Wang, Makarov.

Variants of SLE

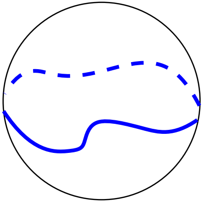


$\rho_- = 0,\ \rho_+ < 0$

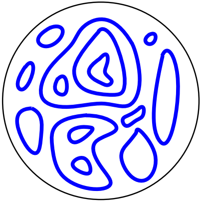


$\rho_- = 0,\ \rho_+ < \kappa/2 - 2$

SLE_κ with two force points $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$



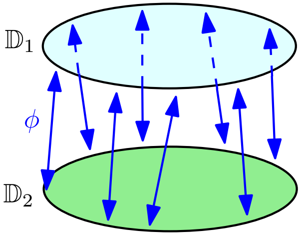
SLE_κ loop



Conformal loop ensemble CLE_κ

The conformal welding problem

- $\mathbb{D}_1, \mathbb{D}_2$ copies of the unit disk; $\phi : \partial\mathbb{D}_1 \to \partial\mathbb{D}_2$ a homeomorphism.
- Conformal welding: a conformal structure on the sphere $\mathbb{S}^2$ obtained by identifying $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_2$ according to ϕ.
  - More precisely, we are interested in a curve η and conformal maps $\psi_j : \mathbb{D}_j \to \widehat{D}_j$, $j = 1, 2$, such that $\phi = \psi_2^{-1} \circ \psi_1|_{\partial\mathbb{D}_1}$.
- Does there exist a conformal welding? If so, is it unique?
- Existence and uniqeness may fail, but sufficient regularity of ϕ or η guarantees the existence of a unique solution (see Younsi's talks).



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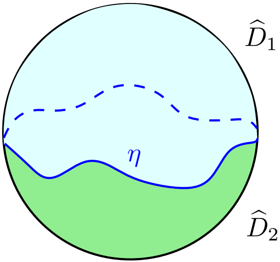
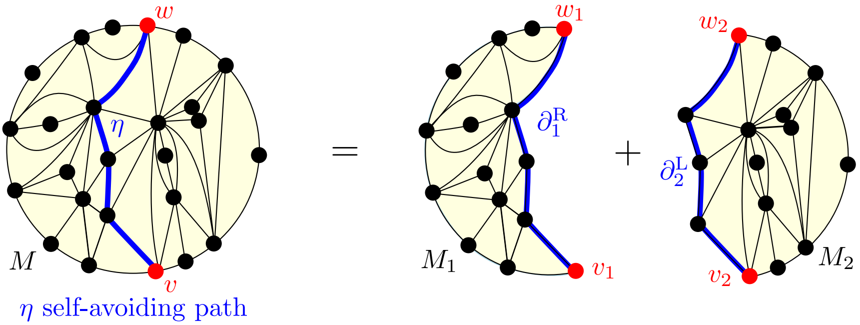


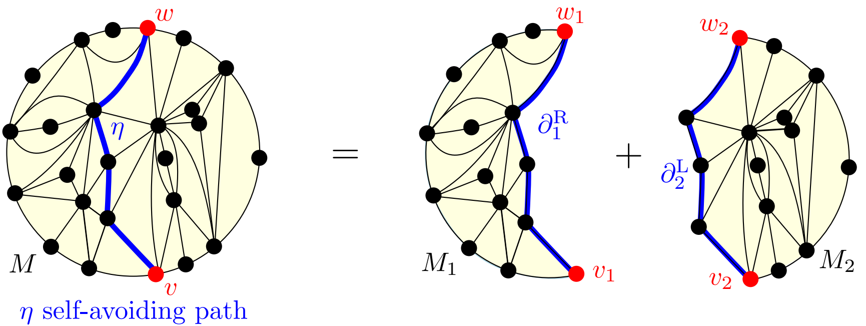
Agenda

Discrete motivation for conformal welding



Bijection: (M, v, w, η) and $\big((M_1, v_1, w_2), (M_2, v_2, w_2)\big), \#\partial_1^{\mathrm{R}} = \#\partial_2^{\mathrm{L}}$

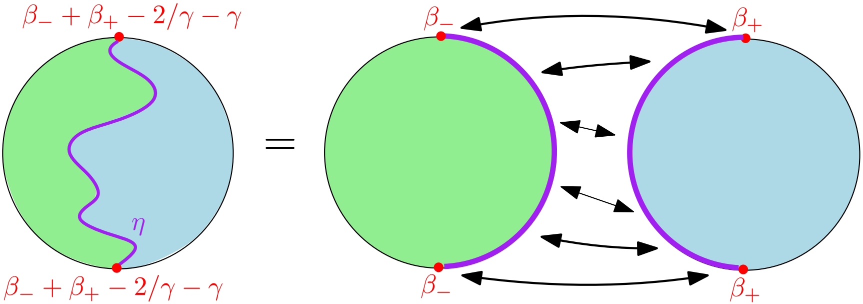
Discrete motivation for conformal welding



Bijection: (M, v, w, η) and $((M_1, v_1, w_2), (M_2, v_2, w_2)), \#\partial_1^{\mathrm{R}} = \#\partial_2^{\mathrm{L}}$

Key strength of conformal welding:
Divide complicated surfaces into simpler pieces

Conformal welding of LQG disks

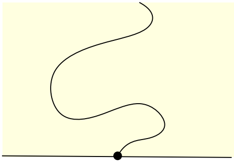


$\beta \in \mathbb{R}$ next to $z \in \partial\mathbb{D}$ means the field looks locally like GFF$+\beta \log|\cdot - z|^{-1}$

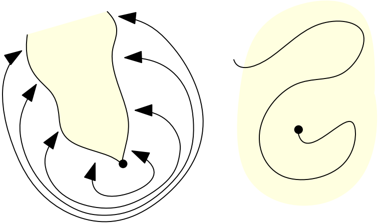
η has law $\mathrm{SLE}_\kappa(\rho_-;\rho_+)$, $\kappa = \gamma^2$, $\rho_\pm = \gamma^2 - \gamma\beta_\pm$

- Green & blue disks **independent** cond. on matching bdy lengths
- SLE and disk in left figure **independent**
- Proof purely continuum, although result inspired by planar maps
- Result can be formulated via either **welding** or **cutting**
- Ang-H.-Sun'20, building on Sheffield'10 & Duplantier-Miller-Sheff.'14

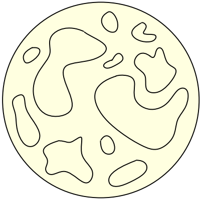
Conformal welding and cutting of LQG surfaces



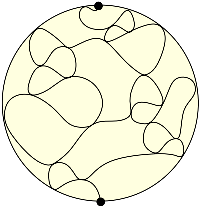
simple chordal SLE



whole-plane SLE from 0 to ∞



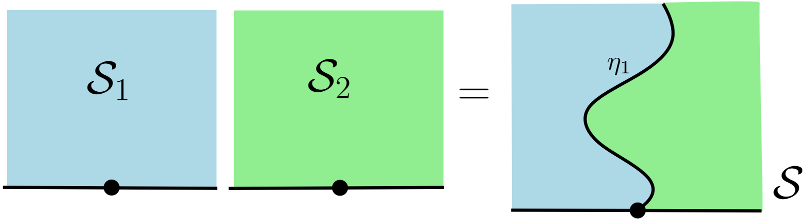
conformal loop ensemble



non-simple chordal SLE

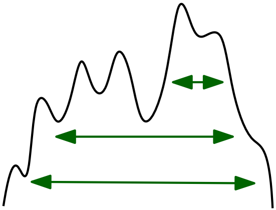
Sheffield'10, Duplantier-Miller-Sheffield'14, Miller-Sheffield-Werner'20

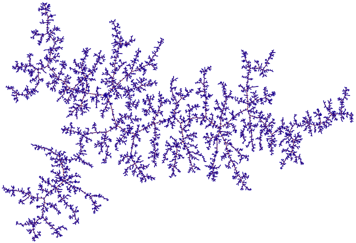
General principles



- Cutting a γ-LQG surface $\mathcal{S}$ by the "right" independent SLE_κ-type curve(s) $\eta_1, \eta_2, \ldots$ gives **independent** surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots$ in the complementary components. Note! Always $\gamma \in \{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$.
- $\mathcal{S}_1, \mathcal{S}_2, \ldots$, plus info about how the surfaces are glued together, determine $\mathcal{S}$ and $\eta_1, \eta_2, \ldots$.
- Discrete analogues on planar maps, although proof continuum.

Mating of trees

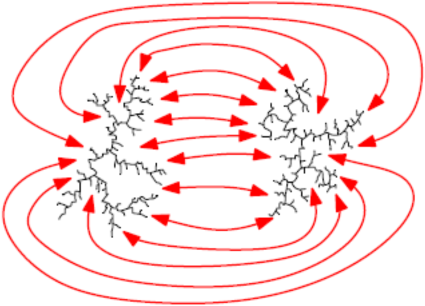




Brownian excursion and continuum random tree

Right figure by Kortchemski

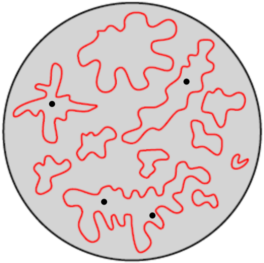
Mating of trees



Duplantier-Miller-Sheffield'14:
Mating/welding two continuum random trees gives
a γ-LQG surface with a space-filling $\mathrm{SLE}_{16/\gamma^2}$

Allows to study LQG and SLE with Brownian motion

LQG disk with marked points



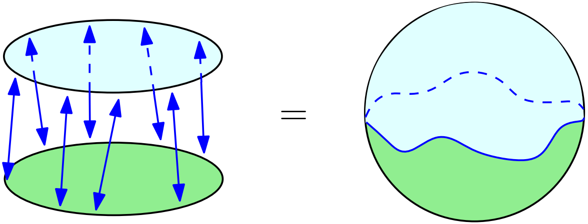
- LQG disk from Liouv. CFT: GFF $+\sum_j \alpha_j \log \frac{1}{|z-z_j|}$ (Rhodes' talks)
  - Location of z_j for $j \in [m]$ random, sampled using partition function.
- Conformal loop ensemble (CLE) weighted by

$$\prod_{\emptyset \neq A \subseteq [m]} e^{\sigma_A N_A}, \qquad \sigma_A \in \mathbb{R},\ N_A = \#\text{loops around } \{z_i \,:\, i \in A\}.$$

- H.-Lehmkuehler'22+: LQG surface inside each loop is indep. LQG disk with the given boundary length and points.
- Application to the partition function of surfaces with marked points.

Disk + disk = sphere + SLE loop

Ang-H.-Sun'21:



Two γ-LQG disks

γ-LQG sphere and SLE_{γ^2} loop

Disk + disk = sphere + SLE loop

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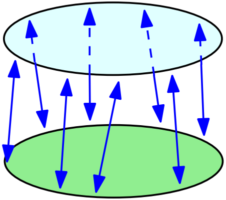

Two γ-LQG disks

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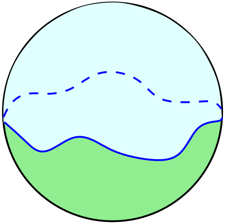


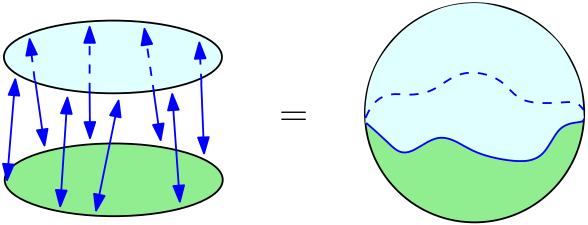
γ-LQG sphere and SLE_{γ^2} loop

The SLE loop:

- Malliavin-Kontsevich-Suhov (MKS) loop measure: satisfies a conformal restriction covariance property.
- Conjecturally there is a unique MKS loop measure for all $\kappa \in (0,4]$.
- Werner'05: Existence and uniqueness for $\kappa = 8/3$ via Brownian loops.
- Kemppainen-Werner'14: Existence for $\kappa \in (8/3,4]$ via CLE_κ.
- Zhan'17: Existence for $\kappa \in (0,8)$ via SLE natural parametrization.

Disk + disk = sphere + SLE loop

Ang-H.-Sun'21:



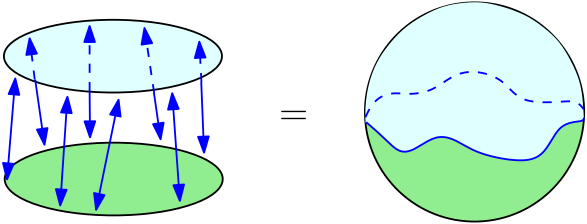
Two γ-LQG disks

γ-LQG sphere and SLE_{γ^2} loop

- Probabilistic analog of action functional identity of Viklund-Wang'19.

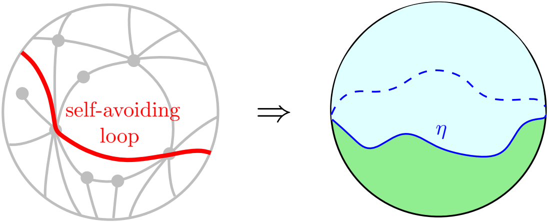
Disk + disk = sphere + SLE loop

Ang-H.-Sun'21:



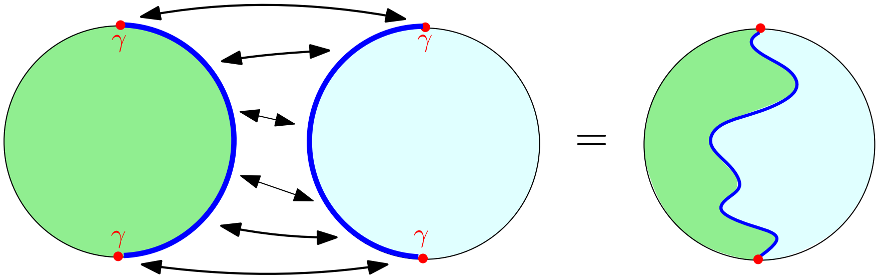
Two γ-LQG disks γ-LQG sphere and SLE_{γ^2} loop

Corollary combining Gwynne-Miller'16 and Ang-H.-Sun'21:



weight $= 12^{-\#\text{faces}} \cdot 54^{-\text{loop length}}$ $\gamma^2 = 8/3$

Disk + disk = sphere + SLE loop: Proof sketch



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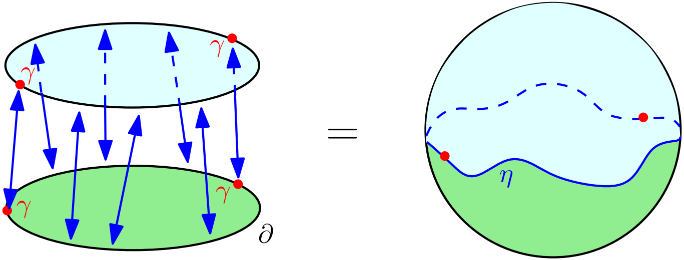
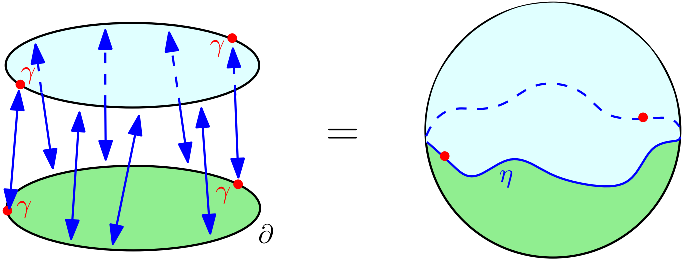


- LHS = LHS in thm, weighted by $\nu(\partial)^2$ & w/points sampled from ν.
- Sufficient to conclude proof:
  RHS = RHS in thm, weighted by $\nu(\eta)^2$ & w/points sampled from ν.

Disk + disk = sphere + SLE loop: Proof sketch



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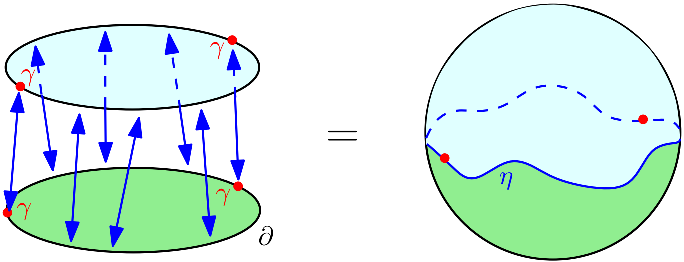
- **Uniform embedding**: Embedd surfaces "uniformly at random" in $\widehat{\mathbb{C}}$.

Disk + disk = sphere + SLE loop: Proof sketch



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- Sufficient to conclude proof:
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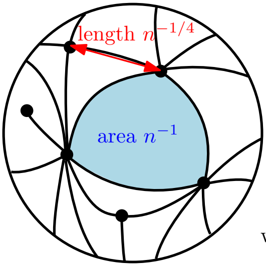
- **Uniform embedding**: Embedd surfaces "uniformly at random" in $\widehat{\mathbb{C}}$.
- Explicit joint law of field, loop, and points under uniform embedding.

Infinite measures on LQG surfaces

- LQG disk & LQG sphere **infinite** measures on the space of LQG surf.
- "Law" of bdy. len. for infinite measure $\mathcal{M}$ on LQG disks is $c\ell^{-7/2}\,d\ell$.
- $\mathcal{M} = \int \mathcal{M}_\ell\, d\ell$, where $\mathcal{M}_\ell$ is a **finite** meas. on disks w/bdy. len. ℓ.
  - The measures $\mathcal{M}_\ell$ often less natural in setting of conformal welding.
- Infinite measures natural from planar map point of view. Example:
  - Weight of triang. M is $n^{3/2}(12\sqrt{3})^{-\#\mathcal{V}_M} \to$ meas. μ_n on triang.
  - Rescale areas (resp. distances) of M by n (resp. $n^{1/4}$).
  - μ_n converges to the natural infinite measure on $\sqrt{8/3}$-LQG spheres.
- We typically work with **finite** measures on **infinite** volume surfaces.



weight $= n^{3/2}(12\sqrt{3})^{-\#\mathcal{V}_M}$

Applications of welding results

Ideas from conformal welding play an essential role for a number of results in random conformal geometry, e.g.

- convergence of random planar maps to LQG
  - three topologies: metric, conformal, mating-of-trees
- LQG distance function (metric) for $\gamma = \sqrt{8/3}$
- stat. phys. models on planar maps (scaling limits, exponents, etc.)
  - random walk, self-avoiding walk, DLA, percolation, FK-percolation, etc.
- integrability results for SLE, LQG and Liouville CFT
- SLE results (regularity for $\kappa \in \{4, 8\}$, dimensions&KPZ, topology, arm exponents, CLE percolations, etc.)
- many other results (e.g. planar map distance exponent)

See works of Ang, Duplantier, Gwynne, H., Kavvadias, Lehmkuehler, Miller, Pfeffer, Schoug, Sheffield, Sun, Werner, etc.

Thanks for attending!