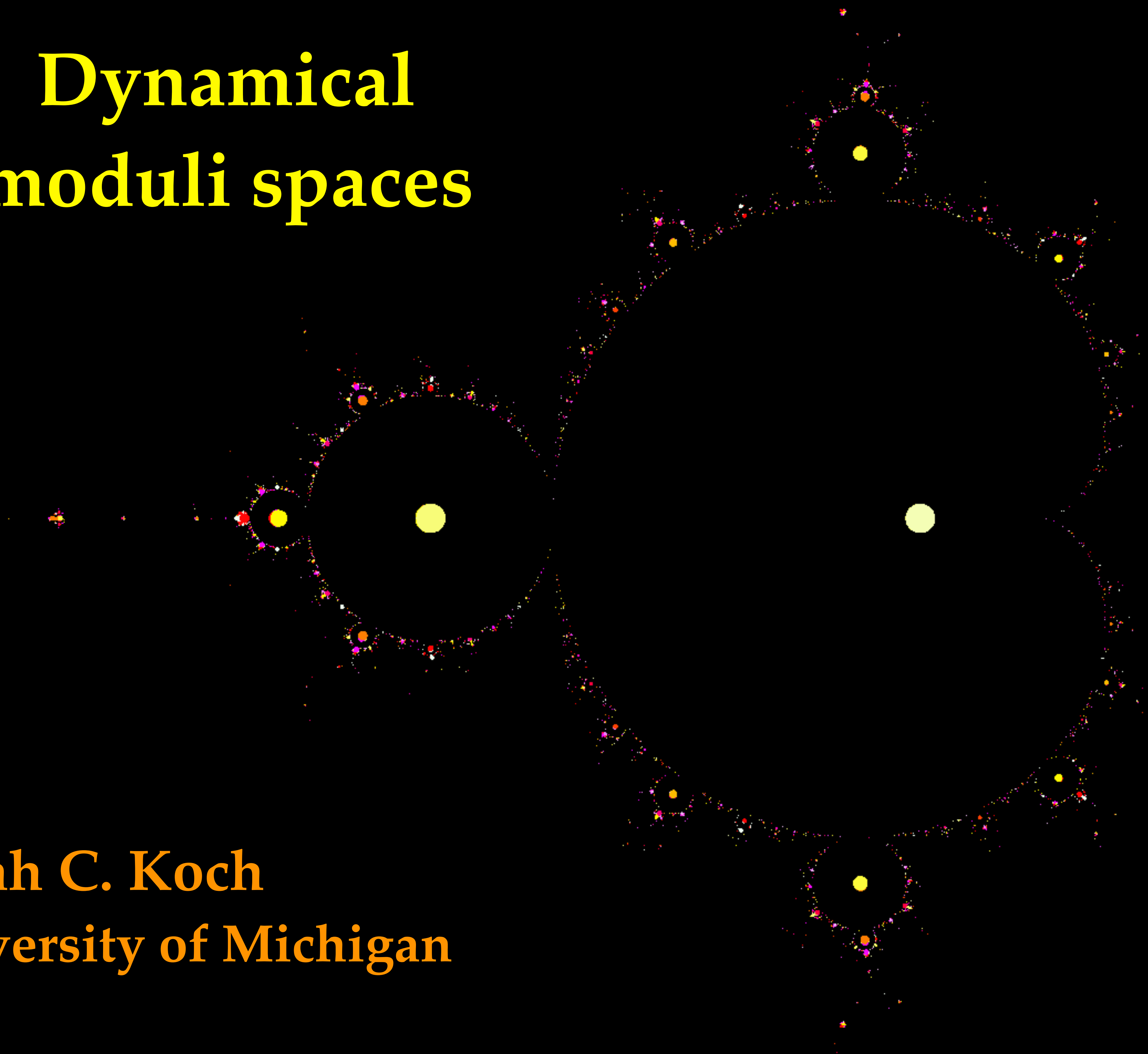
Dynamical moduli spaces



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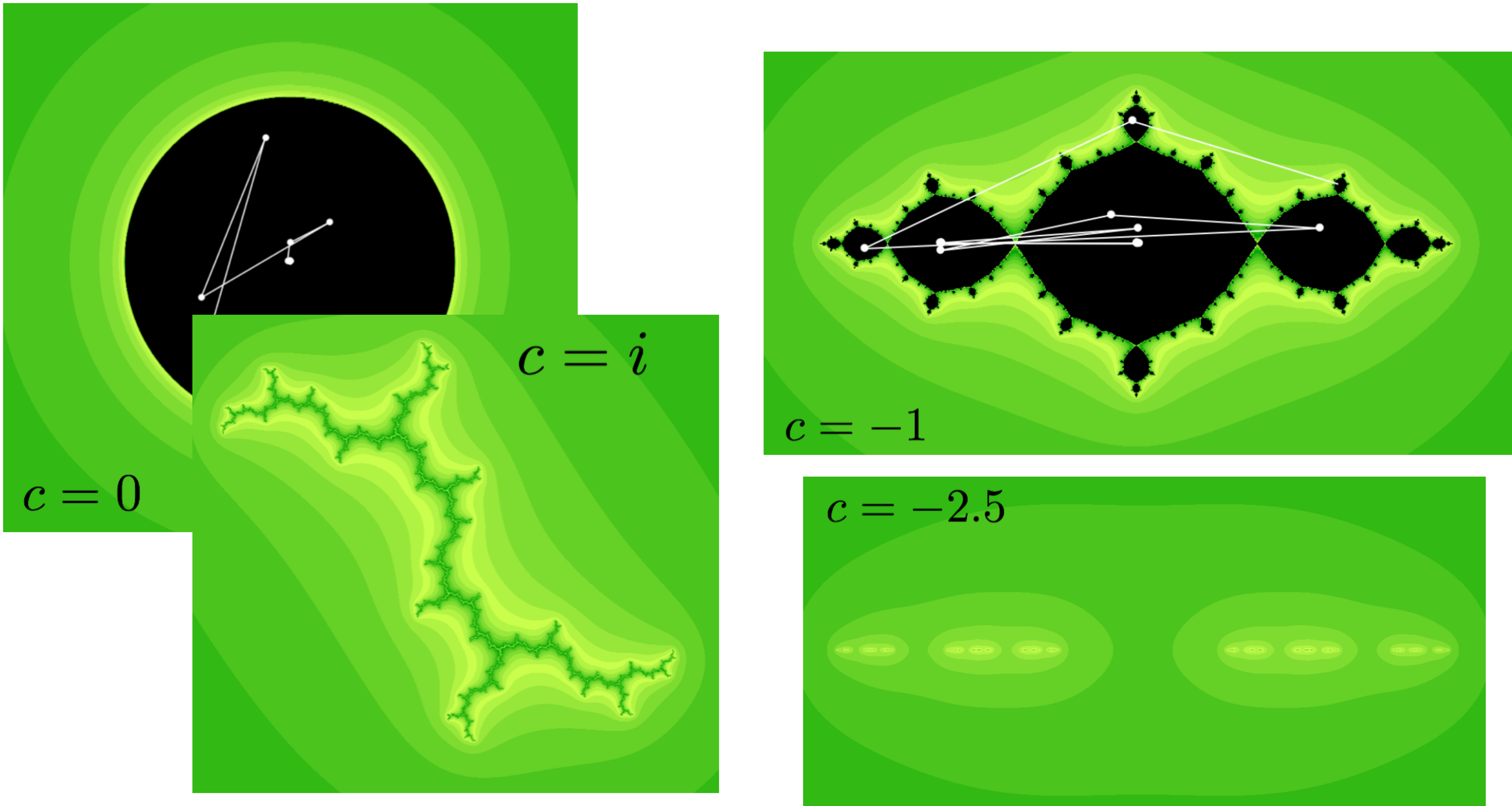
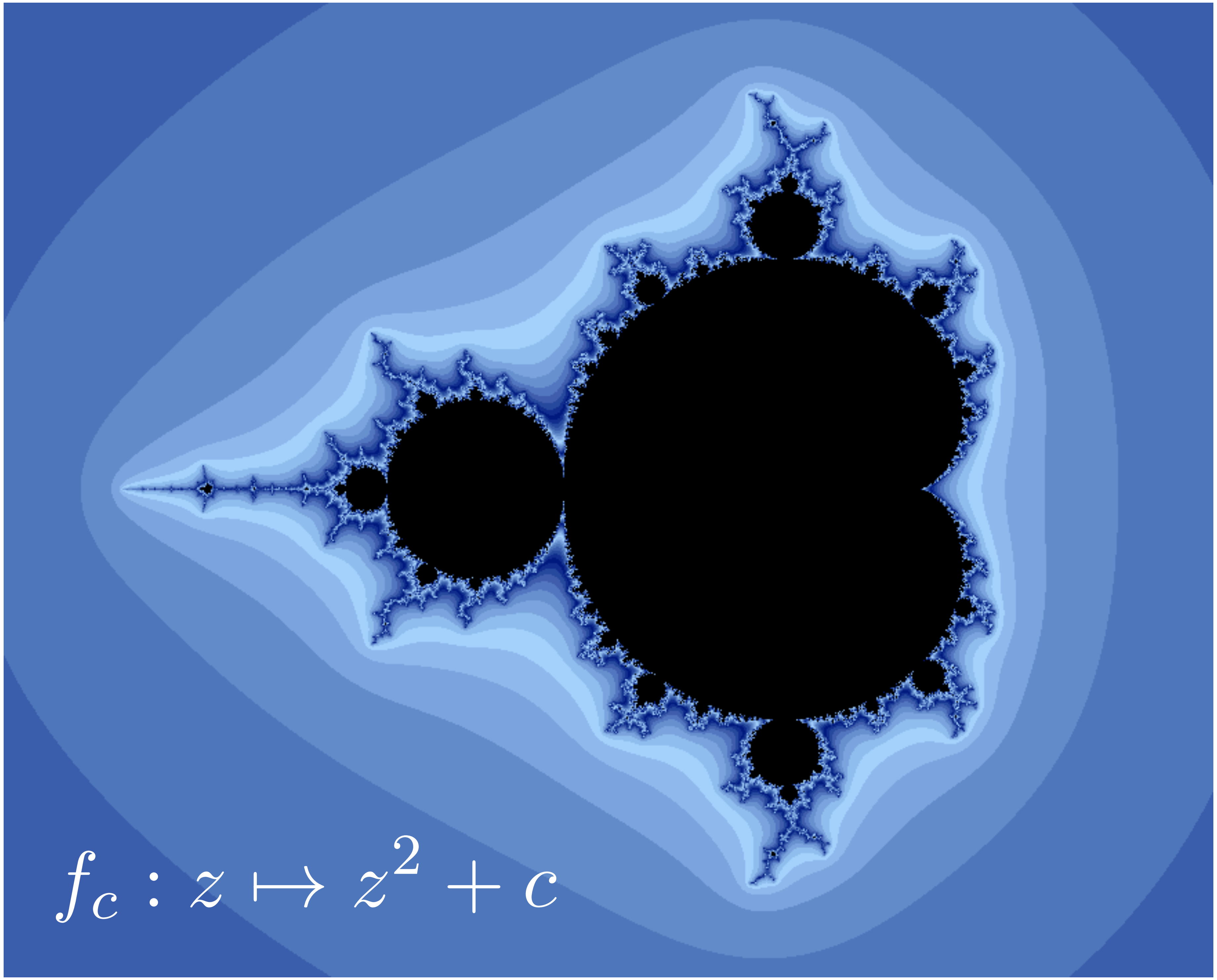
Quadratic polynomials

$f_c : \mathbb{C} \to \mathbb{C}$ $\qquad f_c : z \mapsto z^2 + c$

Filled Julia Set $\quad K_c := \{z \in \mathbb{C} \mid n \mapsto (f_c)^n(z) \text{ is bounded}\}$

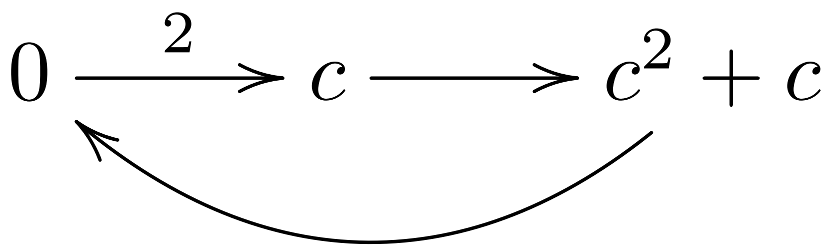


The moduli space of quadratic polynomials



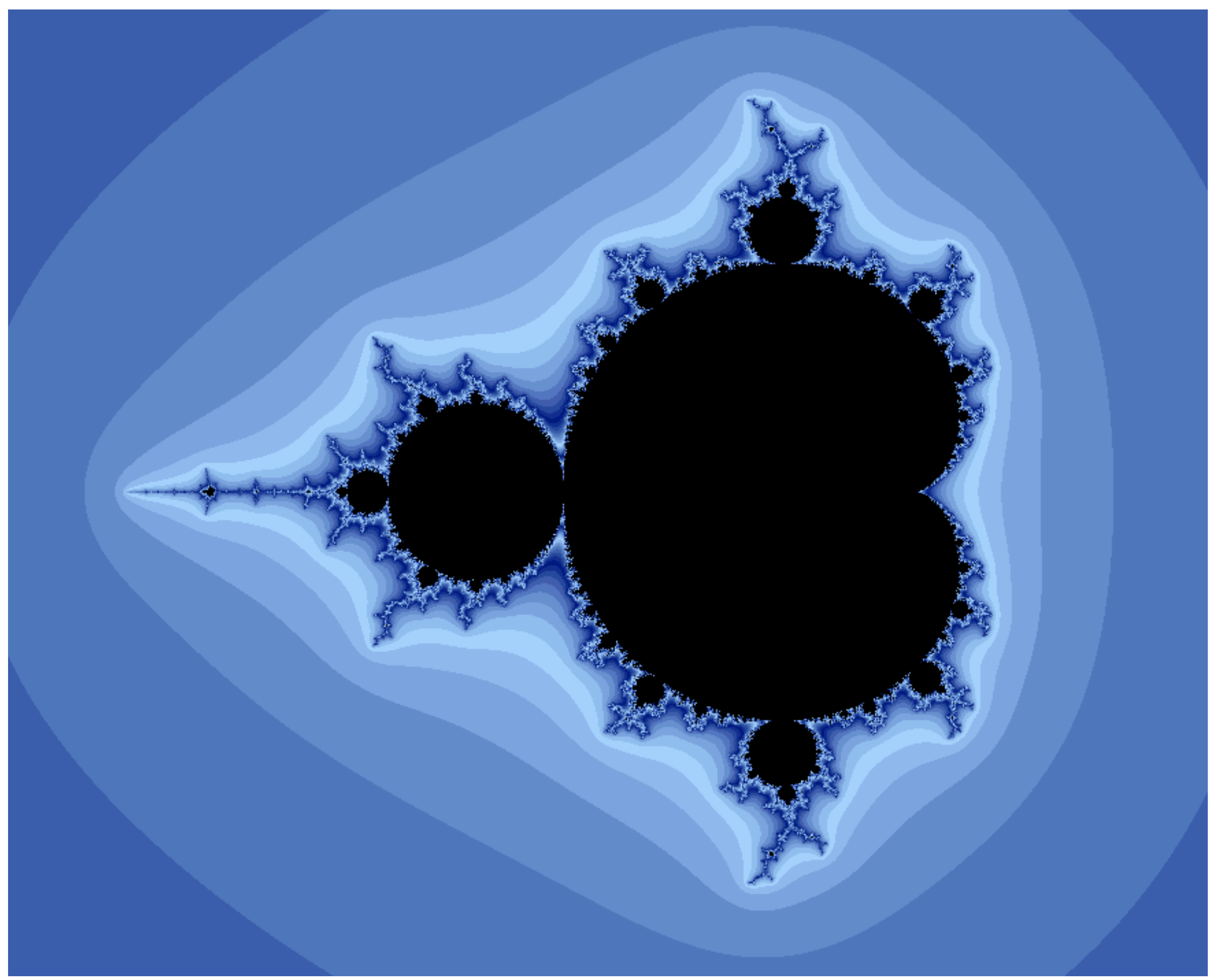
We obtain interesting subsets by constraining the critical orbit

Exercise: Find all c so that 0 is periodic of period 3 for f_c.



$$f_c : z \mapsto z^2 + c$$

$$c(c^3 + 2c^2 + c + 1) = 0$$

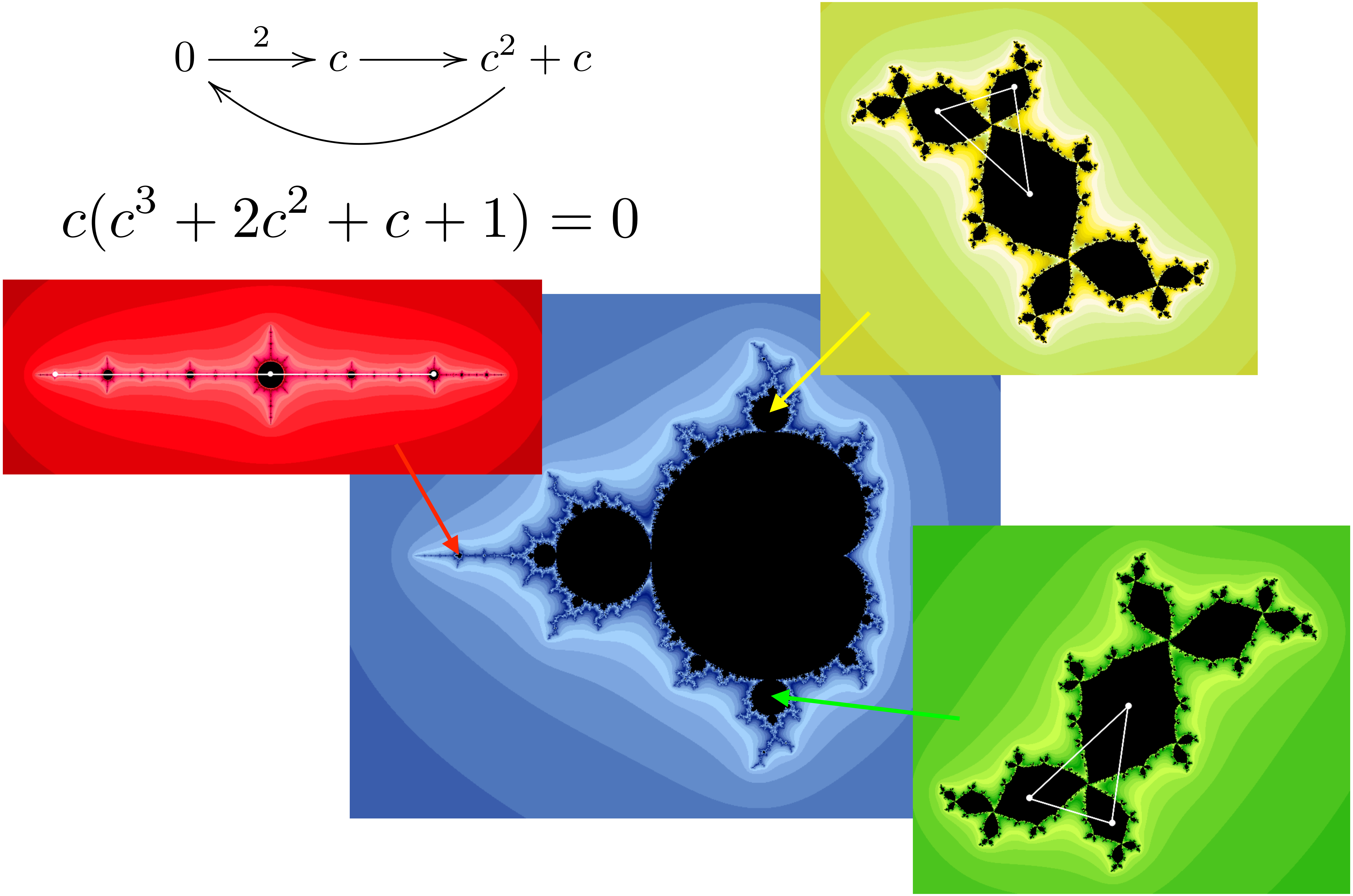


Exercise:

Find all c so that 0 is periodic of period 3 for f_c.

$$0 \xrightarrow{2} c \longrightarrow c^2 + c \longrightarrow 0$$

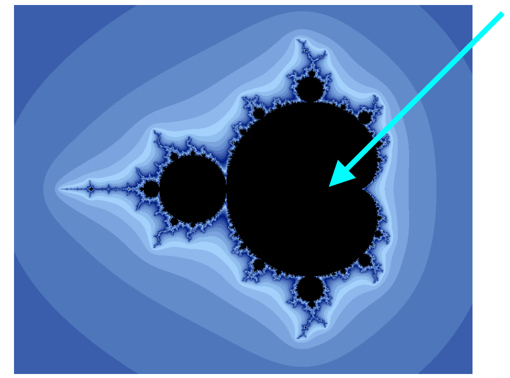
$$c(c^3 + 2c^2 + c + 1) = 0$$



Subvarieties in moduli space

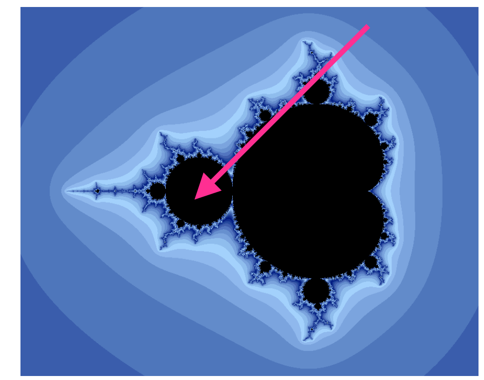
$$0 \circlearrowright 2$$

$$G_1 = c$$



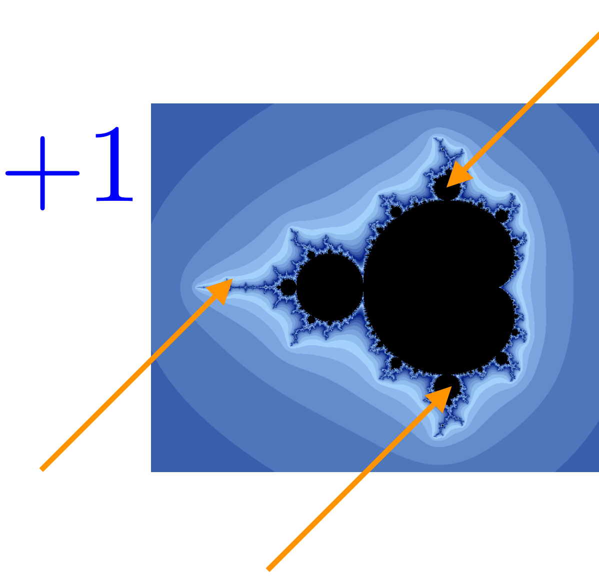
$$0 \xrightarrow{2} c \;(\text{and } c \to 0)$$

$$G_2 = c+1$$



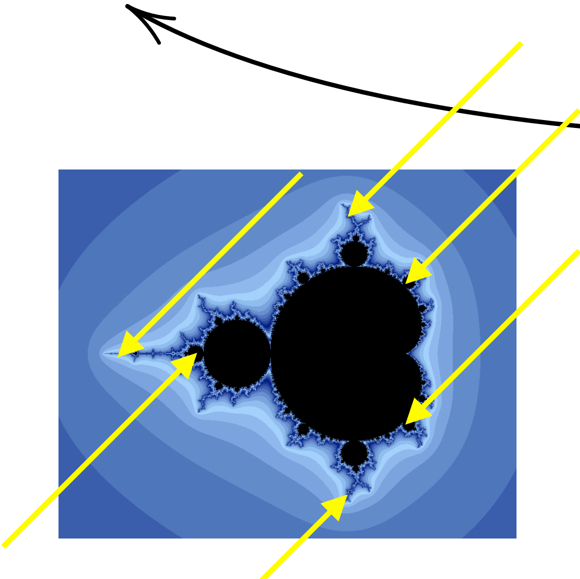
$$0 \xrightarrow{2} c \longrightarrow c^2 + c \;(\text{and } c^2+c \to 0)$$

$$G_3 = c^3+2c^2+c+1$$



$$0 \xrightarrow{2} c \longrightarrow c^2 + c \longrightarrow (c^2+c)^2 + c \;(\text{and } (c^2+c)^2+c \to 0)$$

$$G_4 = c^6+3c^5+3c^4+3c^3+2c^2+1$$



For which *n* is G_n irreducible?

Gleason polynomials

$$G_1 = c$$

$$G_2 = c+1$$

$$G_3 = c^3+2c^2+c+1$$

$$G_4 = c^6+3c^5+3c^4+3c^3+2c^2+1$$

$$G_5 = c^{15}+8c^{14}+28c^{13}+60c^{12}+94c^{11}+116c^{10}+114c^9 +94c^8+69c^7+44c^6+26c^5+14c^4+5c^3+2c^2+c+1$$

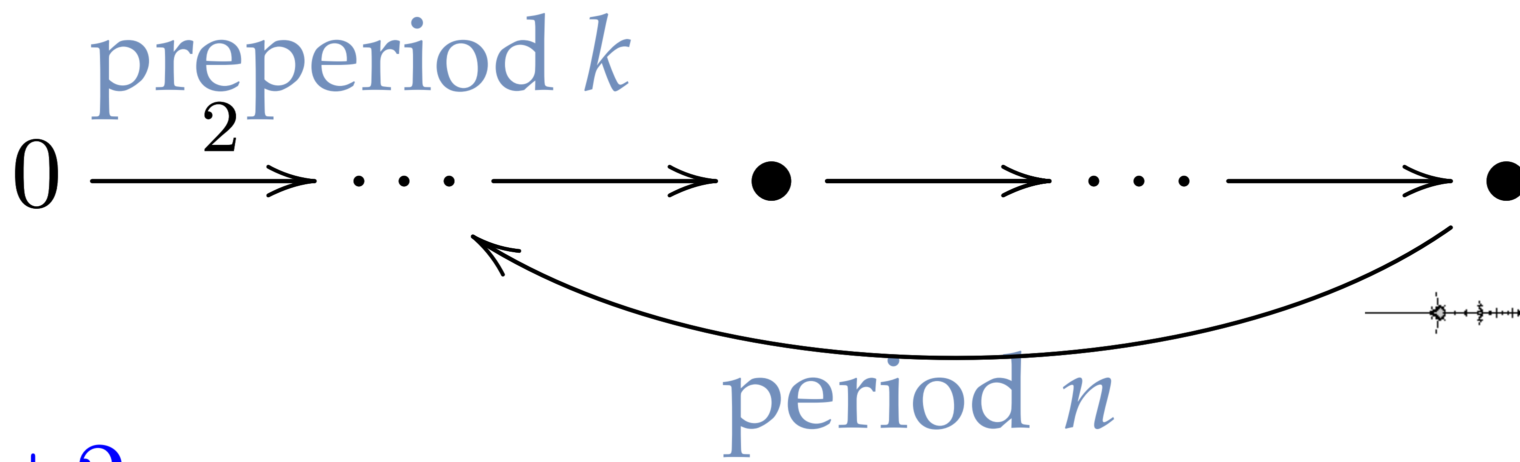
⋮

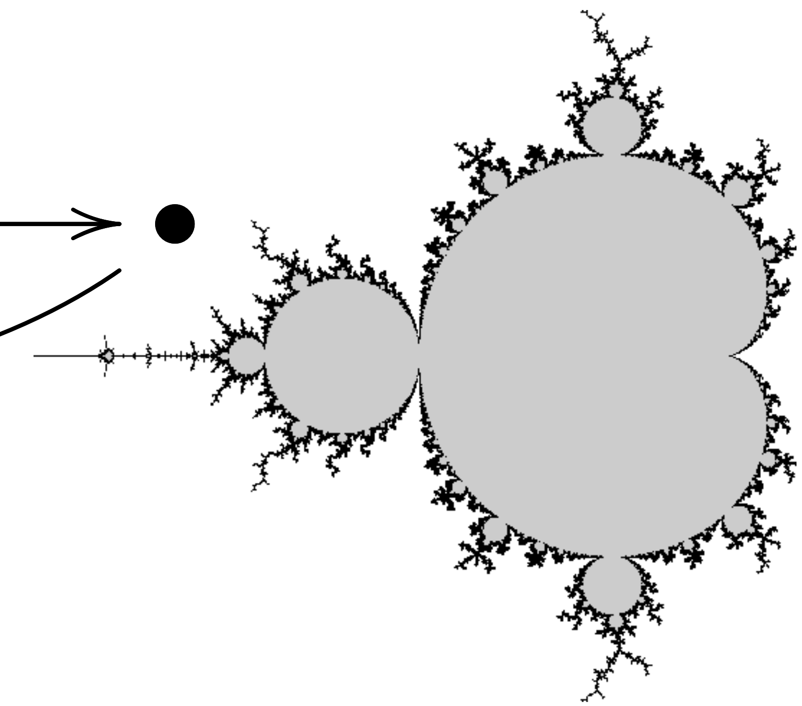
Count the number of hyperbolic components of period *n*

Theorem (Buff, Floyd, K, Parry)

The number of real roots **=** **The number of irreducible factors mod 2**

Subvarieties in moduli space



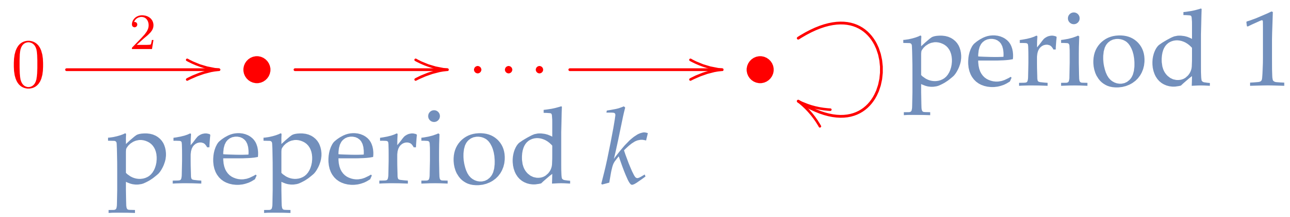


$$M_{2,1} = c+2$$

$$M_{2,2} = c^2+1$$

For which (k,n) is $M_{k,n}$ irreducible?

$$\begin{aligned} M_{4,3} = {} & c^{48} + 24\,c^{47} + 276\,c^{46} + 2036\,c^{45} + 10890\,c^{44} + 45276\,c^{43} + 153142\,c^{42} \\ & + 435210\,c^{41} + 1064138\,c^{40} + 2279440\,c^{39} + 4338224\,c^{38} + 7419308\,c^{37} \\ & + 11508550\,c^{36} + 16318092\,c^{35} + 21290910\,c^{34} + 25708386\,c^{33} + 28869914\,c^{32} \\ & + 30278528\,c^{31} + 29764224\,c^{30} + 27506264\,c^{29} + 23958072\,c^{28} + 19711336\,c^{27} \\ & + 15349808\,c^{26} + 11336496\,c^{25} + 7957154\,c^{24} + 5320536\,c^{23} + 3398212\,c^{22} \\ & + 2079772\,c^{21} + 1224054\,c^{20} + 695364\,c^{19} + 382430\,c^{18} + 203902\,c^{17} \\ & + 105317\,c^{16} + 52552\,c^{15} + 25268\,c^{14} + 11700\,c^{13} + 5242\,c^{12} + 2308\,c^{11} \\ & + 1022\,c^{10} + 470\,c^{9} + 220\,c^{8} + 96\,c^{7} + 36\,c^{6} + 8\,c^{5} + 1 \end{aligned}$$



use Eisenstein, *p*=2

$$M_{2,1} = c + 2$$

$$M_{3,1} = c^3 + 2c^2 + 2c + 2$$

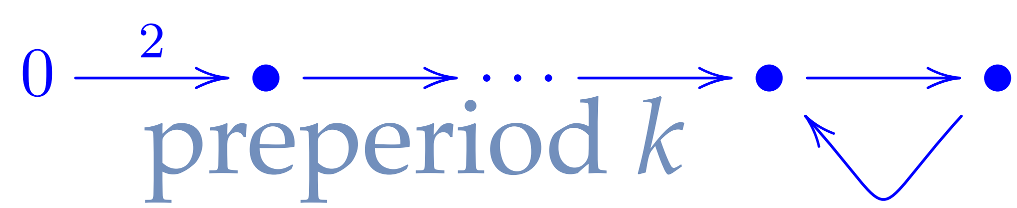
$$M_{4,1} = c^7 + 4c^6 + 6c^5 + 6c^4 + 6c^3 + 4c^2 + 2c + 2$$

$$M_{5,1} = c^{15} + 8c^{14} + 28c^{13} + 60c^{12} + 94c^{11} + 116c^{10} + 114c^9 + 94c^8 + 70c^7 + 48c^6 + 32c^5 + 20c^4 + 10c^3 + 4c^2 + 2c + 2$$

$$M_{6,1} = c^{31} + 16c^{30} + 120c^{29} + 568c^{28} + 1932c^{27} + 5096c^{26} + 10948c^{25} + 19788c^{24} + 30782c^{23} + 41944c^{22} + 50788c^{21} + 55308c^{20} + 54746c^{19} + 49700c^{18} + 41658c^{17} + 32398c^{16} + 23462c^{15} + 15872c^{14} + 10096c^{13} + 6096c^{12} + 3528c^{11} + 1976c^{10} + 1072c^9 + 564c^8 + 290c^7 + 144c^6 + 68c^5 + 28c^4 + 10c^3 + 4c^2 + 2c + 2$$

Polynomials in parameter space

Theorem. (Goksel; 2018) For all $k \geqslant 2$, the polynomials $M_{k,1}$ and $M_{k,2}$ are irreducible over $\mathbb{Z}$.



Theorem. (Buff, Epstein, K; 2018) For all $k \geqslant 2$, the polynomial $M_{k,3}$ is irreducible over $\mathbb{Z}$.

Quadratic rational maps

$$z \mapsto \frac{a_2 z^2 + a_1 z + a_0}{b_2 z^2 + b_1 z + b_0} \longrightarrow [a_0 : a_1 : a_2 : b_0 : b_1 : b_2] \in \mathbb{P}^5$$

$$\mathrm{Rat}_2 \approx \mathbb{P}^5 \setminus \Delta$$

mod out by conjugation by Möbius transformations

The quotient is the *moduli space of quadratic rational maps*; it is isomorphic to $\mathbb{C}^2$ (Milnor).

$$\mathcal{Q}$$

how do we understand this moduli space?

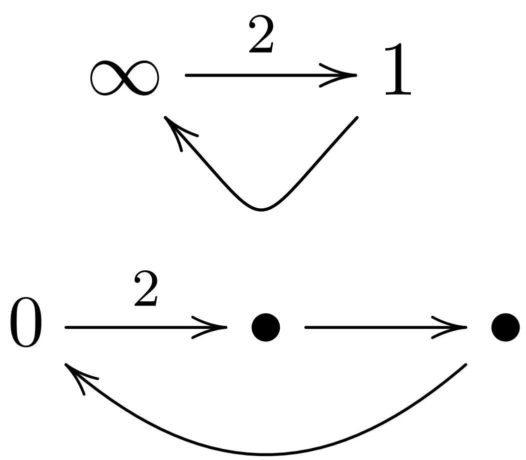
$$f_{a,b}(z) = \frac{z^2 + a}{z^2 + b}$$

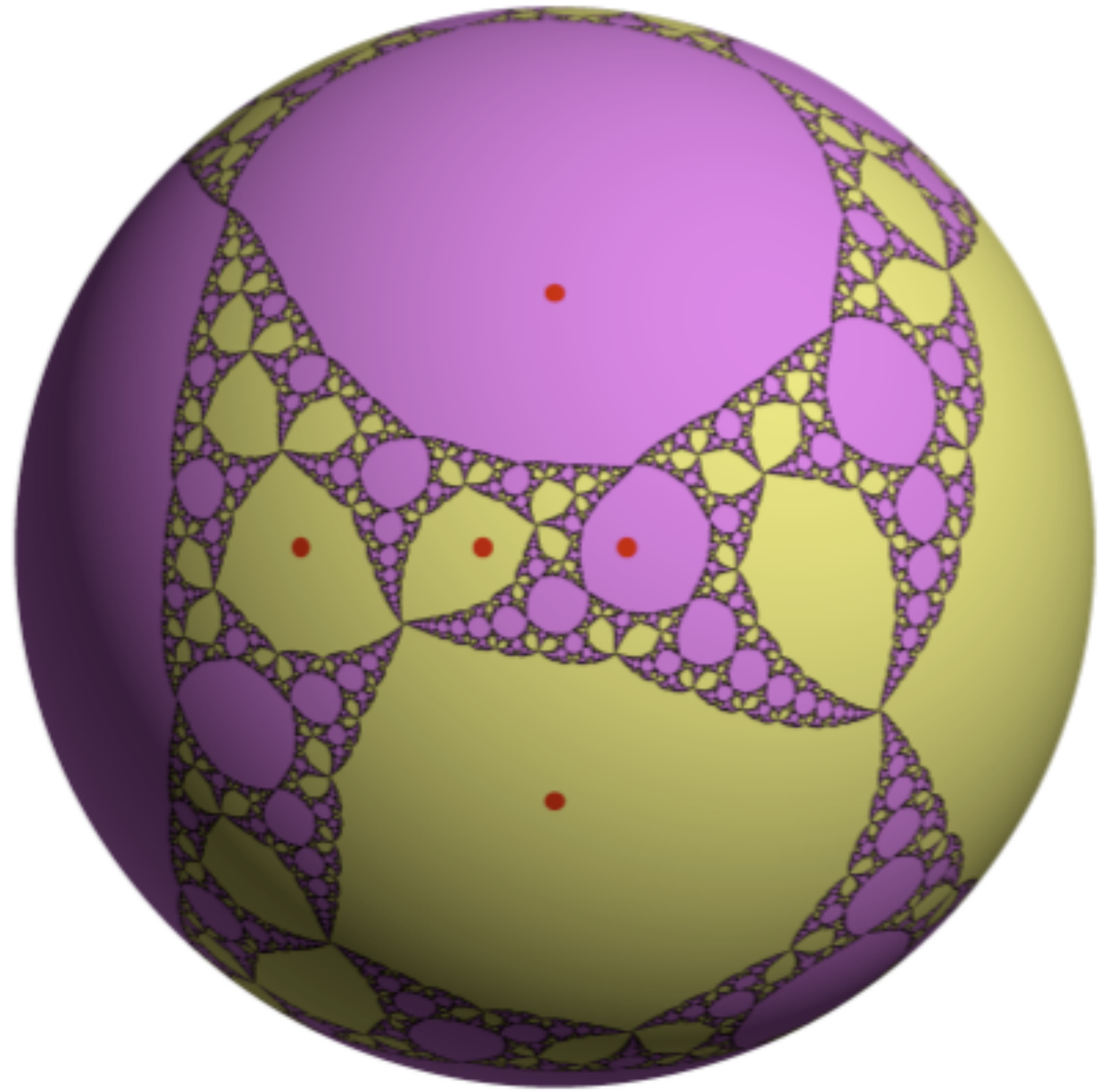
$$0 \xrightarrow{2} \cdots$$

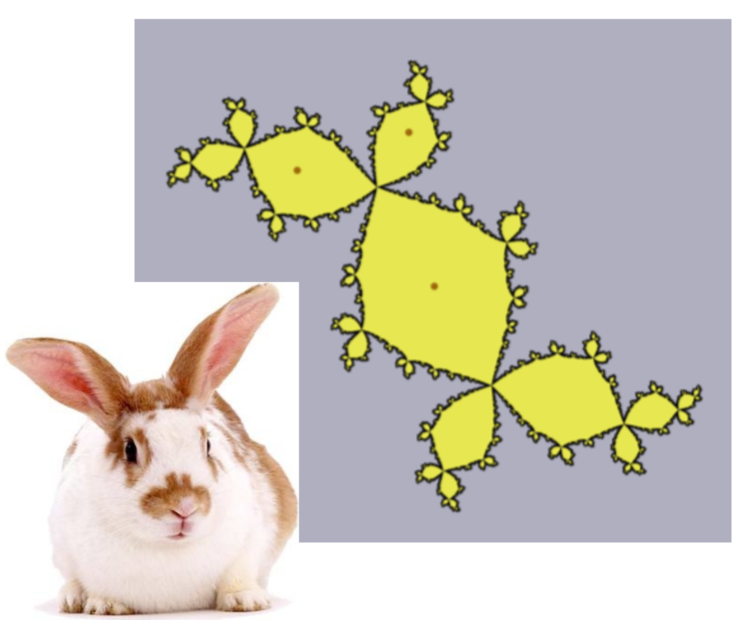
$$\infty \xrightarrow{2} 1 \longrightarrow \cdots$$

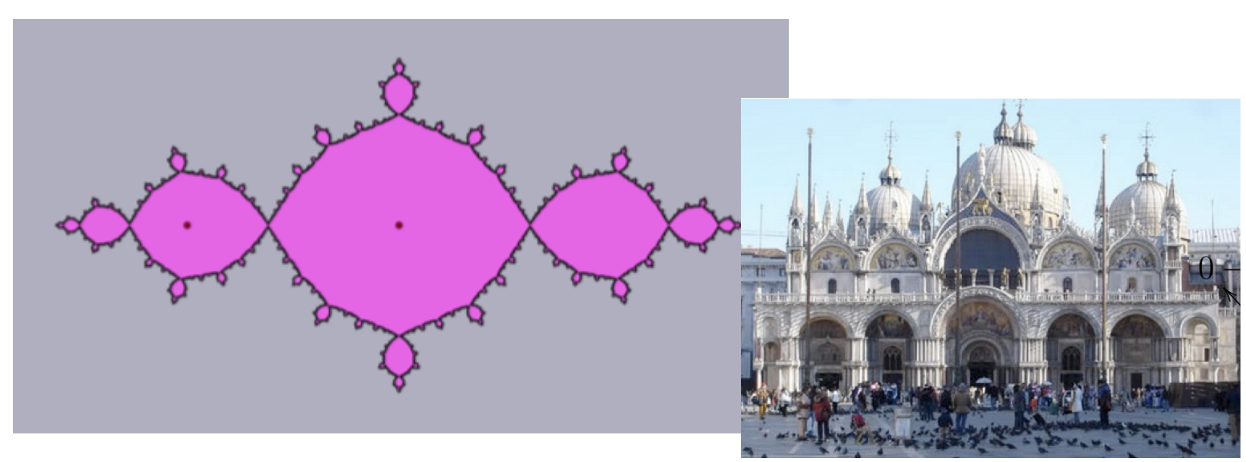
Example:

$$f(z) = \frac{z^2 - e^{-2\pi i/3}}{z^2 - 1}$$









Quadratic rational maps

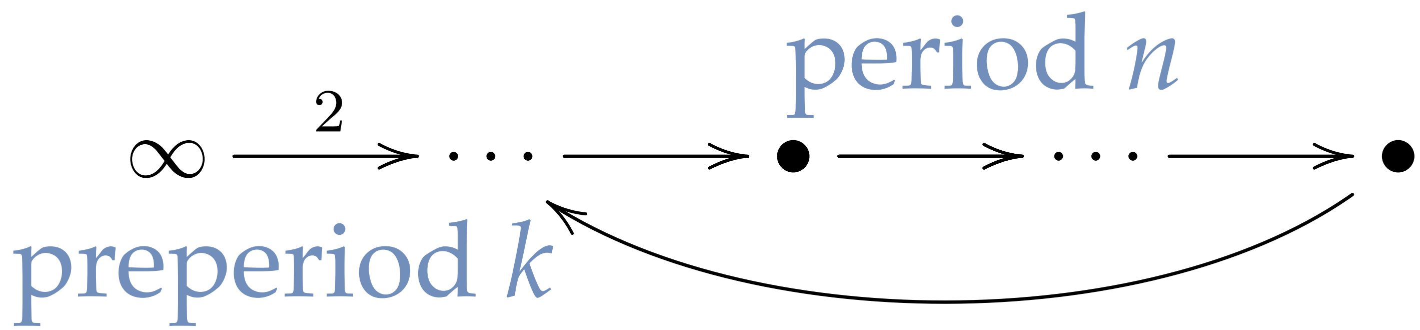
The moduli space $\mathcal{Q}$ is isomorphic to $\mathbb{C}^2$.

these curves are smooth

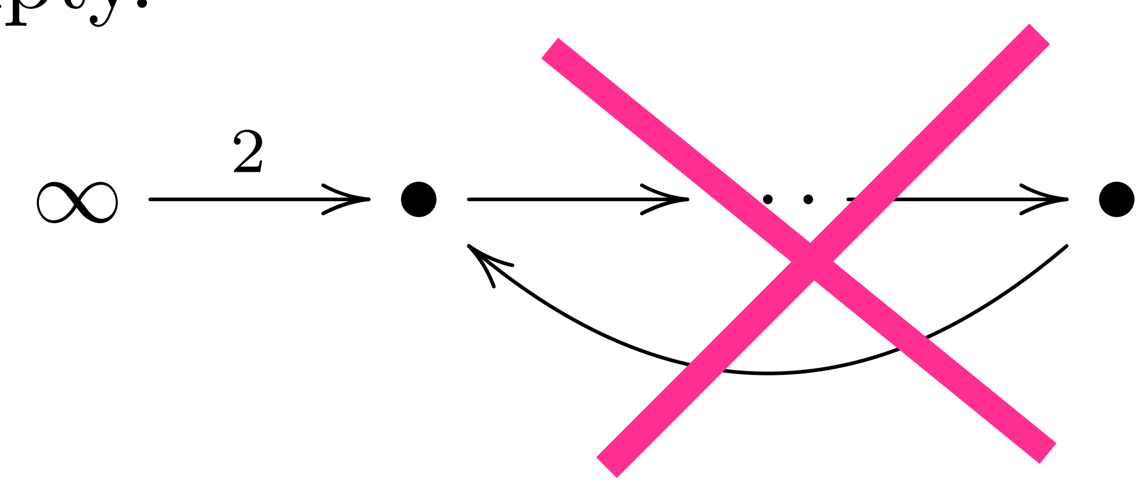
The *Milnor curve* of type (k, n) is

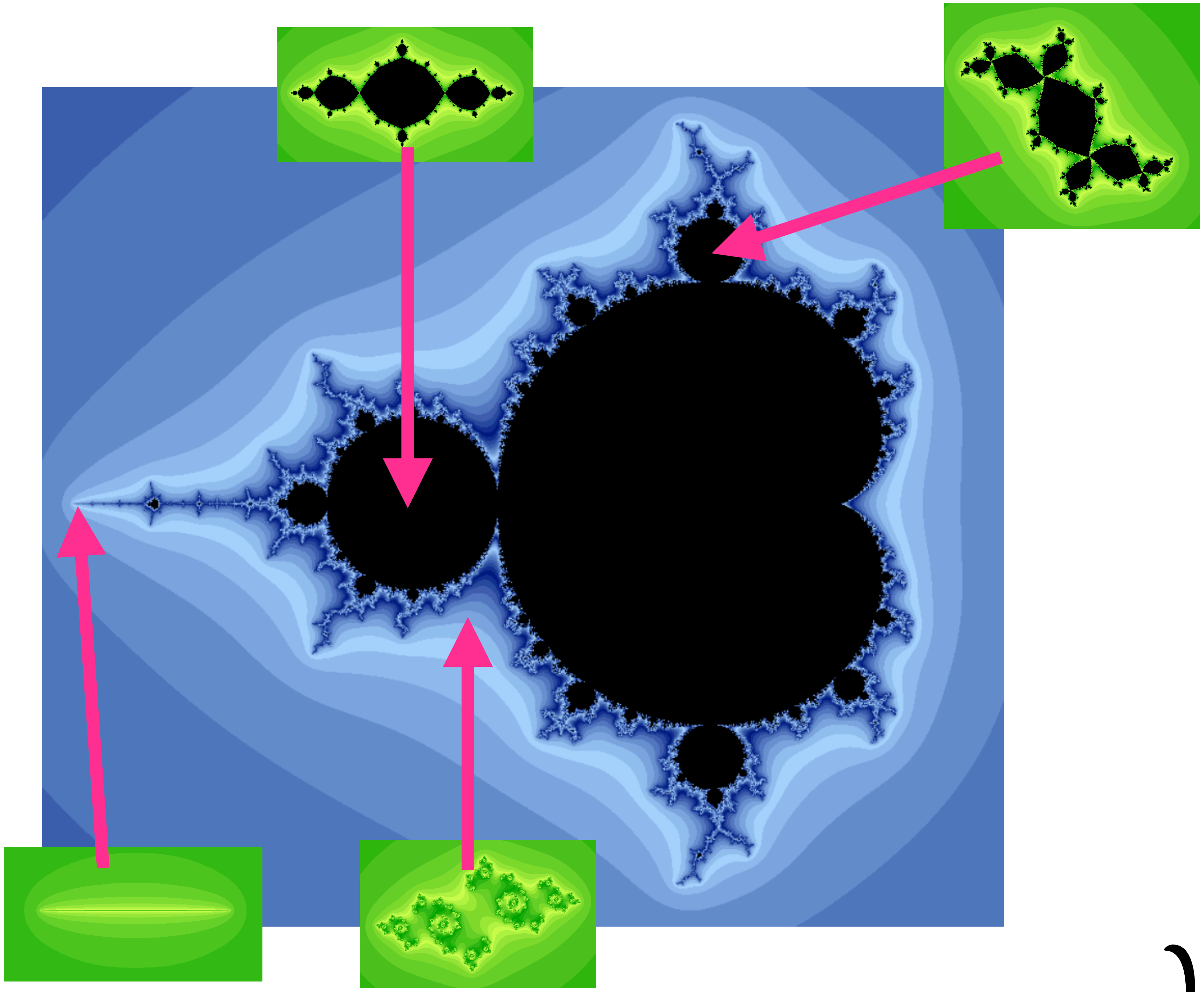
$$\mathcal{V}_{k,n} := \{[f] \in \mathcal{Q} \mid f \text{ has a critical point that is}$$

$$\text{preperiodic, of preperiod } k\text{, to a periodic cycle of period } n\}.$$



Take $k \neq 1$ since $\mathcal{V}_{1,n}$ is empty.

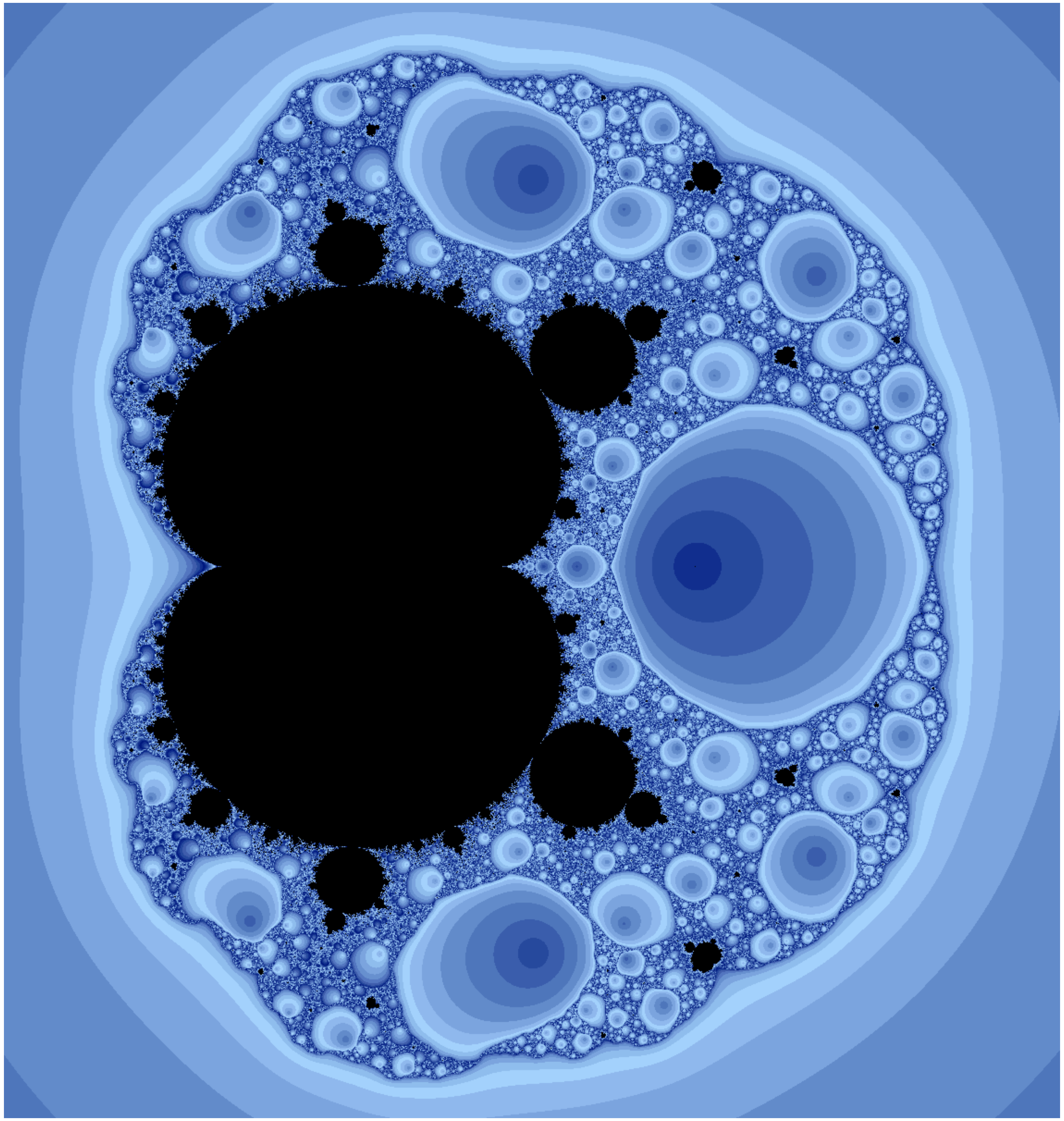


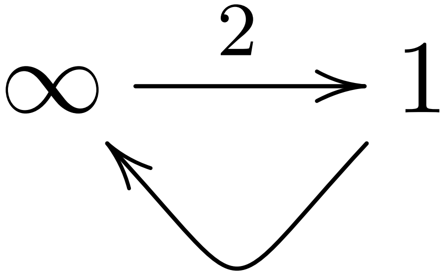


Polynomial slice

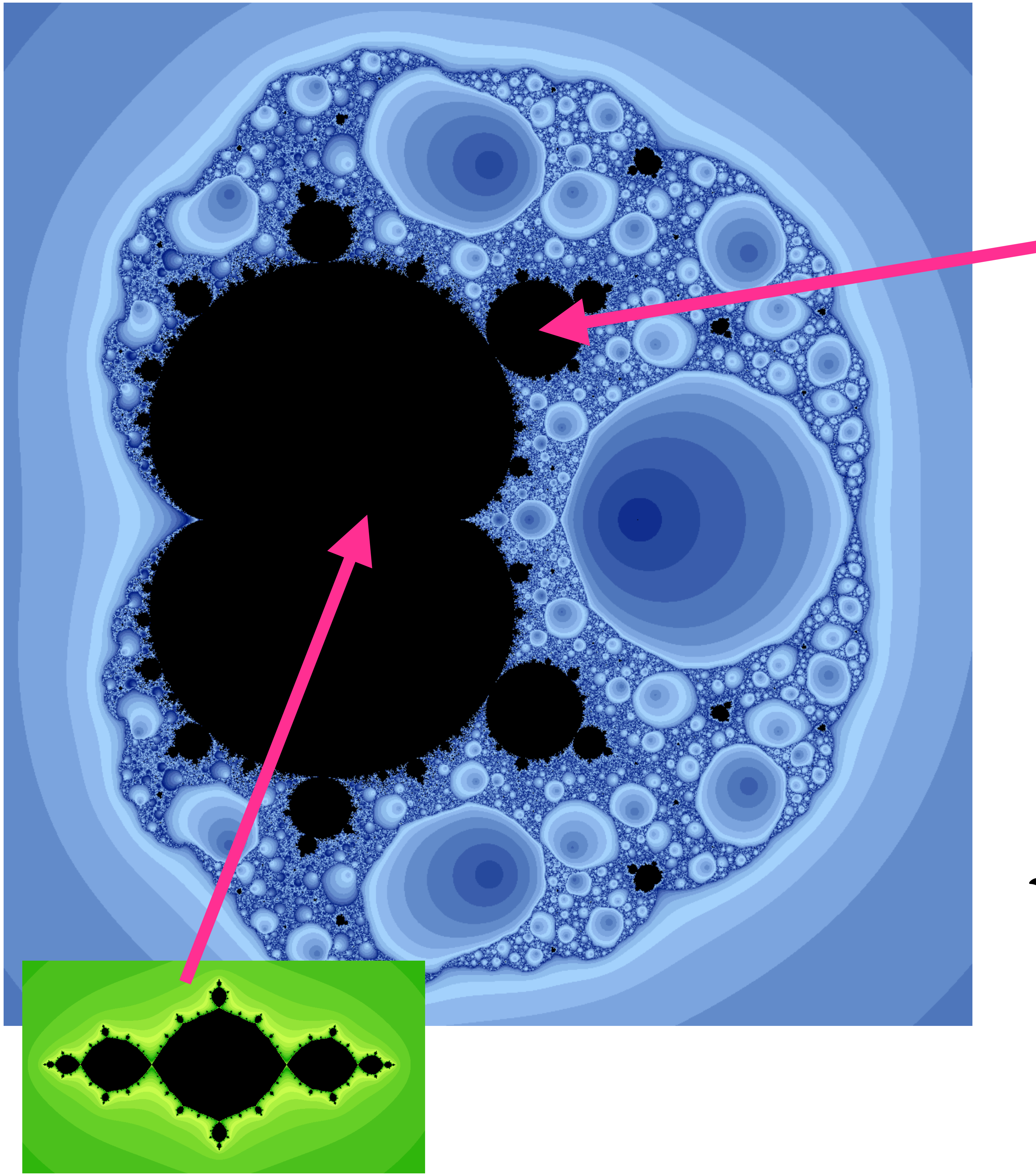
$\infty \circlearrowright^2$

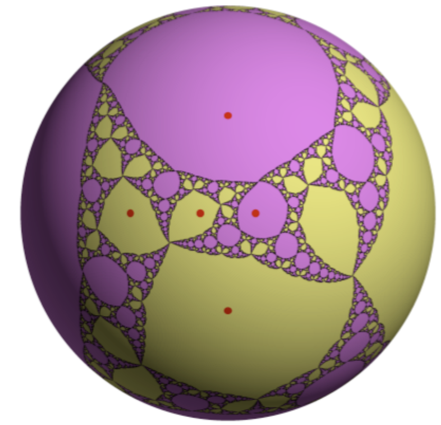
$\mathcal{V}_{0,1}$

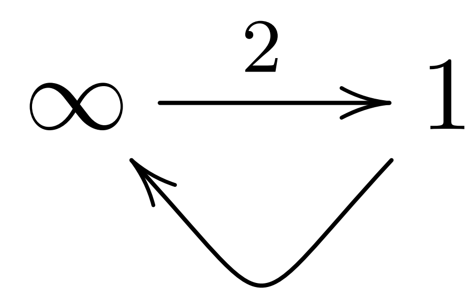




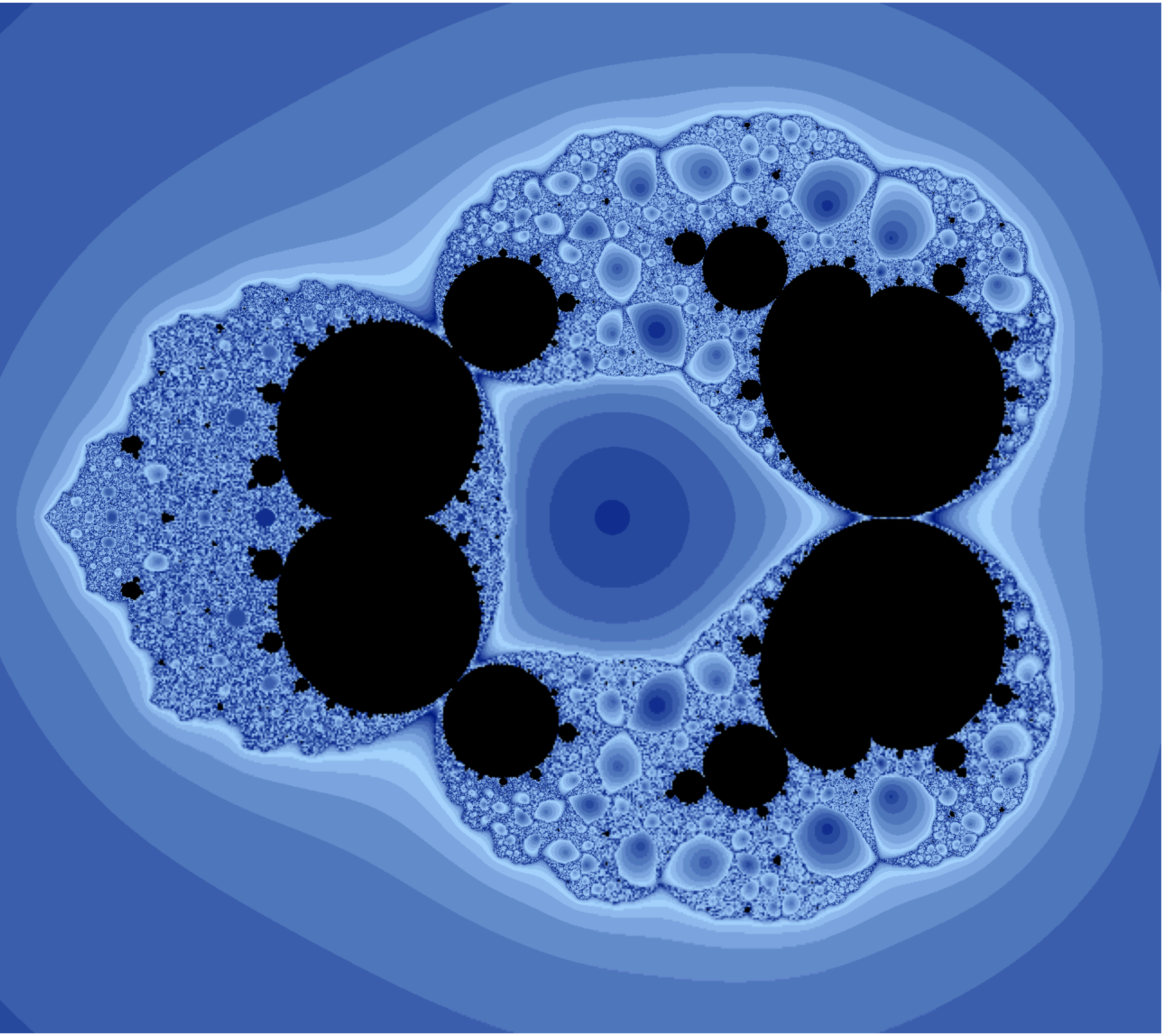
$\mathcal{V}_{0,2}$



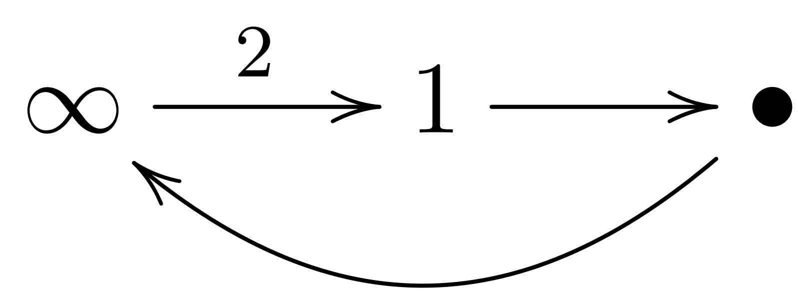


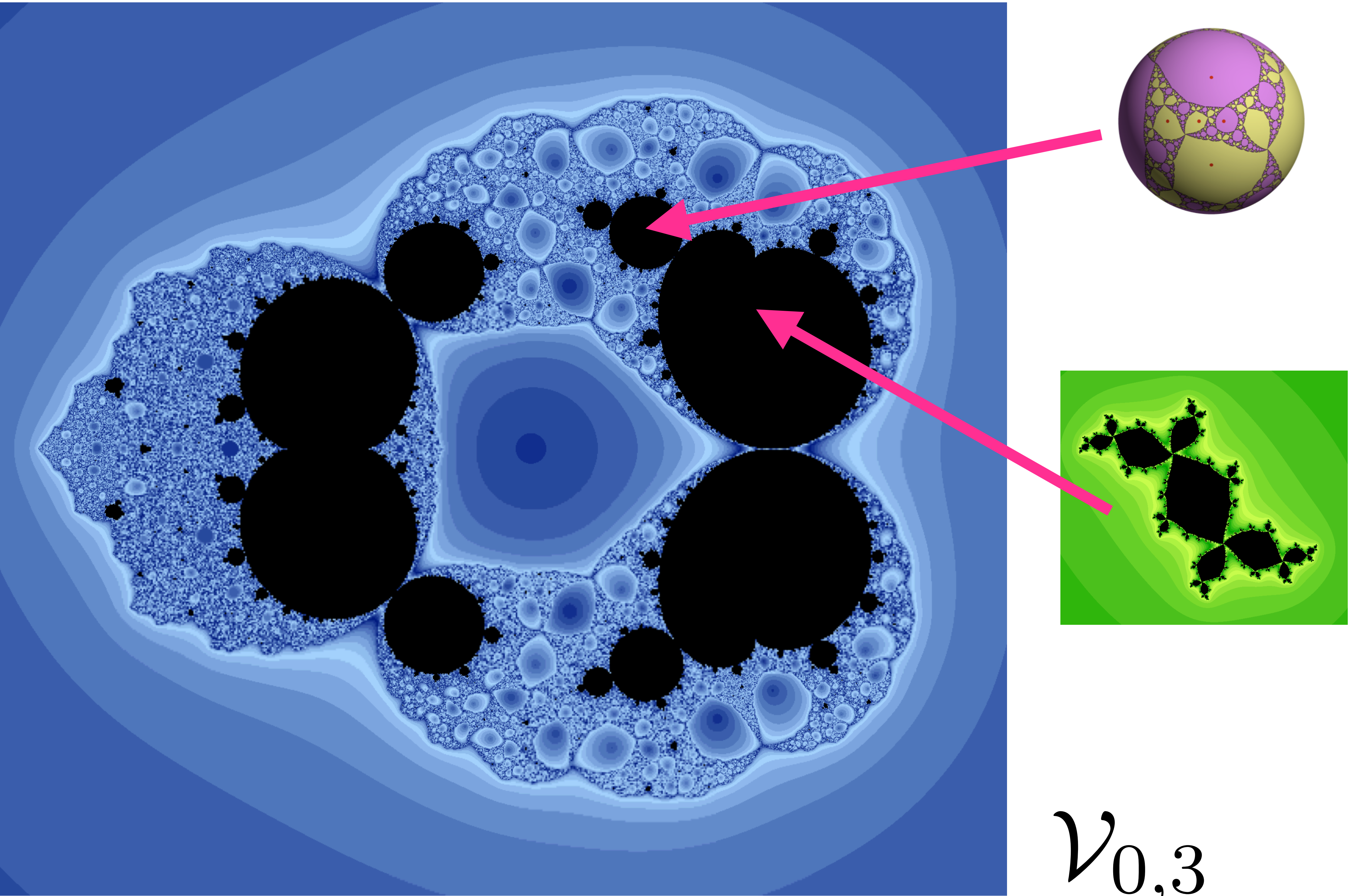


$\mathcal{V}_{0,2}$

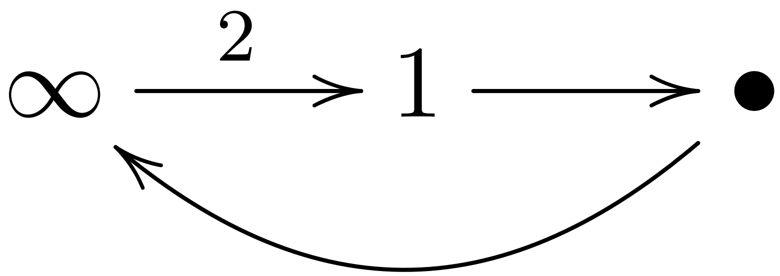


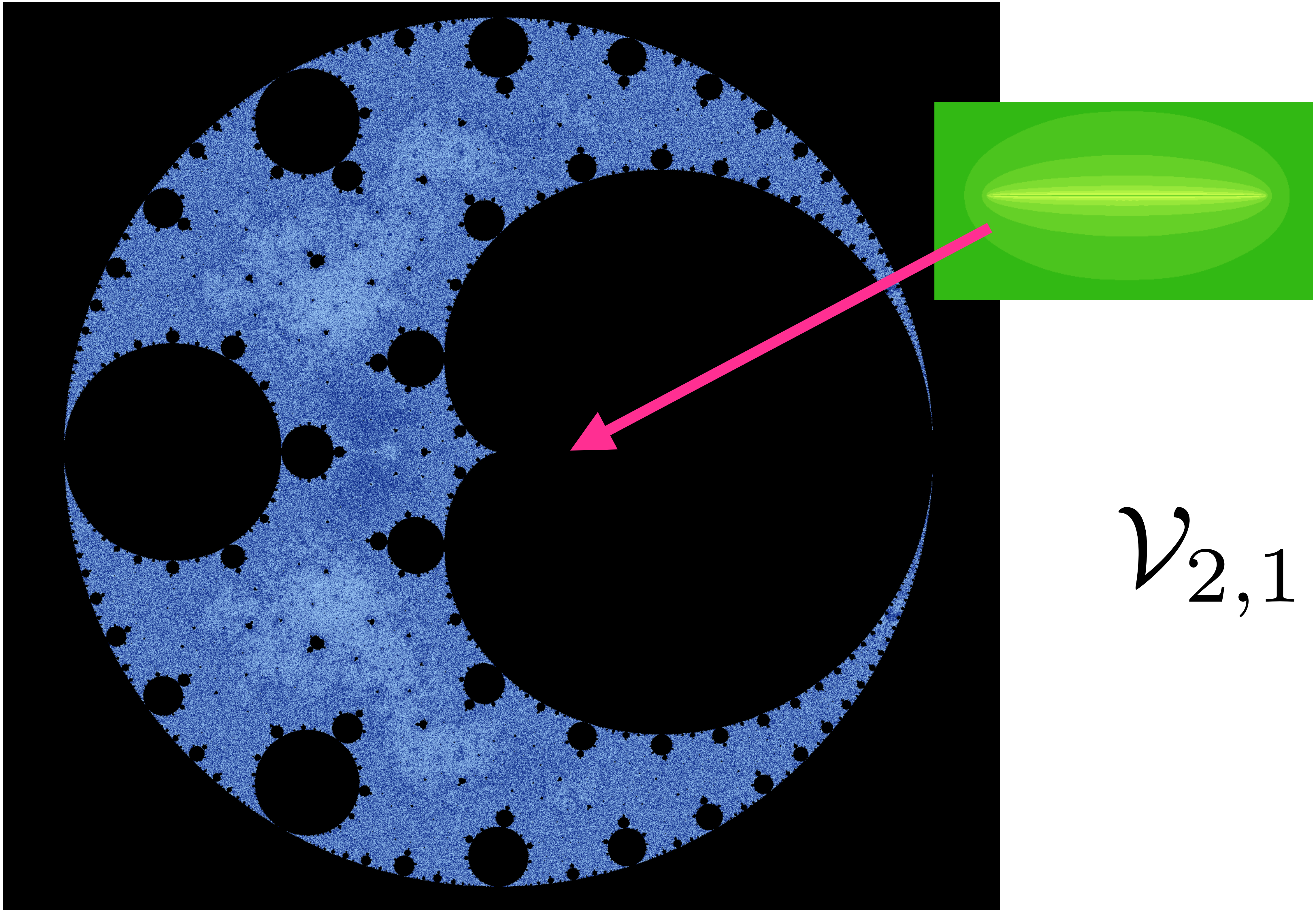
$\mathcal{V}_{0,3}$





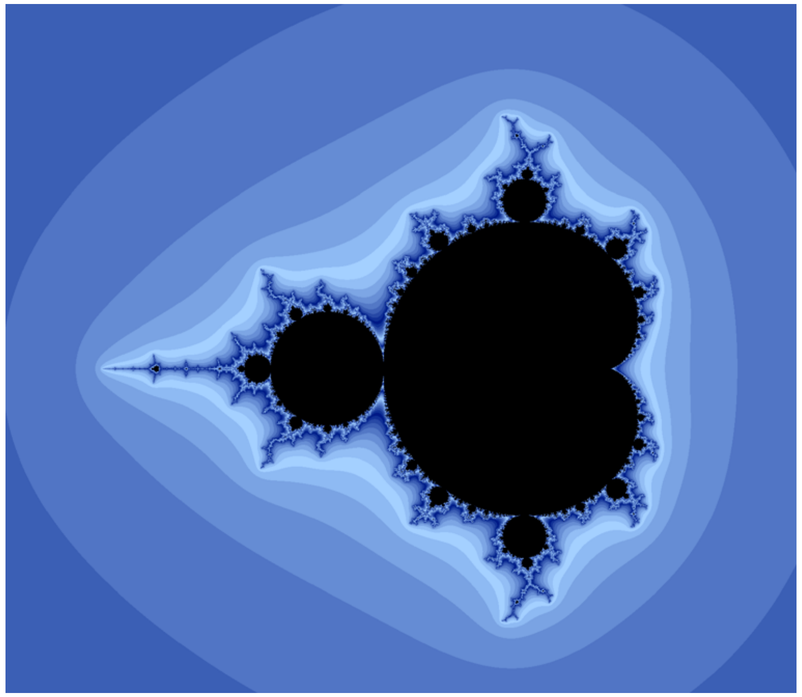
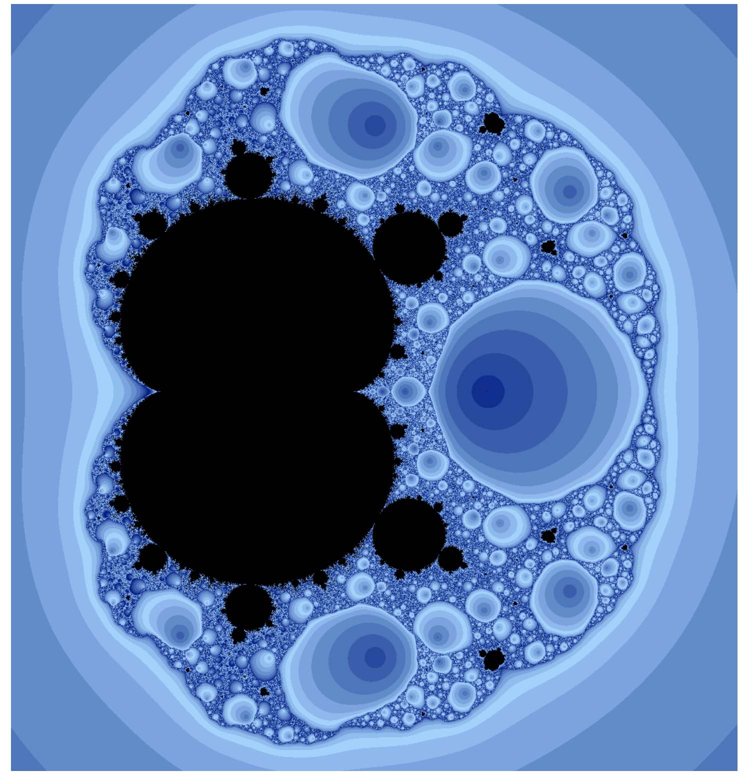
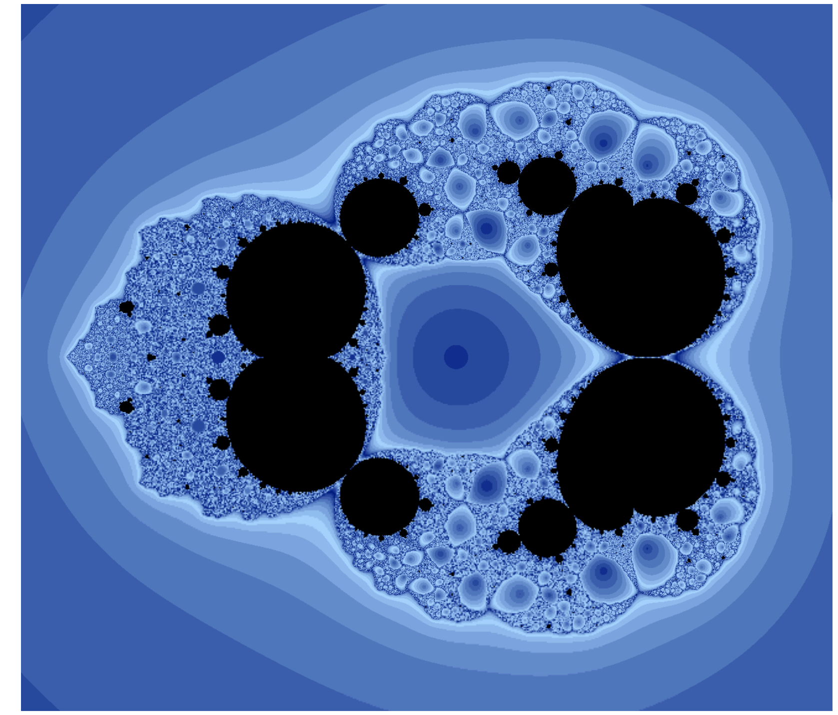
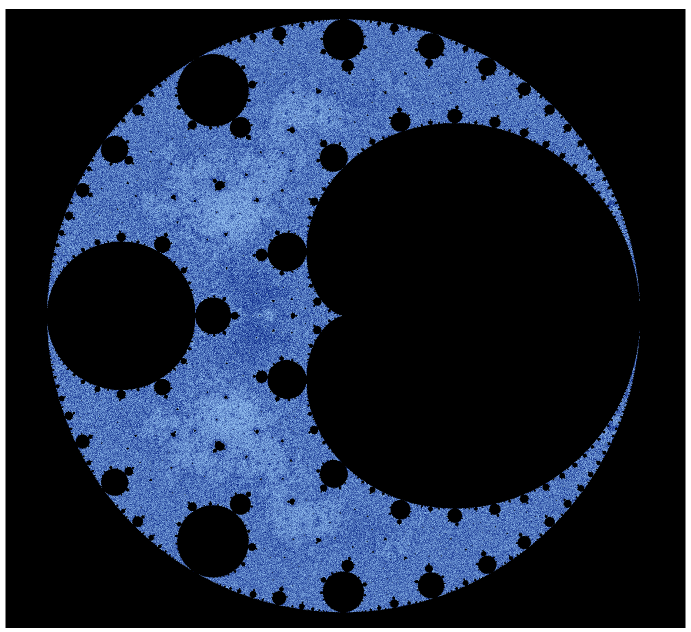
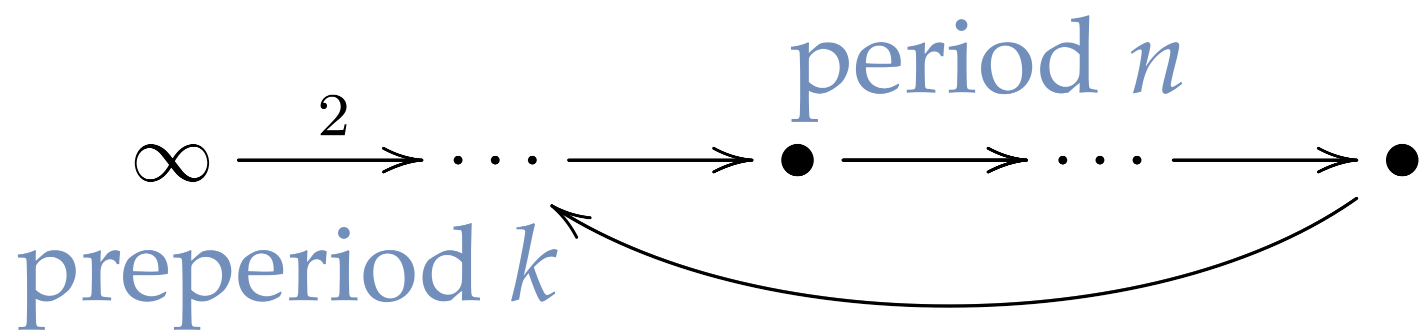
$\mathcal{V}_{0,3}$





$$\mathcal{V}_{2,1}$$

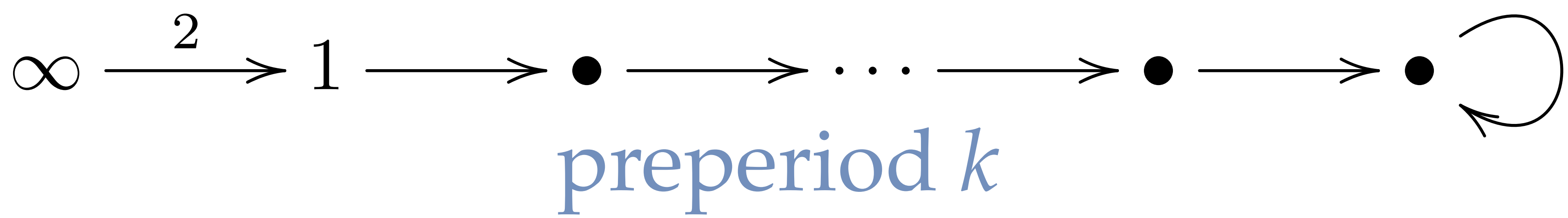
$$\infty \xrightarrow{\;2\;} 1 \longrightarrow \bullet \circlearrowleft$$

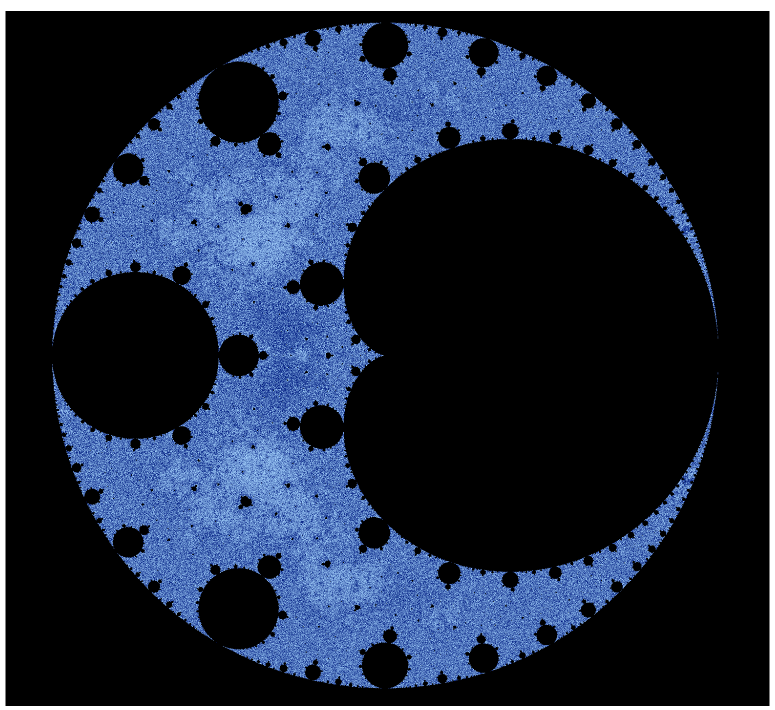






Conjecture. For all $k \in \{0, 2, 3, 4, \ldots\}$ and $n \geqslant 1$, the curve $\mathcal{V}_{k,n}$ is irreducible over $\mathbb{C}$.

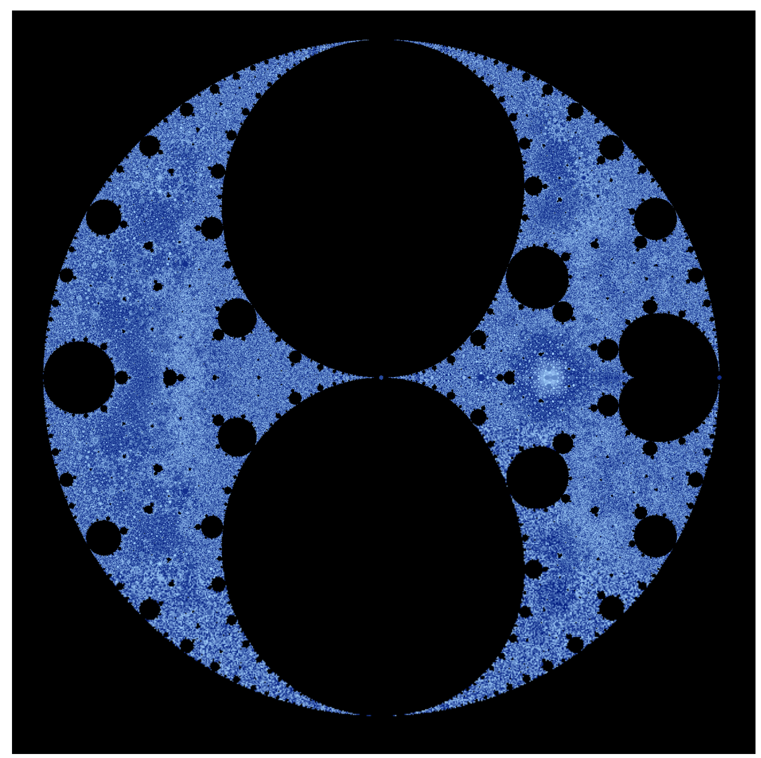
Because these curves are smooth, this is equivalent to asking if they are **connected**.

Theorem. (Buff, Epstein, K) For all $k \geqslant 2$, the curve $\mathcal{V}_{k,1}$ is irreducible (over $\mathbb{C}$).

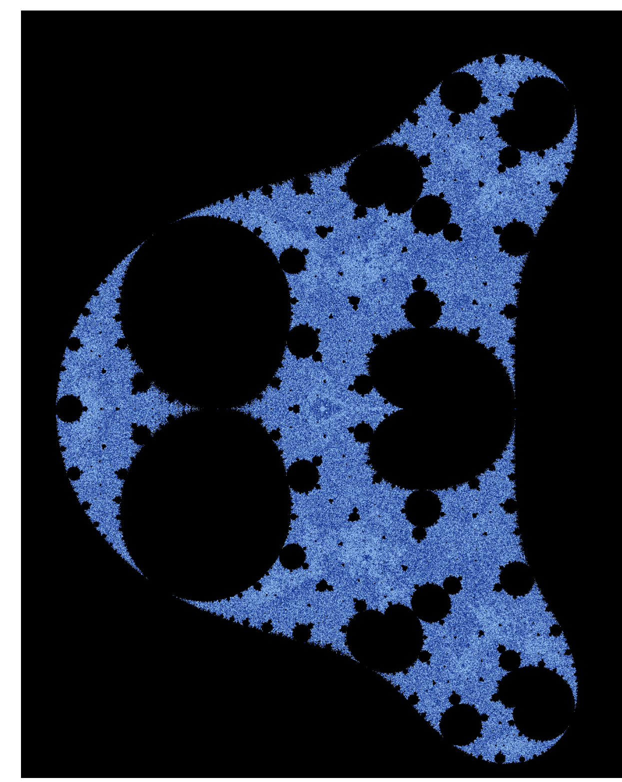




$\mathcal{V}_{2,1}$

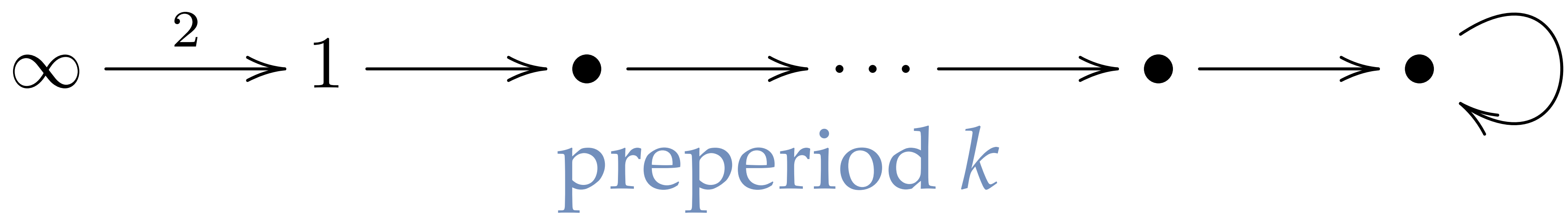


$\mathcal{V}_{3,1}$



$\mathcal{V}_{4,1}$

Theorem. (Buff, Epstein, K) For all $k \geqslant 2$, the curve $\mathcal{V}_{k,1}$ is irreducible (over $\mathbb{C}$).



Main ingredient in proof

Lemma. Let $R \in \mathbb{Q}[a,b]$, and suppose that the affine curve $R = 0$ contains a non-singular point $(a_0, b_0) \in \mathbb{Q}^2$. Then R is irreducible over $\mathbb{Q}$ if and only if it is irreducible over $\mathbb{C}$.

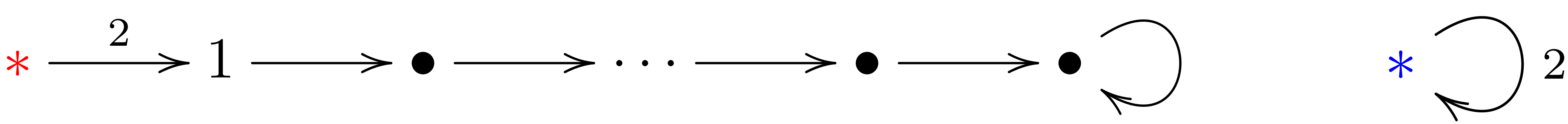
Step 0. Observe that $\mathcal{V}_{k,1}$ is given by some $R \in \mathbb{Q}[a,b]$.

Step 1. Find non-singular $(a_0, b_0) \in \mathbb{Q}^2$ on $\mathcal{V}_{k,1}$.

Step 2. Show $R \in \mathbb{Q}[a,b]$ that defines $\mathcal{V}_{k,1}$ is irreducible over $\mathbb{Q}$.

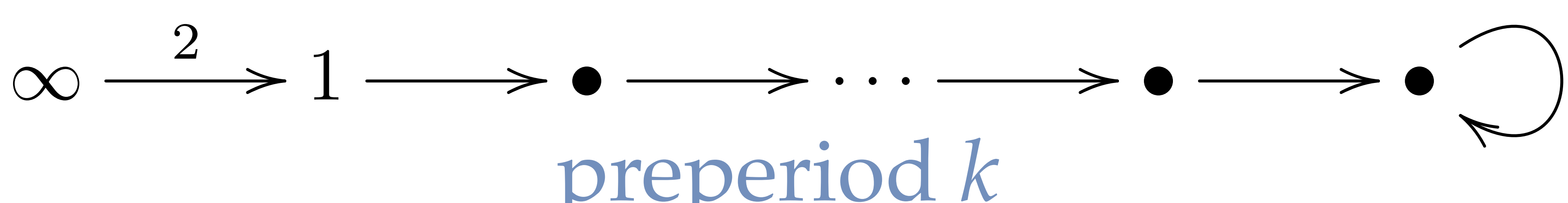
Step 2. Show $R \in \mathbb{Q}[a,b]$ that defines $\mathcal{V}_{k,1}$ is irreducible over $\mathbb{Q}$.

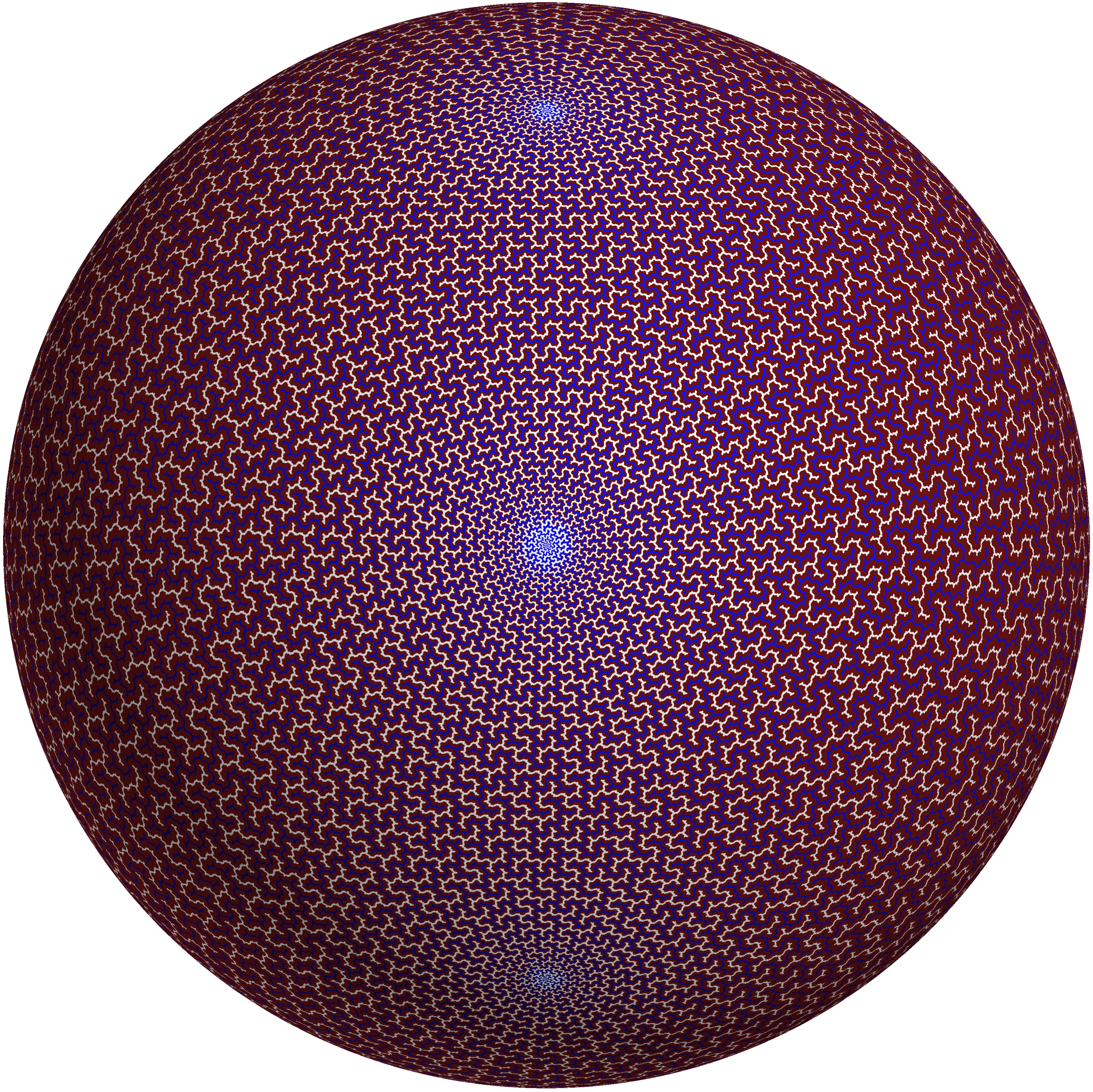
Do this by intersecting with polynomial slice $\mathcal{V}_{0,1}$ and use *Misiurewicz polynomials*.



$$f_c : z \mapsto z^2 + c$$

Theorem. (Buff, Epstein, K) For all $k \geqslant 2$, the curve $\mathcal{V}_{k,1}$ is irreducible (over $\mathbb{C}$).





Thank you!