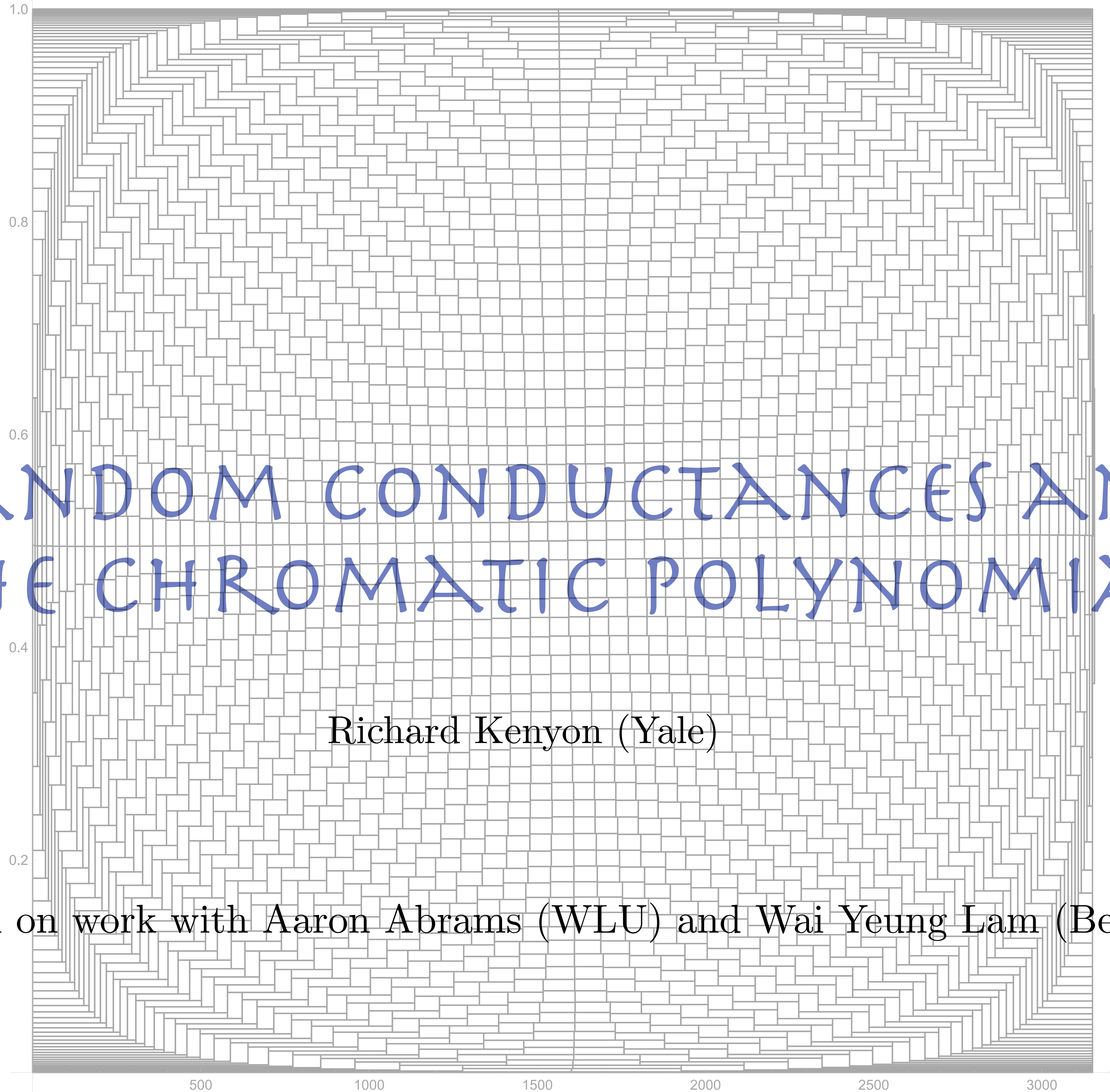

RANDOM CONDUCTANCES AND THE CHROMATIC POLYNOMIAL

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based on work with Aaron Abrams (WLU) and Wai Yeung Lam (Beijing)

1. Dirichlet problem
2. Chromatic polynomial
3. Compatible orientations
4. Statement
5. Enharmonic functions
6. Rectangle tilings

Stat Mech:

$$Z : \{\text{Graphs}\} \to \mathbb{R} \qquad \text{“partition function”}$$

e.g.

$$Z_{\text{tree}}(G) = \#\text{ spanning trees}$$

$$Z_{\text{Ising}}(G) = \sum_{\sigma \in \{\pm 1\}^V} \exp\left(\beta \sum_{u \sim v} J \sigma(u) \sigma(v)\right)$$

These functions are most interesting when we put *arbitrary* edge weights/interactions

$$Z_{\text{tree}}(G) = \sum_{\text{sp. trees } T} \prod_{e \in T} c_e$$

$$Z_{\text{Ising}}(G) = \sum_{\sigma \in \{\pm 1\}^V} \exp\left(\beta \sum_{u \sim v} J_{uv} \sigma(u) \sigma(v)\right)$$

(Discrete) Dirichlet problem

$G = (V, E)$

$V = V_\partial \cup V_{\text{int}}$

$c : E \to \mathbb{R}_{>0}$ edge conductances

$u : V_\partial \to \mathbb{R}$ fixed boundary values

The Dirichlet problem is to find a harmonic extension of u:

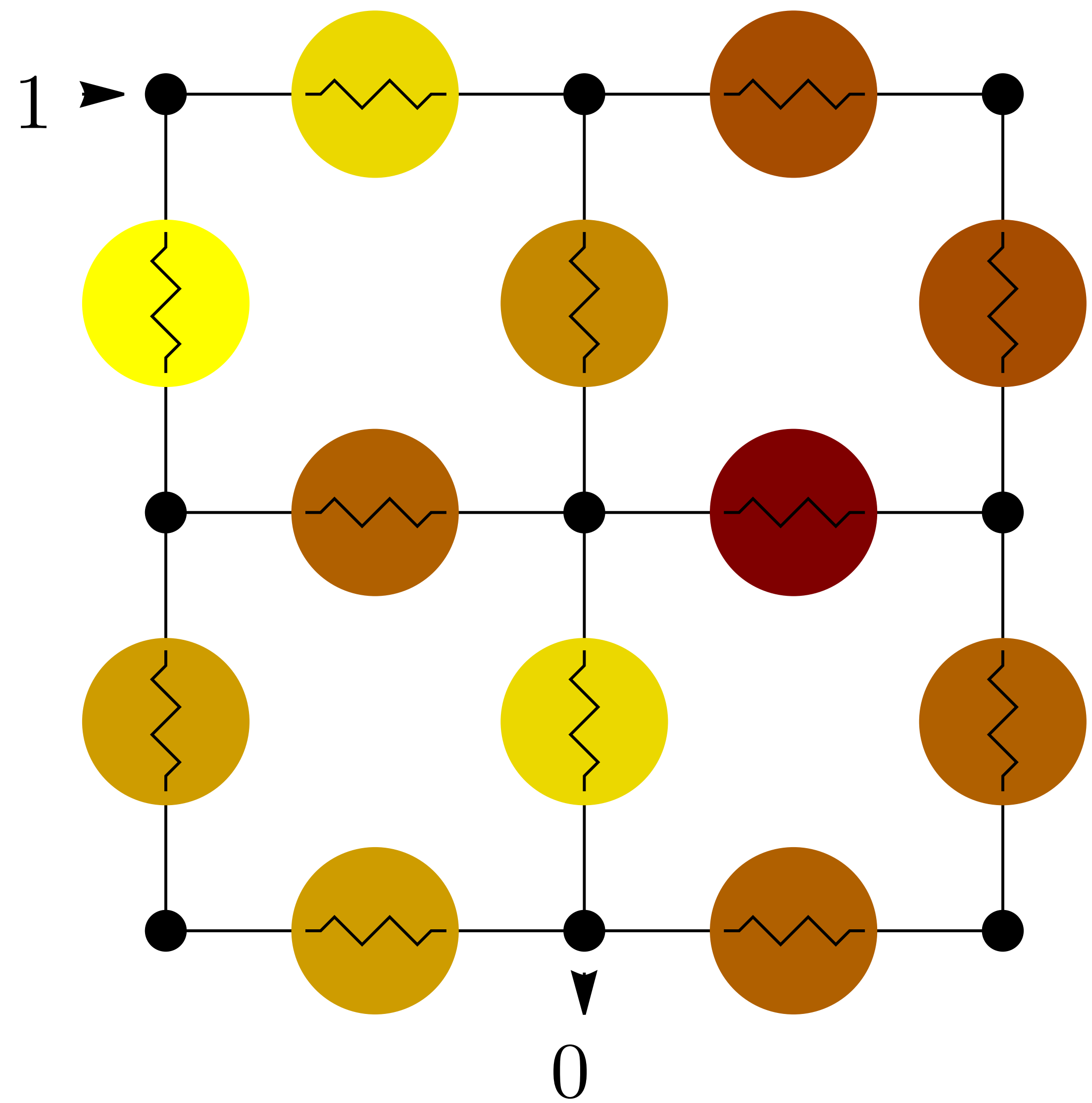
Find $f : V \to \mathbb{R}$ such that

$$f|_{V_\partial} = u$$
$$\Delta f|_{V_{int}} = 0$$

To a harmonic extension is associated an energy

$$\mathcal{E} : E \to \mathbb{R}_{\geq 0}$$

$$\mathcal{E}_{uv} = c_{uv}(f(u) - f(v))^2$$



This talk is about the map (for fixed u)

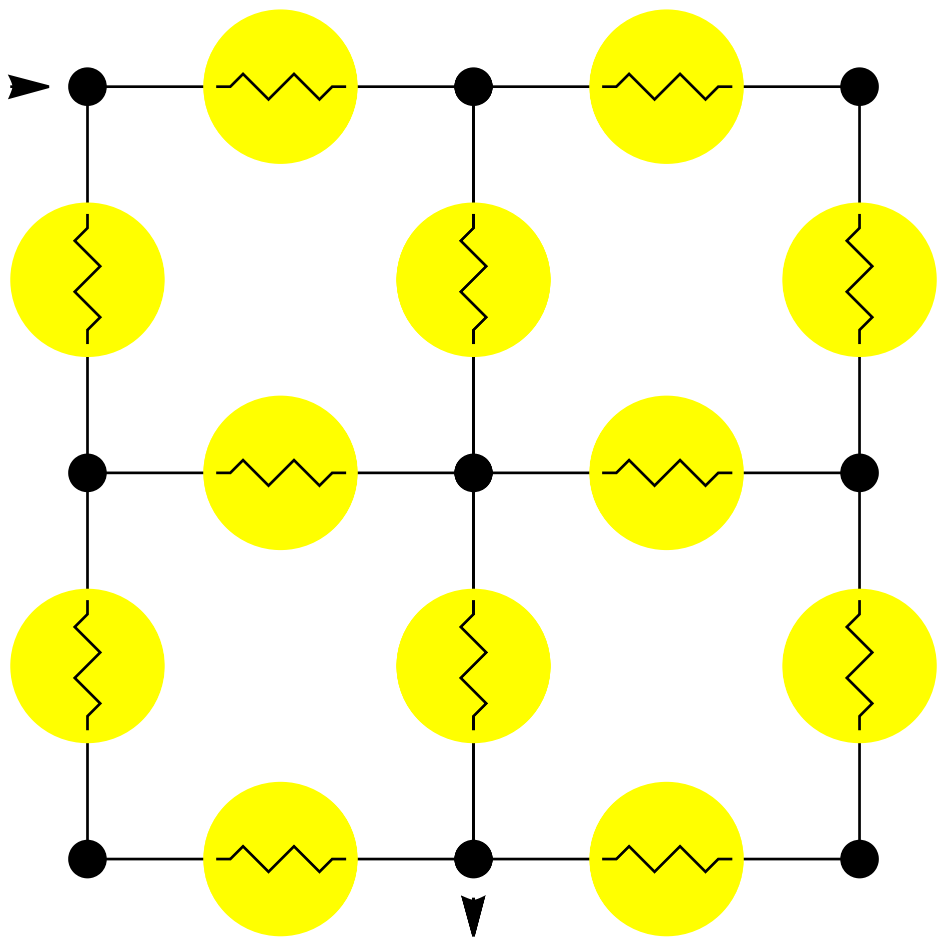
$$\Psi : \{\text{conductances}\} \to \{\text{edge energies}\}$$

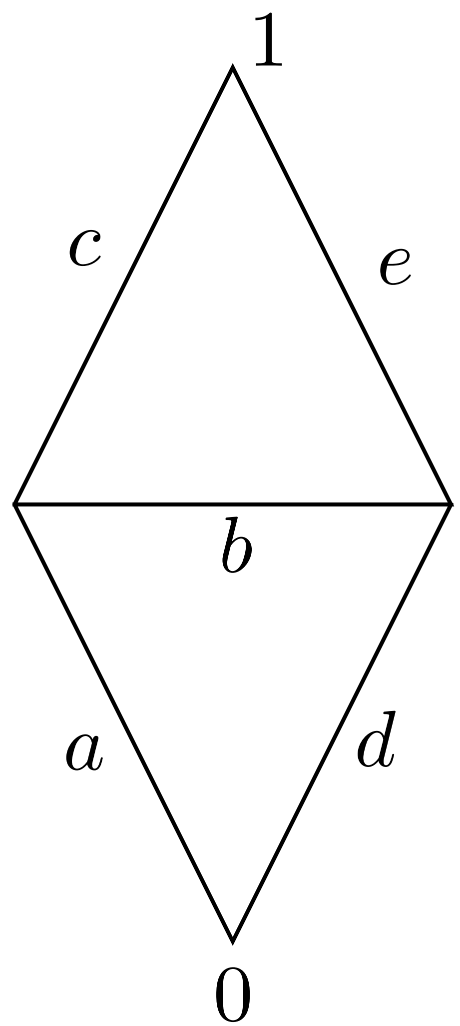
$$\{c_e\} \to \{\mathcal{E}_e\}$$

Q. Is Ψ surjective? Injective?

yes

no, but fiber
has constant size





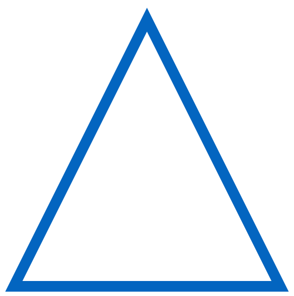
$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \xrightarrow{\Psi} \begin{pmatrix} \frac{a(bc+be+cd+ce)^2}{(ab+ad+ae+bc+bd+be+cd+ce)^2} \\ \frac{b(cd-ae)^2}{(ab+ad+ae+bc+bd+be+cd+ce)^2} \\ \frac{c(ab+ad+ae+bd)^2}{(ab+ad+ae+bc+bd+be+cd+ce)^2} \\ \frac{d(ae+bc+be+ce)^2}{(ab+ad+ae+bc+bd+be+cd+ce)^2} \\ \frac{e(ab+ad+bd+cd)^2}{(ab+ad+ae+bc+bd+be+cd+ce)^2} \end{pmatrix}$$

conductances energies

Chromatic polynomial

Def: For a graph G, $\chi_G(k)$ is the number of proper colorings of G with k colors.

adjacent vertices have distinct colors

Ex. 

$$\chi(k) = k(k-1)(k-2)$$

Prop: χ_G is a polynomial in k.

Prop: $|\chi_G(-1)|$ is the number of *acyclic orientations* of G.

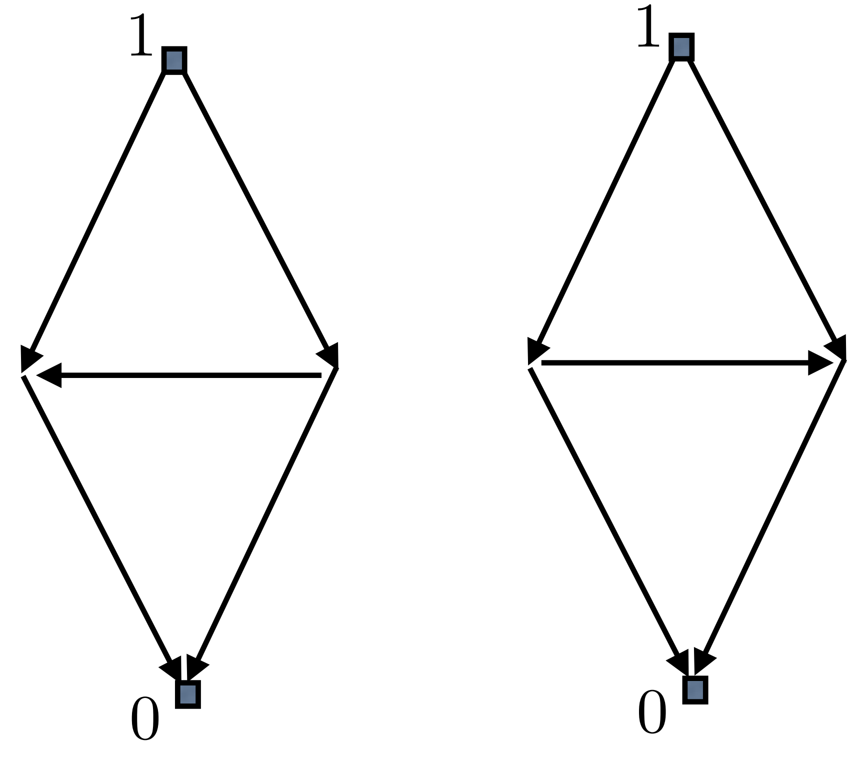
Thm:(Stanley) For $k < 0$, $|\chi_G(k)|$ counts certain kinds of acyclic orientations of G.

Compatible orientations

When we solve the Dirichlet problem on G_k, there is a current flow on edges which is (generically) nonzero. It induces an orientation of edges of G.

Def: A *compatible orientation* is an acyclic orientation for which:

1. No interior sources or sinks
2. no oriented path from lower boundary value to an equal or higher boundary value

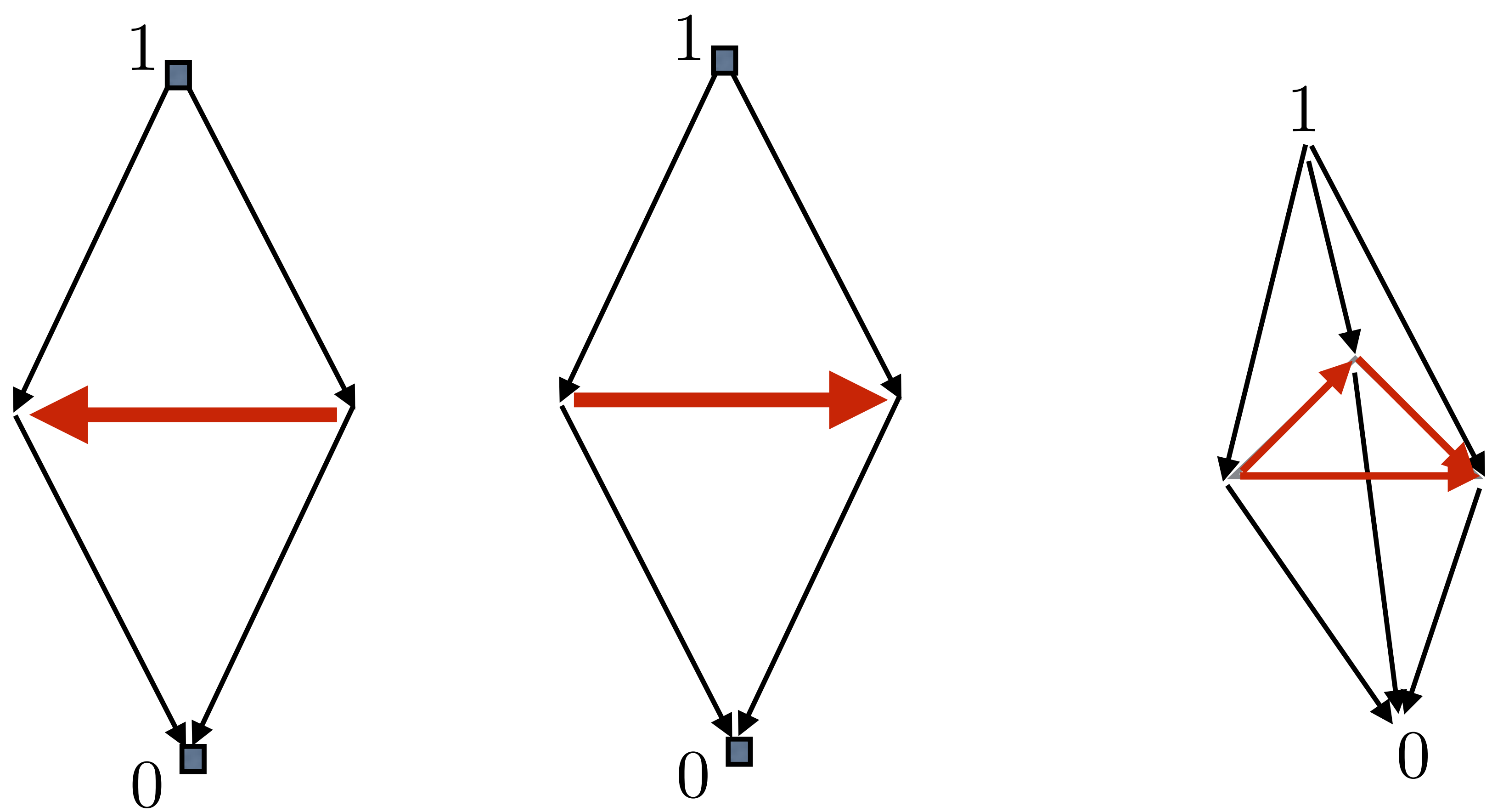


Lemma: Compatible orientations are exactly the orientations arising from the Dirichlet problem.

Thm: Let G_k be obtained from G by adding $k+1$ additional vertices $v_0, \ldots, v_k$, attached to all vertices of G. Fix distinct boundary values u_i on v_i. Then

$$|\chi_G(-k)| = \#\{\text{compatible orientations of } G_k\}.$$

Cor: $|\chi_G(-1)| = \#\{\text{acyclic orientations of } G\}$



Let G_k be obtained from G by adding $k+1$ additional vertices $v_0, \ldots, v_k$, attached to all vertices of G. Fix distinct boundary values u_i on v_i.

Thm: Ψ is surjective and of degree $|\chi_G(-k)|$.

In other words, for each choice of $\{\mathcal{E}_e > 0\}$ there are exactly $|\chi_G(-k)|$ choices of conductances giving those energies.

Thm: For each compatible orientation and tuple $\mathcal{E} \in (0, \infty)^E$ there is a unique choice of positive conductances so that the harmonic extension has those energies and orientations.

What if conductances are random?

Thm: Let G_k be obtained from G by adding $k+1$ additional vertices $v_0, \ldots, v_k$, attached to all vertices of G. Fix distinct boundary values u_i on v_i. Then

$$|\chi_G(-k)| = \mathbb{E}\left[\prod_e \frac{(f(u)-f(v))^2}{Z}\right]$$

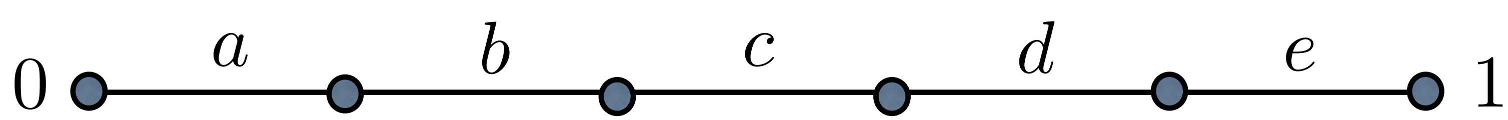
where the expectation is over the simplex of random conductances summing to 1, f is the harmonic extension, and Z is the total Dirichlet energy.

Proof: The degree of Ψ can be computed by integrating the Jacobian determinant over the simplex. □

Lemma: $$\det(D\Psi) = \prod_e \frac{\mathcal{E}_e}{c_e}.$$

Is this theorem useful?

Application 1.



$$1 = n! \int_{\Delta^n} \frac{(e_n)^{n-2}}{(e_{n-1})^n} dvol$$

$$1 = 5! \int_{\Delta_5} \frac{(abcde)^3}{(abcd + abce + abde + acde + bcde)^5} \, da \, db \, dc \, dd$$

Application 2.

How to exactly sample a uniform random acyclic orientation

1. Choose uniform random conductances on G_2.
2. Find harmonic extension h and its compatible orientation.
3. Keep this with probability $\varepsilon \prod_e \frac{(h(x)-h(y))^2}{Z}$.

Application 3???

Can we estimate the RHS on a large regular graph, eg. $n \times n$ grid?

Def: An *enharmonic function* is a solution to the Dirichlet problem which gives constant (or predetermined) energies on the edges. (conductances *not* fixed)

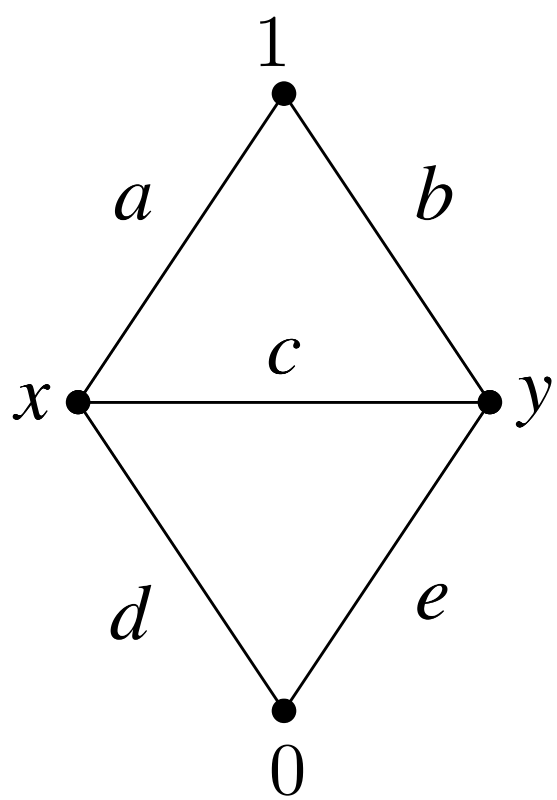
Thm: Enharmonic functions with energies $\mathcal{E} \in (0,\infty)^E$ are precisely the local maxima of the functional

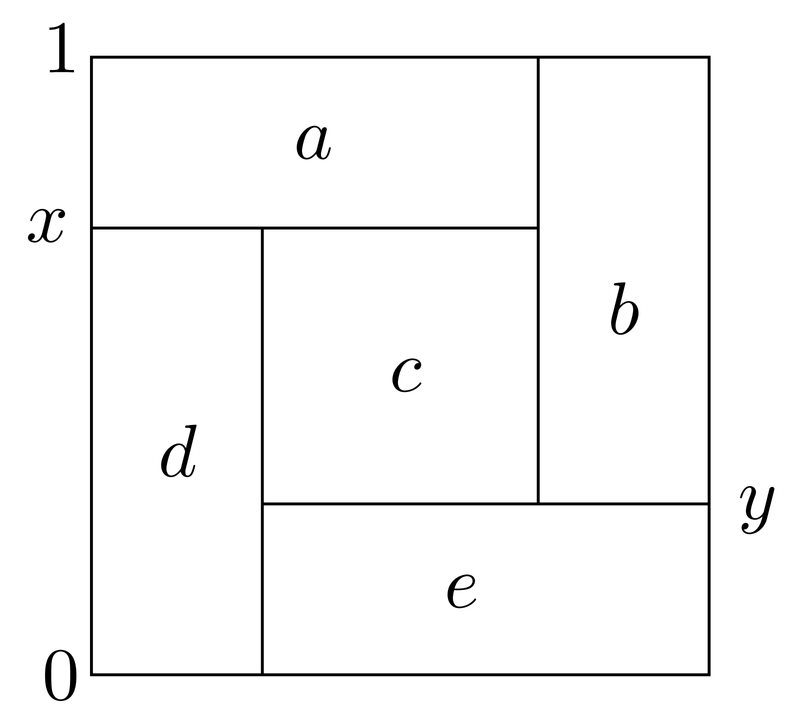
$$M(h) = \prod_{e=xy\in E} |h(x) - h(y)|^{\mathcal{E}_e}.$$

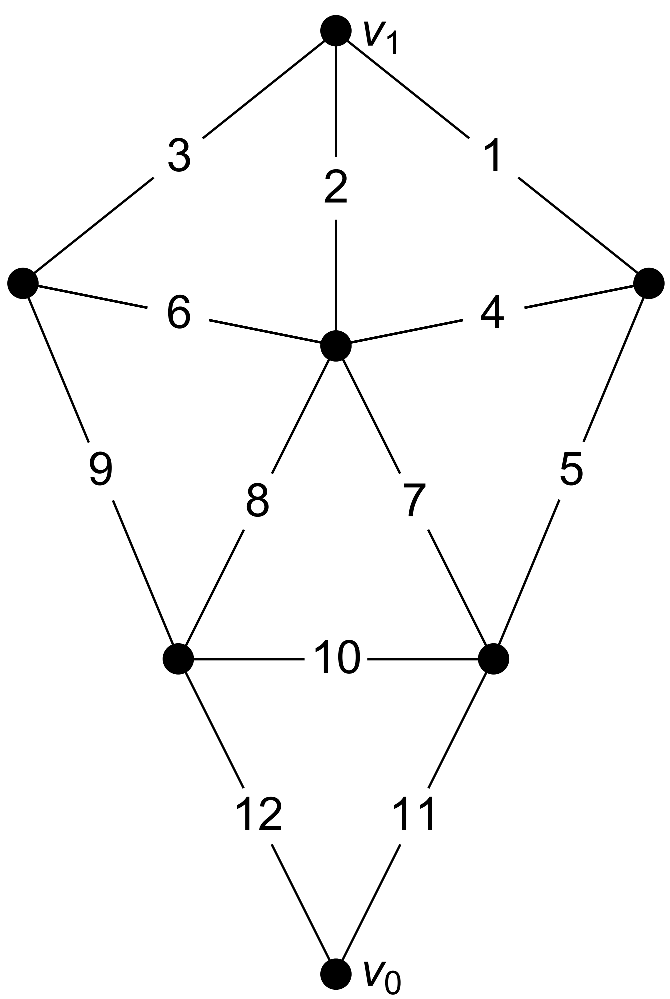
Lemma: $\log M(h)$ is strictly convex on each cell corresponding to a compatible orientation, with boundary values $-\infty$.

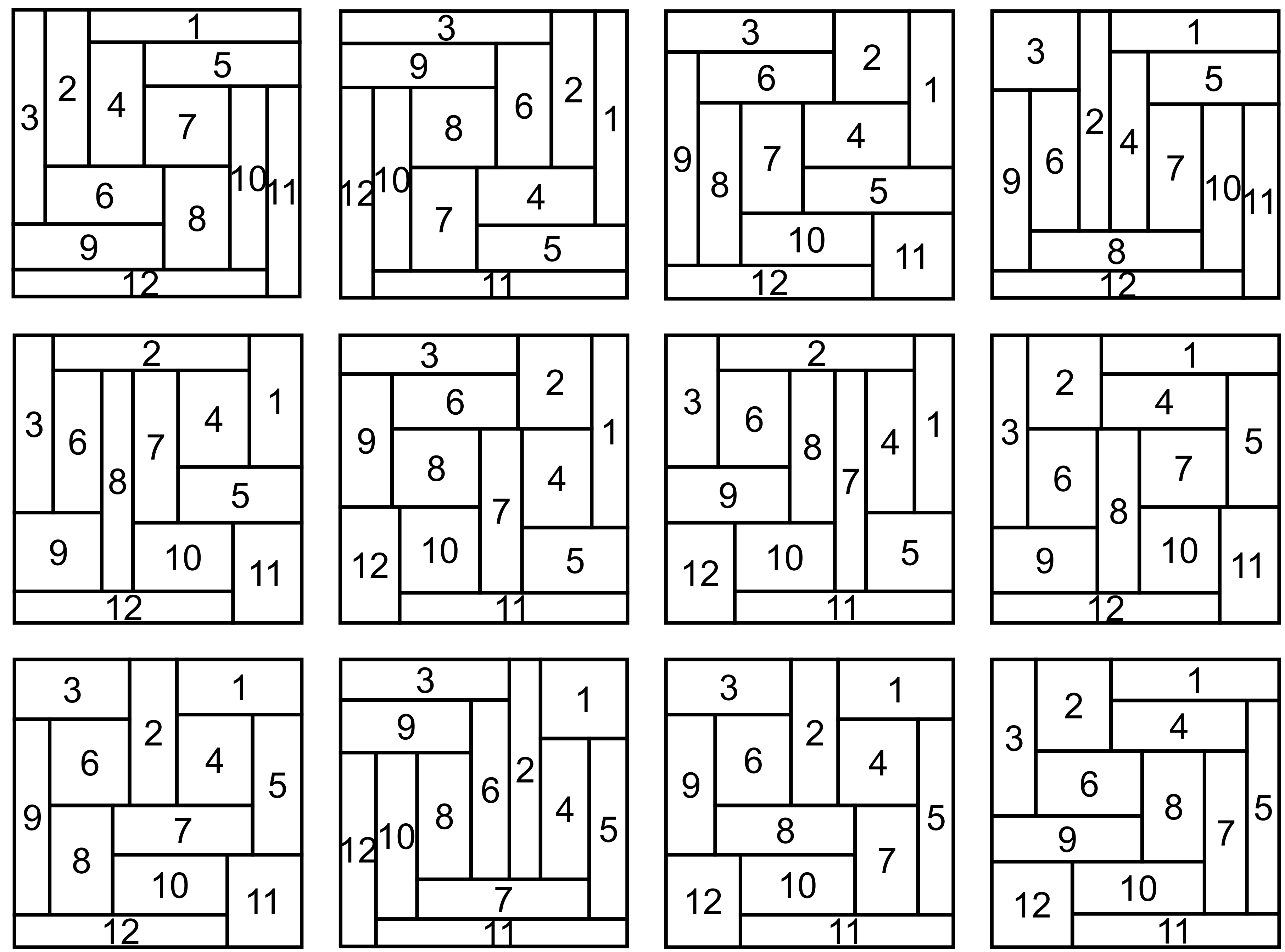
Fixed-area rectangle tilings

[Brooks/Smith/Stone/Tutte 1939]
For a planar graph, with two boundary nodes on the outer face, a solution to the Dirchlet problem gives a rectangle tiling of a rectangle.



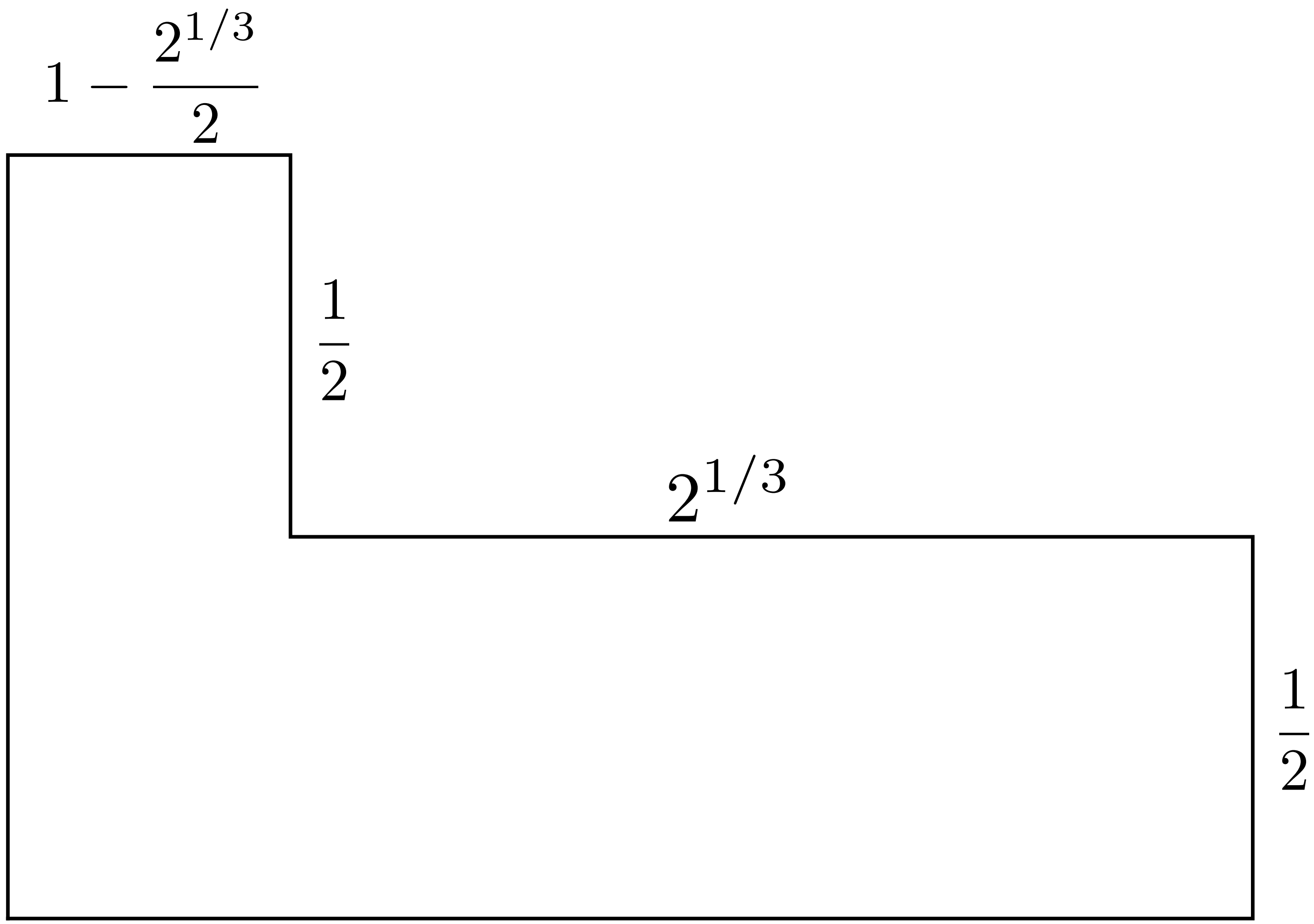






This graph has 12 compatible orientations, and so 12 enharmonic functions:

Thm: A polygon can be tiled with rational-area rectangles only if the set of vertical edge lengths is in a totally-real extension field of the horizontal edge lengths.



Not tileable with rational-area rectangles:

THANK YOU