Liouville CFT: probabilistic construction and bootstrap solution

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Based on joint works with G. Baverez, F. David, C. Guillarmou, A. Kupiainen, V. Vargas

Context

- Euclidean Quantum Field Theories (QFT) model statistical physics.
- Physical content is encoded in expectation values of observables (fields), which are called correlation functions.
- At critical temperature $\Rightarrow$ further conformal symmetries (i.e. Conformal Field Theories CFT).
- Belavin-Polyakov-Zamolodchikov (Conformal Bootstrap, 1984) observed that these extra symmetries constrain the system strongly and used it to classify CFTs. They gave explicit expressions for the correlation functions of several CFTs in 2D (minimal models, e.g. critical Ising model).
- In 3D, Conformal Bootstrap has recently led to spectacular numerical predictions (e.g. 3D Ising model) by Rychkov and collaborators.

Conformal Bootstrap

Axiomatic approach:

- Wightman's or Osterwalder-Schader's axioms
- Segal's axioms
- Vertex Operator Algebras in representation theory
- ...

Constructive approach:

- path integral
- perturbative or approximative
- stochastic quantization
- ...

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This talk: Liouville CFT

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Plan of the talk

Path integral for Liouville CFT

Conformal bootstrap

Structure of Liouville CFT

- Structure constants and the DOZZ formula
- Segal's axioms or CFT Lego game
- Spectrum of Liouville CFT

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Path integral for Liouville CFT

On Riemannian surface (Σ, g), a functional measure

$$\boxed{F \mapsto \int F(\Phi) e^{-S_\Sigma(\Phi, g)} D\Phi}$$

physics def / formal

where $D\Phi$ is the formal Lebesgue measure on $L^2(\Sigma, \mathrm{v}_g)$ and the Liouville action is

$$\boxed{S_\Sigma(\Phi, g) = \frac{1}{4\pi} \int_\Sigma (|d\Phi|_g^2 + QK_g\Phi + \mu e^{\gamma\Phi}) \mathrm{dv}_g}$$

with $\mu > 0$, $Q = 2/\gamma + \gamma/2$ and $\gamma \in (0, 2)$, K_g = scalar curvature of g

- Critical points of $S_\Sigma(g, \Phi)$ are related to finding Φ_0 s.t. $K_{e^{\gamma\Phi_0} g}$ = negative constant.

Probabilistic construction (David-Guillarmou-Kupiainen-R-Vargas 14-16')

Gaussian Free Field on (Σ, g): let X_g be the GFF on Σ in the metric g

$$X_g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\geq 1} \frac{\alpha_n}{\sqrt{\lambda_n}} e_n(x)$$

with

- $(\alpha_n)_n$ iid standard Gaussian r.v.
- $(e_n)_n$ orthonormal basis of eigenfunctions of Laplacian Δ_g with eigenvalues $(\lambda_n)_n$ and b.c. $\int_\Sigma e_n \mathrm{dv}_g = 0$
- the series converges a.s. in the Sobolev spaces $H^s(\Sigma, g)$ for $s < 0$.
- Covariance $\mathbb{E}[X_g(x)X_g(x')] = G_g(x, x')$ Green function of the Laplacian.

Gaussian integral:

$$\boxed{\int F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g} D\Phi = (\det{}'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[F(c + X_g)\Big]\, dc}$$

maths

Probabilistic construction (David-Guillarmou-Kupiainen-R-Vargas 14-16')

Liouville path integral

$$\langle F \rangle_{\Sigma,g} := \int F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma (|d\Phi|_g^2 + QK_g\Phi + e^{\gamma\Phi})\mathrm{dv}_g} D\Phi \qquad \text{physics def / formal}$$

$$\boxed{\langle F \rangle_{\Sigma,g} := (\det'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[F(c+X_g) e^{-\frac{1}{4\pi}\int_\Sigma (QK_g(c+X_g) + \mu e^{\gamma(c+X_g)})\mathrm{dv}_g}\Big] dc} \qquad \text{maths}$$

where

- $\mu > 0$, $\gamma \in (0,2)$ and $Q = 2/\gamma + \gamma/2$
- X_g be the GFF on Σ in the metric g
- $e^{\gamma X_g}\mathrm{dv}_g$ is a random measure (Gaussian multiplicative chaos, Eero's talk)

$$e^{\gamma X_g(x)}\mathrm{dv}_g(x) := \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{g,\epsilon}(x)}\mathrm{dv}_g(x)$$

with $X_{g,\epsilon}$ a regularization of X_g

Correlation functions

Fix arbitrarily some points $x_1, \dots, x_n \in \Sigma$ and some weights $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

$$\langle \prod_{j=1}^{n} V_{\alpha_j}(x_j) \rangle_{\Sigma,g} = \int e^{\alpha_1 \Phi(x_1)} \dots e^{\alpha_n \Phi(x_n)} e^{-S_\Sigma(\Phi,g)} D\Phi \qquad \text{physics def / formal}$$

and the math definition

$$\langle \prod_{j=1}^{n} V_{\alpha_j}(x_j) \rangle_{\Sigma,g} \overset{def}{=} (\det{}'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[\prod_{j=1}^{n} e^{\alpha_j (c + X_g(x_i))} e^{-\frac{1}{4\pi} \int_\Sigma (Q K_g (c+X_g) + e^{\gamma(c+X_g)}) \mathrm{dv}_g} \Big] dc$$

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Theorem (David-Guillarmou-Kupiainen-Rhodes-Vargas '14-'16)

The correlation functions are non trivial iff the Seiberg bounds are satisfied

$$\forall i \quad \alpha_i < Q \qquad \textit{and} \qquad \sum_{i=1}^{n} \alpha_i > \chi(\Sigma) Q$$

with $\chi(\Sigma)$ the Euler characteristics.

The probabilistic construction of the Liouville model is a CFT

- **Diffeomorphism invariance:** for $\psi : \Sigma \to \Sigma$ diffeo

$$\langle \prod_i V_{\alpha_i}(\psi(x_i)) \rangle_{\Sigma, g} = \langle \prod_i V_{\alpha_i}(x_i) \rangle_{\Sigma, \psi^* g}$$

- **Weyl covariance:** for $\varphi : \Sigma \to \mathbb{R}$ smooth

$$\langle \prod_i V_{\alpha_i}(z_i) \rangle_{\Sigma, e^{\varphi} g} = e^{\frac{\mathbf{c}}{96\pi} \int_\Sigma |d\varphi|_g^2 + 2K_g \varphi} \Big(\prod_i e^{-\Delta_{\alpha_i} \varphi(z_i)} \Big) \langle \prod_i V_{\alpha_i}(z_i) \rangle_{\Sigma, g}$$

where

- $\mathbf{c} \in \mathbb{C}$ is the central charge (which classifies CFTs).
- Δ_α is called conformal weight of the primary field V_α. For Liouville CFT

$$\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$$

Riemann sphere and structure constants

If $\Sigma = \hat{\mathbb{C}}$ is Riemann equipped with round metric g_0, the 3 point correlation function

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3)\rangle_{\hat{\mathbb{C}},g_0}$$

depends trivially on x_1, x_2, x_3 by **diffeomorphism invariance** and **Weyl covariance**. One can then take $x_1 = 0, x_2 = 1, x_3 = +\infty$.

The quantity

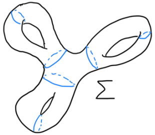
$$\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\hat{\mathbb{C}},g_0}$$

is then called the structure constant of the CFT.

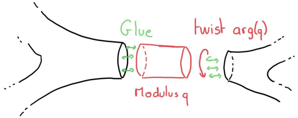
Moduli space and plumbing coordinates

Let $\mathcal{M}_{h,n}$ be the moduli space of Riemann surfaces, $h = \mathrm{genus}(\Sigma)$ and n marked points.

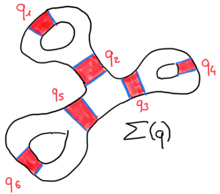
- Choose a pant decomposition of the surface



- At each splitting curve, glue a annulus of modulus $|q|$ with twist $\arg(q)$



- The mapping $q \in \mathbb{D}^{3h-3+n} \mapsto \Sigma(q)$ provides a system of complex local coordinates on $\mathcal{M}_{h,n}$ (for well chosen curves, see Hinich-Vaintrob)



General physics conjecture for CFTs

For a closed Riemannian surface (Σ, g) with n marked points $x = (x_1, \dots, x_n) \in \Sigma^n$

$$\langle V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n)\rangle_{\Sigma,g} = C_g \int_{\mathcal{S}^{3h-3+n}} \rho(p,\alpha)|\mathcal{F}_{c,p,\Delta_\alpha}(q)|^2 m(\mathrm{d}p)$$

- $\rho(p, \alpha)$ is a product of structure constants
- $q \mapsto \mathcal{F}_{c,p,\alpha}(q)$ = conformal blocks holomorphic in $q = (q_1, \dots, q_{3h-3+n})$, plumbing (complex) coordinates on the moduli space $\mathcal{M}_{h,n}$, $h = \mathrm{genus}(\Sigma)$.
- m is a measure over a subset $\mathcal{S}$ (called spectrum) of primary fields of the CFT
- $C_g > 0$ an explicit constant depending on g.

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Remarks (first round):

- structure constants are model dependent, Conformal blocks universal
- the measure m can be finite sum of Dirac masses (minimal models or rational CFTs) or diffuse (non compact CFTs)
- Conformal blocks can be expanded as power series in q with coefficients depending only on the commutation relations of the Virasoro algebra. Convergence of the series is unknown

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Remarks (second round):

- Vertex Operator Algebras (Borcherds, Frenkel,...) make sense of CFTs via the right-hand side
- consistency conditions (crossing symmetry/modular invariance) hard to check
- in the VOA context, the case of rational CFTs about to be treated by Y-Z. Huang, B. Gui and collaborators

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Theorem (Guillarmou-Kupiainen-Rhodes-Vargas '21)

The conformal bootstrap holds for the probabilistic construction of the Liouville CFT.

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Theorem (Guillarmou-Kupiainen-Rhodes-Vargas '21)

The conformal bootstrap holds for the probabilistic construction of the Liouville CFT.

- Convergence (and maths def) of the conformal blocks is an output of our proof
- Probabilistic representation of the conformal block for torus 1-point correlation function (Ghosal-Rémy-Sun-Sun '20).

Solving CFTs

$$\boxed{\langle V_{\alpha_1}(x_1)\dots V_{\alpha_n}(x_n)\rangle_{\Sigma,g} = C_g \int_{\mathcal{S}^{3h-3+n}} \rho(p,\alpha)|\mathcal{F}_{c,p,\Delta_\alpha}(q)|^2 m(\mathrm{d}p)}$$

3 key steps to solve CFTs

- show that the above factorization holds
- determine the structure constants
- determine the spectrum $(\mathcal{S}, m)$

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3 key steps to solve CFTs (Liouville CFT)

- show that the above factorization holds (Segal's axioms)
- determine the structure constants (DOZZ formula)
- determine the spectrum $(\mathcal{S}, m)$ (Scattering theory)

DOZZ formula

Theorem (Kupiainen Rhodes Vargas, '17)

Assume the Seiberg bounds $\forall i,\ \alpha_i < Q$ *and* $\sum_{i=1}^{3} \alpha_i > 2Q$. *Then*

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}, g_0} = C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1, \alpha_2, \alpha_3)$$

with $C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1, \alpha_2, \alpha_3) :=$

$$\left(\pi\, \mu \, \frac{\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})} \, (\frac{\gamma}{2})^{2-\gamma^2/2}\right)^{\frac{2Q-\bar{\alpha}}{\gamma}} \times \frac{\Upsilon'_{\frac{\gamma}{2}}(0) \Upsilon_{\frac{\gamma}{2}}(\alpha_1) \Upsilon_{\frac{\gamma}{2}}(\alpha_2) \Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2Q}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_1}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_2}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_3}{2})}$$

with $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and the function $\Upsilon_{\frac{\gamma}{2}}$ defined as analytic continuation of the following integral defined for $0 < \Re(z) < \Re(Q)$

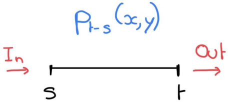
$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z\right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4}) \sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

Warm-up: amplitudes for semigroups

Consider a semigroup $(P_t)_{t\geq 0}$ acting on some space $L^2(\pi)$ with integral kernel

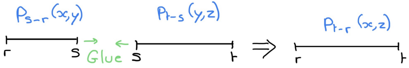
$$P_t f(x) = \int p_t(x,y) f(y) \pi(\mathrm{d}y)$$

Amplitudes: assign the kernel p_{t-s} to each segment $[s,t]$



Gluing segments maps to composition of operators (gluing kernels):

$$p_{t-r}(x,z) = \int p_{s-r}(x,y) p_{t-s}(y,z) \pi(dy)$$



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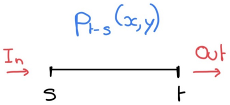

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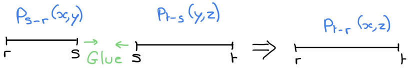
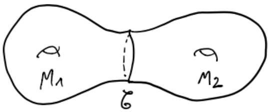


Segal's axioms are a generalization of this picture to Riemann surfaces

Segal axioms (physics heuristics)



Disintegration of path integral using conditioning on $\mathcal{C} = \partial\Sigma_1 = \partial\Sigma_2$: if the action is local

$$S_\Sigma(\Phi, g) = S_{\Sigma_1}(\Phi|_{\Sigma_1}, g) + S_{\Sigma_2}(\Phi|_{\Sigma_2}, g)$$

one should have

$$\int_{\{\Phi:\Sigma\to\mathbb{R}\}} e^{-S_\Sigma(\Phi,g)} D\Phi = \int_{\{\varphi:\mathcal{C}\to\mathbb{R}\}} \Big(\int_{\substack{\{\Phi:\Sigma_1\to\mathbb{R},\\ \Phi|_{\mathcal{C}}=\varphi\}}} e^{-S_{\Sigma_1}(\Phi|_{M_1},g)} D\Phi \Big) \Big(\int_{\substack{\{\Phi:\Sigma_2\to\mathbb{R},\\ \Phi|_{\mathcal{C}}=\varphi\}}} e^{-S_{\Sigma_2}(\Phi|_{\Sigma_2},g)} D\Phi \Big) D\varphi$$

$$= \int_{E(\mathcal{C})} \mathcal{A}_{\Sigma_1}(\varphi)\mathcal{A}_{\Sigma_2}(\varphi) D\varphi$$

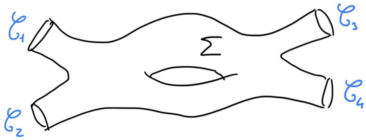
A_{Σ_j} is called amplitude of Σ_j.

Segal axioms

A Conformal Field Theory is

- Object: $\mathcal{H}$ a Hilbert space attached to $\mathbb{S}^1$ (for us: $\mathcal{H} = L^2(H^{-s}(\mathbb{S}^1), \mu_0)$)
- Morphism: to each Riemannian surface (Σ, g) with parametrized boundary $\partial\Sigma = \sqcup_{i=1}^{b} \mathcal{C}_i$, we associate an amplitude

$$\mathcal{A}_{\Sigma,g} \in L^2(H^{-s}(\mathbb{S}^1)^b, \mu_0^b) = \otimes^b \mathcal{H}$$



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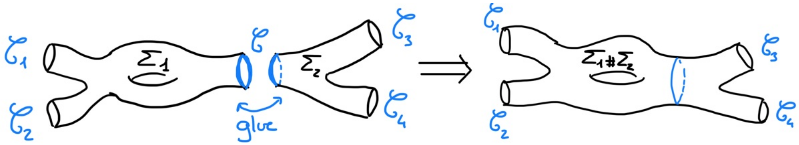
- natural behaviour under geometrical moves (diffeomorphism invariance and Weyl covariance)
- Gluing rule: if we glue (Σ_1, g_1) with (Σ_2, g_2) by identifying the circle $\mathcal{C}$ then the amplitude of the glued surface $\Sigma_1 \sharp \Sigma_2$ is obtained by integrating out the $\mathcal{C}$-component of A_{Σ_1,g_1} against the $\mathcal{C}$ component of A_{Σ_2,g_2}

$$\boxed{\mathcal{A}_{\Sigma,g} = \mathcal{A}_{\Sigma_1,g_1} \circ_{\mathcal{C}} \mathcal{A}_{\Sigma_2,g_2}}$$



Segal's axioms for Liouville CFT

Hilbert space: take $\Omega := (\mathbb{R}^2)^{\mathbb{N}^*}$ equipped with Gaussian measure $\mathbb{P} = \prod_{n\geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2+y_n^2)} dx_n dy_n$,

$$\boxed{\mathcal{H} := L^2(\mathbb{R}_c \times \Omega, dc \otimes \mathbb{P}) = L^2(H^{-\varepsilon}(\mathbb{S}^1), \mu_0)}$$

where μ_0 is pushfoward of $dc \otimes \mathbb{P}$ by the real-valued random field

$$(*) \qquad \varphi = c + \sum_{n\neq 0} \varphi_n e^{in\theta}, \quad \varphi_n = \frac{1}{2}\frac{x_n + iy_n}{\sqrt{n}}, \; n > 0$$

If b disjoint circles, $\mathcal{H}^{\otimes b} = L^2(H^{-\varepsilon}(\mathbb{S}^1)^b, \mu_0^b)$, take b independent copies of φ.

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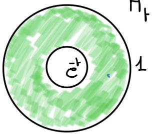
Amplitudes: let (Σ, g) with b parametrized boundary circles and n weighted marked points (x_i, α_i):

$$\mathcal{A}_{\Sigma,g,x,\alpha}(\boldsymbol{\varphi}) = \int_{\substack{\{\Phi:\Sigma\to\mathbb{R},\\ \Phi|_{\mathcal{C}}=\boldsymbol{\varphi}\}}} \prod_{j=1}^{n} e^{\alpha_j \Phi(x_j)} e^{-S_\Sigma(\Phi,g)} D\Phi \qquad \text{formal def}$$

with $\boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^b) \in H^{-\varepsilon}(\mathbb{S}^1)^b$.
Rigorous definition similar to Liouville path integral with further conditioning related to the GFF.

Hamiltonian of Liouville CFT

Consider the annulus



$A_t = \{z \in \mathbb{C} ; e^{-t} \leqslant |z| \leqslant 1\}$

with metric $g = \frac{|dz|^2}{|z|^2}$

Define the operator $S(t) : \mathcal{H} \to \mathcal{H}$:

$$\forall \varphi \in \mathcal{H}, \quad \boxed{(S(t)F)(\varphi) := \int_{\mathcal{H}} A_{\mathbb{A}_t}(\varphi, \varphi')F(\varphi')d\mu_0(\varphi')}$$

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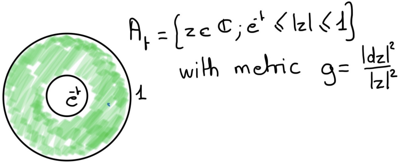

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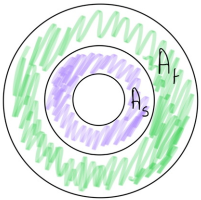
idea 1: gluing two annuli produces bigger annuli

$\Longrightarrow S(t)$ should be a semi-group.



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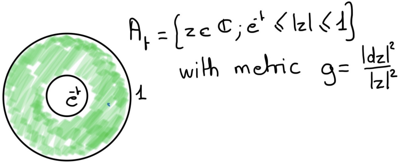

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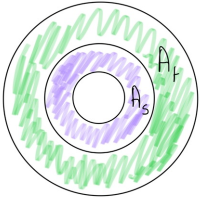
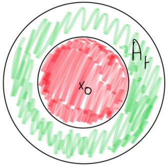


idea 2: gluing annulus $\mathbb{A}_t$ with a disk $\mathbb{D}$ with one marked point at 0 produces a bigger disk

$\Longrightarrow S(t)A_{\mathbb{D},0,\alpha} = e^{\lambda t}A_{\mathbb{D},0,\alpha}$.



Recall $\Omega := (\mathbb{R}^2)^{\mathbb{N}^*}$ with measure $\mathbb{P} = \prod_{n\geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2+y_n^2)} dx_n dy_n$, and

$$(*) \qquad \varphi = c + \sum_{n\neq 0} \varphi_n e^{in\theta}, \quad \varphi_n = \frac{1}{2}\frac{x_n + iy_n}{\sqrt{n}}, \; n > 0$$

Proposition (Guillarmou-Kupiainen-Rhodes-Vargas '20)

The operator $e^{-(\frac{1+6Q^2}{12})t} S(t) = e^{-t\mathbf{H}}$ *is a Markovian contraction semi-group on* $\mathcal{H} = L^2(\mathbb{R} \times \Omega; dc \otimes \mathbb{P})$ *with self-adjoint generator*

$$\boxed{\mathbf{H} = \frac{1}{2}(-\partial_c^2 + Q^2 + 2\mathbf{P} + \mu e^{\gamma c} V)}$$

with $\mathbf{P}$ *the infinite harmonic oscillator and* $V \in L^{\frac{2}{\gamma^2}-}(\Omega)$ *a positive potential/measure:*

$$\mathbf{P} := \sum_{n=1}^{\infty} n[(\partial_{x_n})^* \partial_{x_n} + (\partial_{y_n})^* \partial_{y_n}], \quad V(\tilde{\varphi}) := \frac{1}{2\pi} \int_{\mathbb{S}} e^{\gamma \tilde{\varphi}(\theta)} d\theta$$

where $\tilde{\varphi} = \varphi - c$.

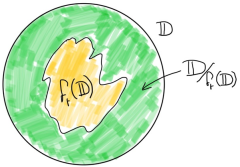
Flow of deformations (GKRV+Baverez, soon)

Take a holomorphic vector field $\mathbf{v} := v(z)\partial_z$ on $\mathbb{D}$ with

$$v(z) = -\sum_{n\geq -1} v_n z^{n+1}$$

If $\mathrm{Re}(\bar{z}v(z)) < 0$ on $\partial\mathbb{D}$ then $\partial_t f_t(z) = v(f_t(z))$ generates a flow of conformal maps

$$f_t : \mathbb{D} \to \mathbb{D}_t \subset \mathbb{D}.$$



Define

$$S^{\mathbf{v}} F(\varphi) = c_t \int_{\mathcal{H}} F(\varphi + Q \ln \frac{|f_t'|}{|f_t|}) A_{\mathbb{D}\setminus\mathbb{D}_t}(\varphi, \varphi')\mu_0(\mathrm{d}\varphi')$$

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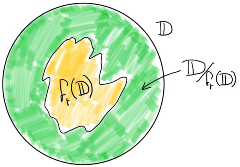

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Theorem

Under some conditions on $\mathbf{v}$, this is a Markovian semigroup $(e^{-t\mathbf{H}_{\mathbf{v}}})_t$ acting on $L^2(\mu_0)$.

Representation of the Virasoro algebra

Consider the following generators of the deformation semigroup

$$\mathbf{L}_n = \frac{1}{2}(\mathbf{H}_{\mathbf{v}} - i\mathbf{H}_{i\mathbf{v}}), \qquad \widetilde{\mathbf{L}}_n = \frac{1}{2}(\mathbf{H}_{\mathbf{v}} + i\mathbf{H}_{i\mathbf{v}}) \quad \text{for} \quad \mathbf{v} := -z^{n+1}\partial_z$$

They form two commuting families of unitary representations of the Virasoro algebra

$$[\mathbf{L}_n, \widetilde{\mathbf{L}}_m] = 0, \qquad \mathbf{L}_n^* = \mathbf{L}_{-n}, \qquad \widetilde{\mathbf{L}}_n^* = \widetilde{\mathbf{L}}_{-n}$$
$$[\mathbf{L}_n, \mathbf{L}_m] = (n-m)\mathbf{L}_{n+m} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m}\mathrm{Id},$$
$$[\widetilde{\mathbf{L}}_n, \widetilde{\mathbf{L}}_m] = (n-m)\widetilde{\mathbf{L}}_{n+m} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m}\mathrm{Id}.$$

with $c_L = 1 + 6Q^2$ the central charge.

Remark: for $\mathbf{v} := -z\partial_z$, the flow f_t is the flow of dilations of the unit disk $f_t(z) = e^{-t}z$. Then $\mathbf{L}_{\mathbf{v}} = \mathbf{L}_0 + \widetilde{\mathbf{L}}_0 = \mathbf{H}$ (generator of semigroup generated by the annulus amplitude)

Spectrum of Liouville CFT (diagonalization of $\mathbf{H}$ using scattering theory)

Recall

$$\mathbf{H} = \frac{1}{2}(-\partial_c^2 + Q^2 + 2\mathbf{P} + e^{\gamma c} V)$$

with $\mathbf{P}$ the infinite harmonic oscillator and $V \in L^{\frac{2}{\gamma^2}-}(\Omega)$ a positive potential/measure

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Young diagrams: decreasing finite sequence $\nu = (\nu_1, \dots, \nu_k)$ with $\nu_j \in \mathbb{N}$. Length $|\nu| = \sum_j \nu_j$.

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Theorem (Guillarmou-Kupiainen-Rhodes-Vargas '20)

Let $\gamma \in (0,2)$, $Q = 2/\gamma + \gamma/2$. There is a complete family of eigenstates $\Psi_{Q+ip,\nu,\tilde{\nu}} \in e^{-\varepsilon c} L^2(\mathbb{R}_c \times \Omega)$ labeled by $p \in \mathbb{R}_+$ and Young diagrams $\nu, \tilde{\nu}$ s.t.

$$\mathbf{H}\Psi_{Q+ip,\nu,\tilde{\nu}} = \Big(\frac{Q^2}{2} + \frac{p^2}{2} + |\nu| + |\tilde{\nu}|\Big)\Psi_{Q+ip,\nu,\tilde{\nu}}.$$

- *Plancherel formula: $\Psi_{Q+ip,\nu,\tilde{\nu}}$ is a complete family diagonalizing $\mathbf{H}$: $\forall u_1, u_2 \in L^2(\mathbb{R} \times \Omega)$*

$$\langle u_1, u_2 \rangle = \sum_{|\nu'|=|\nu|} \sum_{|\tilde{\nu}'|=|\tilde{\nu}|} \int_0^{\infty} \langle u_1, \Psi_{Q+ip,\nu,\tilde{\nu}} \rangle \langle \Psi_{Q+ip,\nu',\tilde{\nu}'}, u_2 \rangle \mathcal{Q}_{Q+ip}^{-1}(\nu,\nu') \mathcal{Q}_{Q+ip}^{-1}(\tilde{\nu},\tilde{\nu}')\, dp$$

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- $\mathcal{Q}_{Q+ip}(\nu,\tilde{\nu})$ is a Gram matrix, called Schapovalov form. Uniquely determined by the commutation relations of the Virasoro algebra.
- $\Psi_{Q+ip,\nu,\tilde{\nu}}$ are not orthonormal! Formally

$$\langle \Psi_{Q+ip,\nu,\tilde{\nu}}, \Psi_{Q+ip',\nu',\tilde{\nu}'}\rangle = \delta_{p=p'}\delta_{|\nu'|=|\nu|}\delta_{|\tilde{\nu}'|=|\tilde{\nu}|}\mathcal{Q}^{-1}_{Q+ip}(\nu,\nu')\mathcal{Q}^{-1}_{Q+ip}(\tilde{\nu},\tilde{\nu}')$$

- Mathematical formulation of the Operator Product Expansion in the physics language.

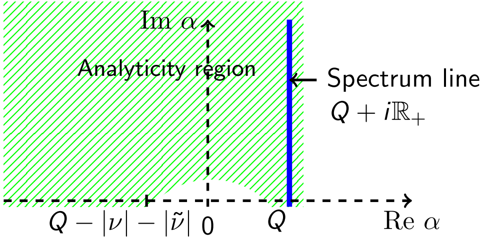
Link with the amplitude of the disk

Proposition (Guillarmou-Kupiainen-Rhodes-Vargas '20)

1) The eigenstates can be analytically continued on some domain of $\mathbb{C}$

$$\alpha \mapsto \Psi_{\alpha,\nu,\tilde{\nu}}$$

The spectrum corresponds to $\alpha \in Q + i\mathbb{R}_+$.



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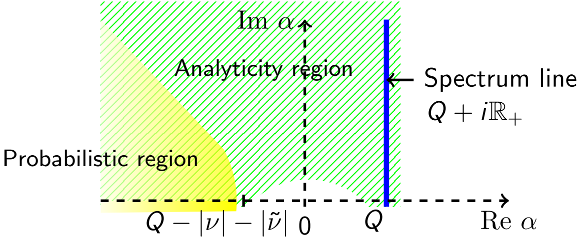
The spectrum corresponds to $\alpha \in Q + i\mathbb{R}_+$.
2) For $\alpha < Q$ *real, we have a probabilistic representation of the eigenstates. In particular,*

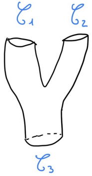
- $\Psi_{\alpha,0,0}$ *is the amplitude of the disk with one marked point at* $x = 0$ *and weight* α.
- $\Psi_{\alpha,\nu,\tilde{\nu}} = \mathbf{L}_{-\nu_1} \dots \mathbf{L}_{-\nu_k} \widetilde{\mathbf{L}}_{-\tilde{\nu}_1} \dots \widetilde{\mathbf{L}}_{-\tilde{\nu}_{\tilde{k}}} \Psi_{\alpha,0,0}$



Pant amplitude

Consider the amplitude of a pant A_{pant}.
We want to evaluate

$$\langle A_{\mathrm{pant}}, \otimes_{j=1}^{3} \Psi_{Q+ip_j,0,0} \rangle_{\mathcal{H}^3}$$



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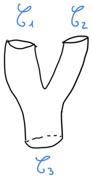

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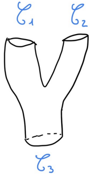

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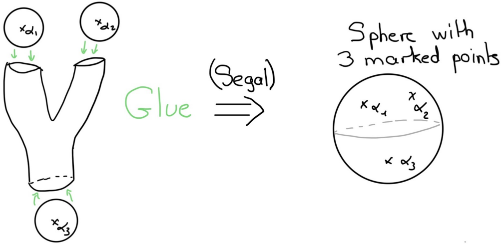
Playing with Segal yields

$$\langle A_{\mathrm{pant}}, \otimes_{j=1}^{3} \Psi_{\alpha_j,0,0} \rangle_{\mathcal{H}^3} = C_{\gamma,\mu}^{\mathrm{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$$



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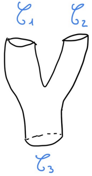

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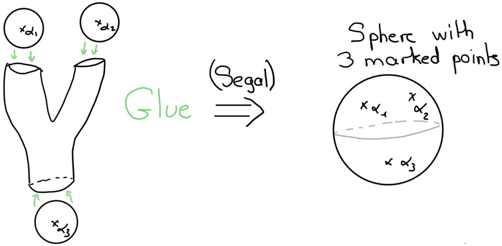
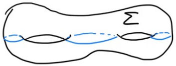


Similar idea, using Ward identities yields

$$\langle A_{\mathrm{pant}}, \otimes_{j=1}^{3} \Psi_{\alpha_j,\nu_j,\tilde{\nu}_j} \rangle_{\mathcal{H}^3} = \text{factor contributing to the conformal blocks} \times C_{\gamma,\mu}^{\mathrm{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$$

Conformal Bootstrap: idea of the proof in genus 2 partition function

- Choose a pant decomposition of the surface



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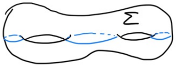
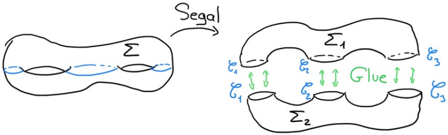


- Use Segal's gluing rule: $\langle 1 \rangle_{\Sigma, g} = \langle A_{\Sigma_1, g}, A_{\Sigma_2, g} \rangle_{\mathcal{H}^{\otimes 3}}$



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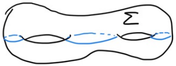

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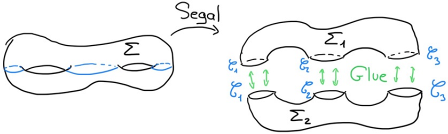

- Use the Plancherel identity

$$\langle A_{\Sigma_1,g}, A_{\Sigma_2,g}\rangle_{\mathcal{H}^{\otimes 3}}$$

$$= \sum_{\text{Young diag.}} \int_{\mathbb{R}_+^3} \langle A_{\Sigma_1,g}, \otimes_{j=1}^3 \Psi_{Q+ip_j,\nu_j,\tilde{\nu}_j}\rangle_{\mathcal{H}^{\otimes 3}} \langle \otimes_{j=1}^3 \Psi_{Q+ip_j\nu_j',\tilde{\nu}_j'}, A_{\Sigma_1,g}\rangle_{\mathcal{H}^{\otimes 3}} \times \text{Schapo}.\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3$$

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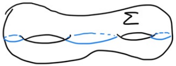

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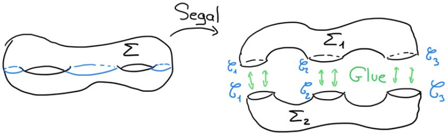

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- Use the pant computations

$$\langle 1 \rangle_{\Sigma_q,g_q} = \int_{(\mathbb{R}^+)^3} C^{\mathrm{DOZZ}}(Q+ip_1, Q+ip_2, Q+ip_3) C^{\mathrm{DOZZ}}(Q-ip_1, Q-ip_2, Q-ip_3) |\mathcal{F}_p|^2 dp$$

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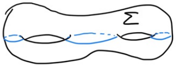

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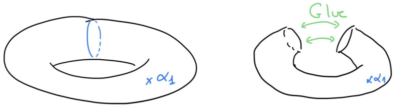
- Change of moduli of surface: glue annuli of moduli $q = (q_1, q_2, q_3) \in \mathbb{D}^3$ between Σ_1 and Σ_2, this only enters the conformal block

$$\langle 1\rangle_{\Sigma_q,g_q} = \int_{(\mathbb{R}^+)^3} C^{\mathrm{DOZZ}}(Q+ip_1, Q+ip_2, Q+ip_3) C^{\mathrm{DOZZ}}(Q-ip_1, Q-ip_2, Q-ip_3)|\mathcal{F}_p(q)|^2 dp$$

Another example: torus 1 point



1-point function on torus $\mathbb{T}^2_\tau = \mathbb{C}/(2\pi\mathbb{Z} + 2\pi\tau\mathbb{Z})$, with $q = e^{2i\pi\tau}$

$$\langle V_{\alpha_1}(x_1)\rangle_{\mathbb{T}^2_\tau} = \frac{|q|^{-\frac{1+6Q^2}{12}}}{2\pi}\int_0^\infty C(Q+ip,\alpha_1,Q-ip)|q|^{-2\Delta_{Q+ip}}|\mathcal{F}_{p,\alpha_1}(q)|^2 dp$$

Remarks:

- first mathematical proof of the full bootstrap formulae proposed by physicists (Knizhnik, Belavin, Sonoda, Polchinski, Teschner ...).
- the bootstrap formula depends on the chosen decomposition into pairs of pants, annuli with 1 marked point/insertion and disks with 1 or 2 marked points/insertions
- proves crossing symmetries: formulas for correlations functions given by bootstrap approach do not depend on the decomposition into geometric blocks (although conformal blocks do)
- implies convergence a.e. $P \in \mathbb{R}$ of conformal block series (this was an open problem)

$$\mathcal{F}_{P,\alpha}(q) = \sum_{k \in \mathbb{N}_0^{3h-3+n}} w_k(\alpha, p) q_1^{k_1} \dots q_{3h-3+n}^{k_{3h-3+n}}$$

for $q = (q_1, \dots, q_{3h-3+n}) \in \mathbb{D}^{3h-3+n}$ Marden-Kra plumbing coordinates; here $w_k(\alpha, p)$ are representation theoretic constants depending only on Virasoro commutation relations.

Perspectives:

- Conformal bootstrap for Liouville CFT on open surfaces (with Baojun Wu).
  Based on recent developments to compute the boundary structure constants (Nina's talk, works by Ang, Holden, Rémy, Sun, Zhu)
- General probabilistic construction of the conformal blocks ? (Ghosal, Rémy, Sun, Sun)
- Representation of Mapping class group in the space of conformal blocks, modular functor, link with Quantum Teichmuller (Teschner,...)
- Use conformal welding (Nina's talk) to bridge Liouville CFT with CFT of CLE ? (Ang, Holden, Rémy, Sun, Wu...)
- (Long run) Develop these techniques for CFTs with extended symmetry algebra, e.g. Toda CFT (Cerclé, Huang)
- Other approaches to solve CFTs: Y.Z. Huang and collaborators on the way to provide the VOA solution for rational CFTs.

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