Dynamic Tessellations Associated with Cubic Polynomials

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Work in Progress.

Connections Workshop
Complex Dynamics
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Introduction

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The set of all such maps $F = F_{a,v}$ will be identified with the ***parameter space****, consisting of all pairs $(a, v) \in \mathbb{C}^2$.*

The Period p Curve

Definition: the **period p curve** $\mathcal{S}_p \subset \mathbb{C}^2$, consists of all maps $F = F_{a,v}$ such that the marked critical point a has period exactly p.

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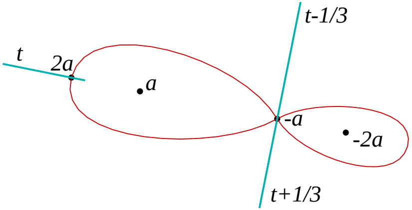

Dynamical plane for $F \in \mathcal{E}_j$.

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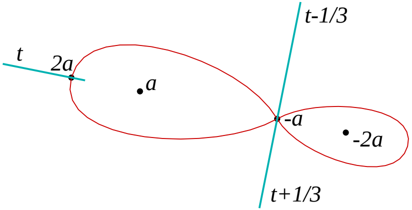

Dynamical plane for $F \in \mathcal{E}_j$. The equipotential through $2a$ and $-a$ is a figure eight curve.

Co-period angles + Escape regions

A rational angle $t = \theta \in \mathbb{Q}/\mathbb{Z}$ will be called **co-periodic** of ***co-period*** q if either $\theta + 1/3$ or $\theta - 1/3$ is periodic of period q under tripling modulo $\mathbb{Z}$.

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The complement $\mathcal{S}_p \smallsetminus X$ is the disjoint union of the open sets $\mathcal{E}_j$.

Dynamical External Rays.

For a cubic polynomial in the connectedness locus, there is a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}\smallsetminus K(f) & \xrightarrow{\cong} & \mathbb{C}\smallsetminus\overline{\mathbb{D}} \\ \Big\downarrow f & & \Big\downarrow (w\mapsto w^3) \\ \mathbb{C}\smallsetminus K(f) & \xrightarrow{\cong} & \mathbb{C}\smallsetminus\overline{\mathbb{D}} \end{array}$$

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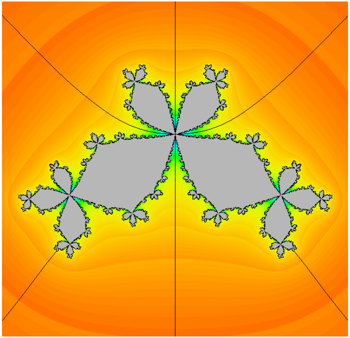

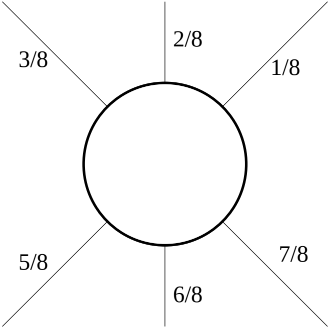

Example in $\mathcal{S}_2$: The rays are labeled by angles in $\mathbb{R}/\mathbb{Z}$.

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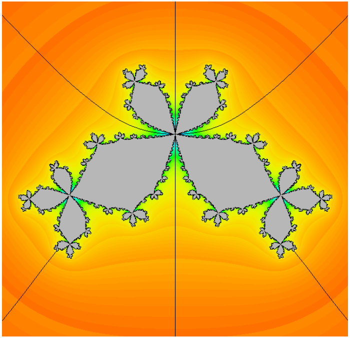

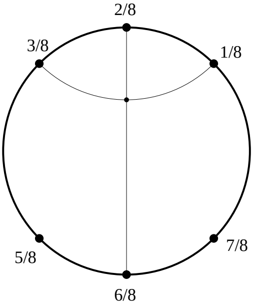

Here $q = 2$, and $1/8 \sim 2/8 \sim 3/8 \sim 6/8$, where

$$1/8 \leftrightarrow 3/8 \,, \quad 1/4 \leftrightarrow 3/4 \,, \quad 5/8 \leftrightarrow 7/8 \,.$$

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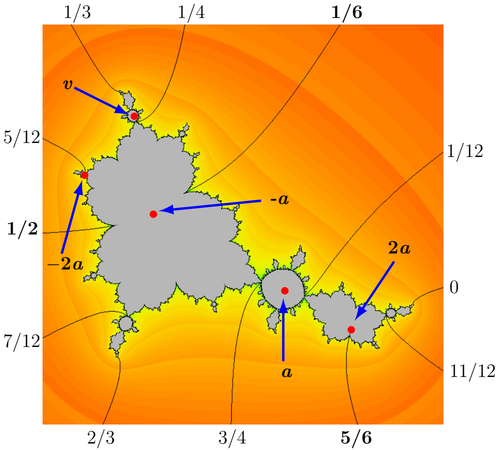

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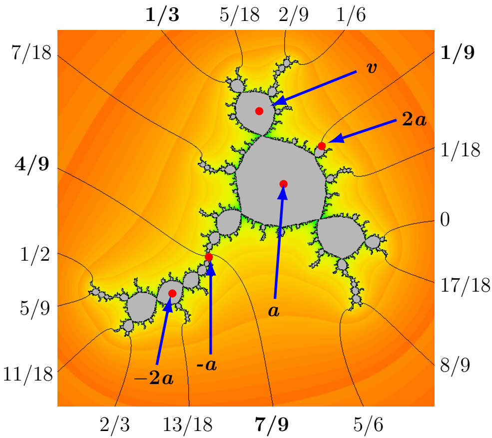

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Conjecture. All faces of the tessellation $\mathbf{Tes}_q(\overline{\mathcal{S}}_p)$ are simply-connected if and only if either $p = q$ or $p = 1$.

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For any $q \geq 1$ and any $p \geq 1$ the parameter rays of co-period q in the compactified parameter space $\overline{\mathcal{S}}_p$ divide it up into a number of **faces** $\mathcal{F}_k$.

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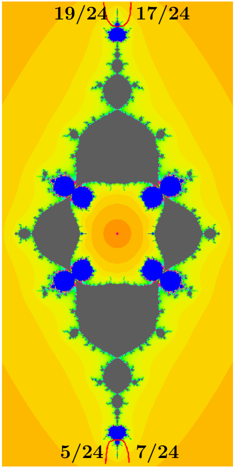

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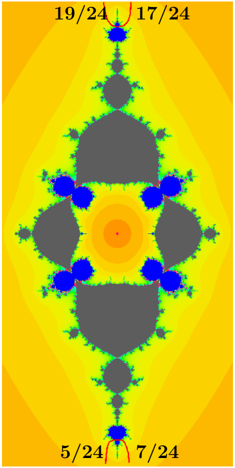
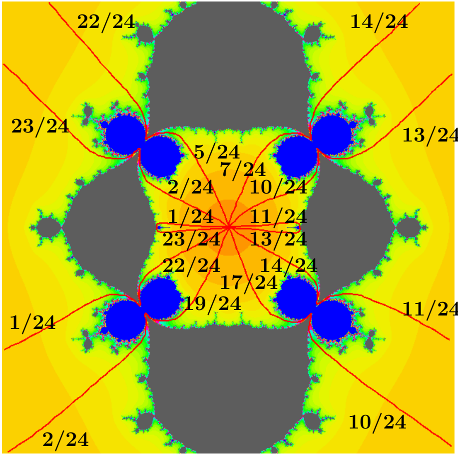
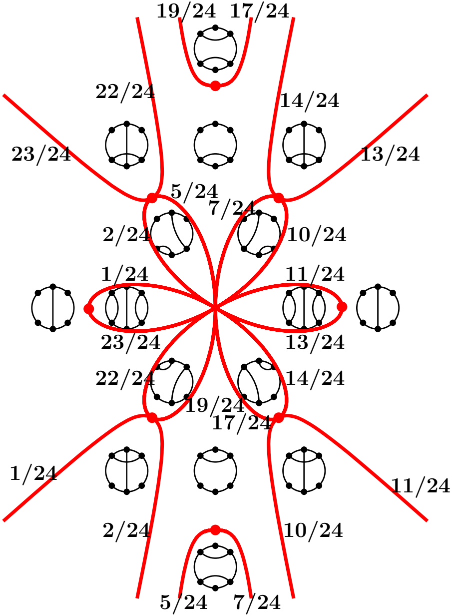




Corresponding Orbit Portraits



Theorem: Persistence of Orbit Portraits

For each $F \in \mathcal{F}_k$, and each angle $\theta_0 \in \mathbb{Q}/\mathbb{Z}$ of period q under tripling, the dynamic ray $\mathcal{R}_F(\theta_0)$ lands at a repelling periodic point $z(F) \in J(F) \subset \mathbb{C}$.

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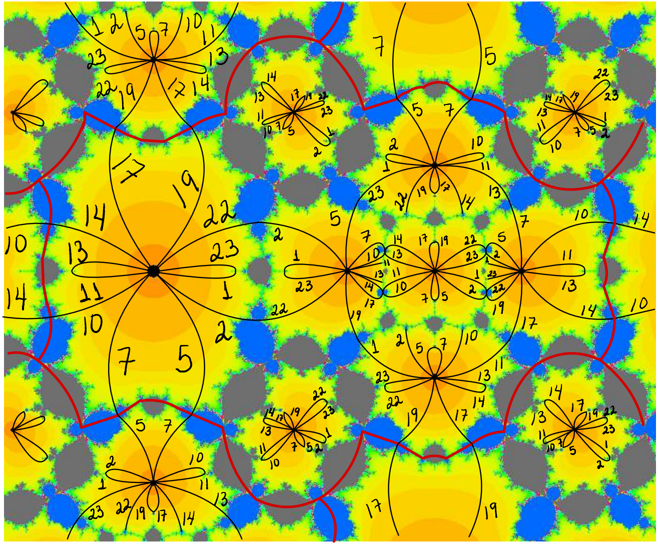
(But faces with an edge in common always seem to have different orbit portraits.)

The Case $q = 2, \quad p = 3.$



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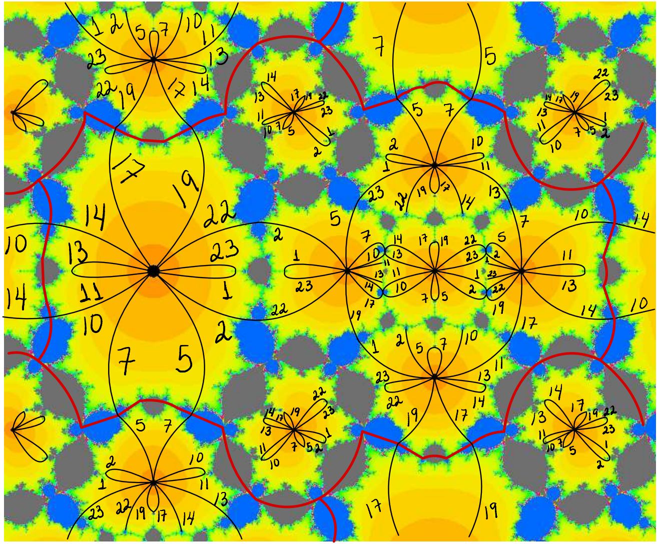
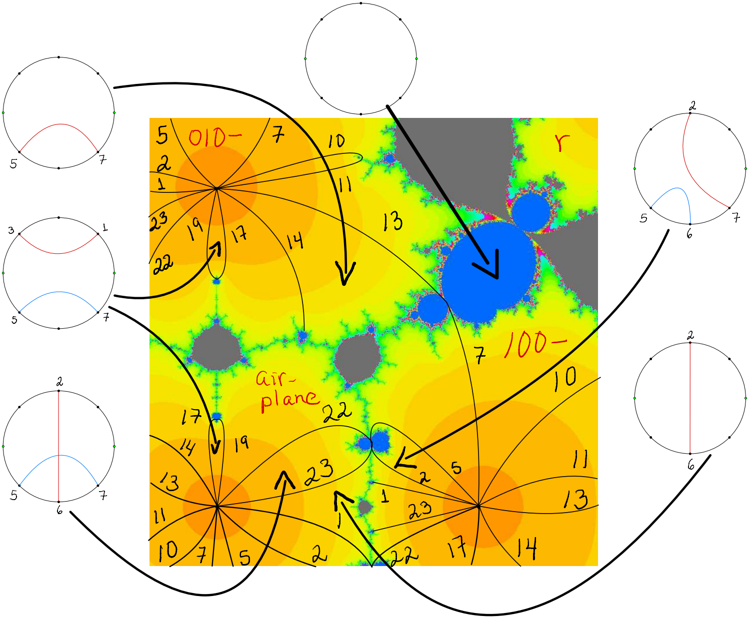


This figure shows the tessellation $\mathrm{Tes}_2(\overline{\mathcal{S}}_3)$, lifted to the universal covering space of the torus $\overline{\mathcal{S}}_3$.

Period 2 Orbit Portraits for Part of $\overline{\mathcal{S}}_3$.



Parabolic Limits:

The correspondence $F \mapsto \mathbf{OP}_q(F)$ is ***upper semi-continuous*** at each parabolic point $F_0 \in \mathcal{S}_p$, in the sense that F_0 has a neighborhood U so that $\mathbf{OP}_q(F) \subset \mathbf{OP}_q(F_0)$ for all $F \in U$.

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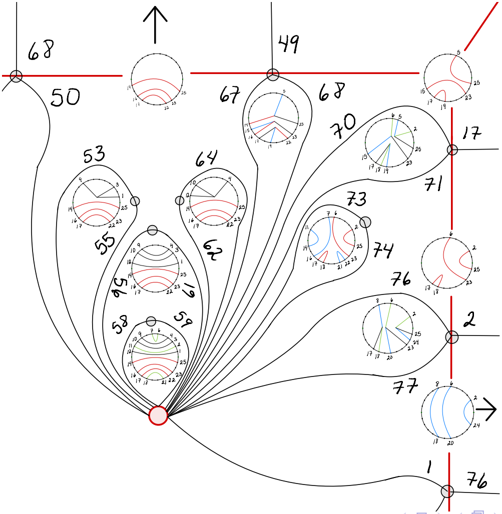

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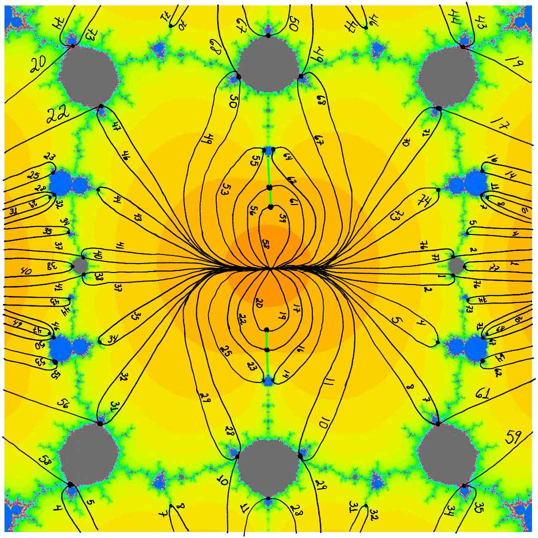

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Proposition. Assuming both, the Mandelbrot Copy Conjecture, and the Wake Conjecture, then it follows that every boundary point of an escape region is the landing point of at most two parameter rays from this region.

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Monotonicity and Non-Monotonicity Conjectures 20.

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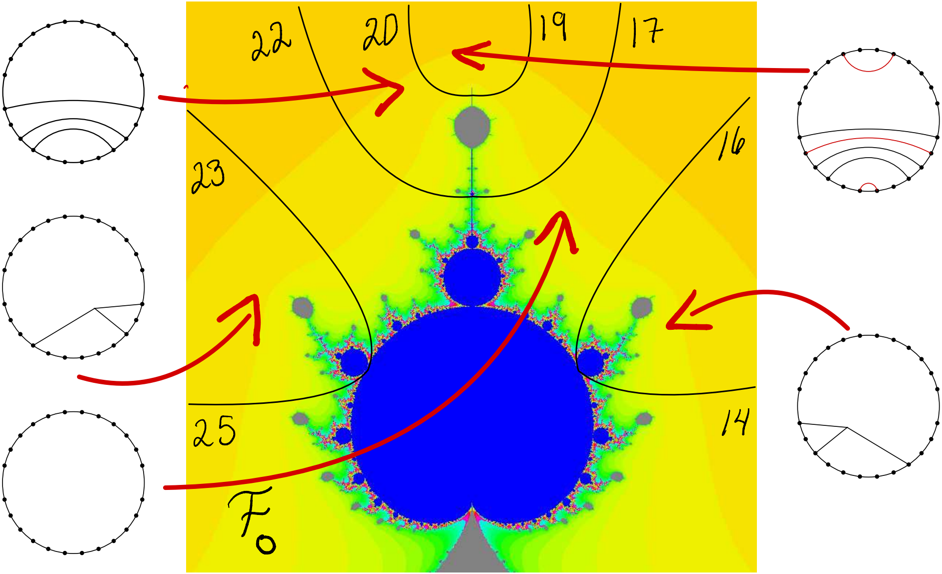
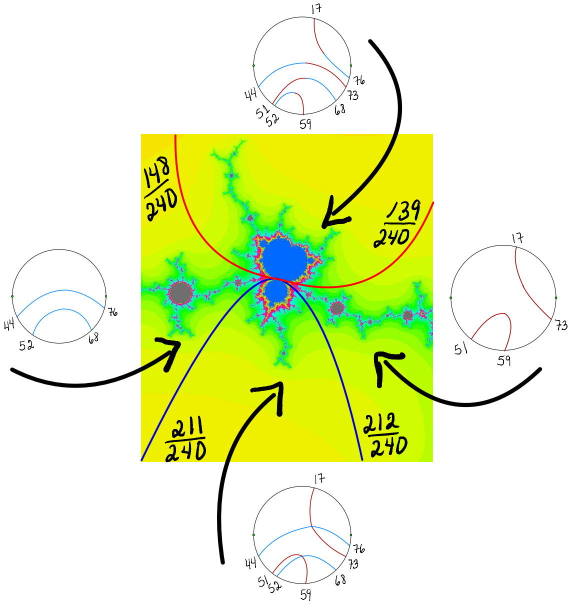
Non-Monotonicity Conjecture. As we cross a secondary ray of co-period q, the period q orbit portrait changes non-monotonically, so that neither of the two orbit portraits contains the other.

Detail Period 3-tessallation of $\mathcal{S}_1$



An Example with $q = 4$ in $\mathcal{S}_3$



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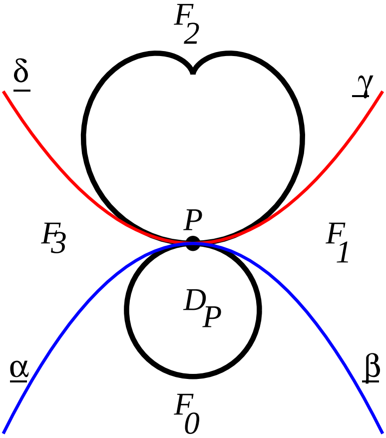
$\theta = 3\underline{\theta}$ corresponding periodic angle.

Hypothesis for the 4-Rays Landing Conjecture



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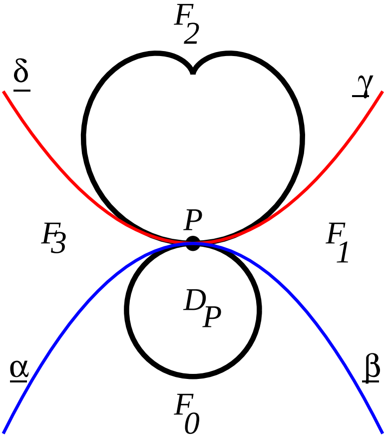

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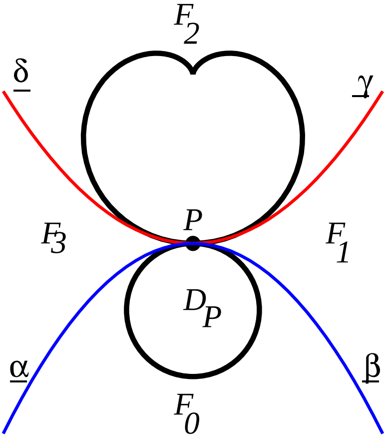

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Furthermore the co-periodic angles $\underline{\alpha}$ and $\underline{\beta}$ are ***consecutive***,
$\underline{\beta} = \underline{\alpha} + 1/3(3^q - 1)$.

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Assuming the above three conjectures, and the previous Hypothesis, the distinguishing equivalence relations for the orbit portraits of the four faces around P are completely determined by the four parameter angles; as follows:

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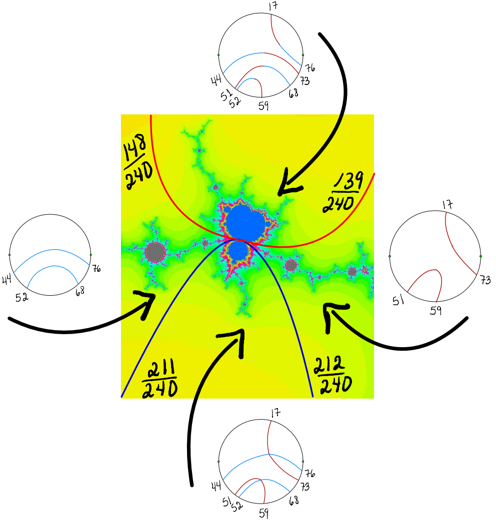
It follows that the orbit portrait for the primary face is the amalgamation of the orbit portraits for the two side faces.

Period 4 Orbit Portraits for 4-Ray Landing Case 24.



Mandelbrot copy across the boundary between 010− and the airplane region.

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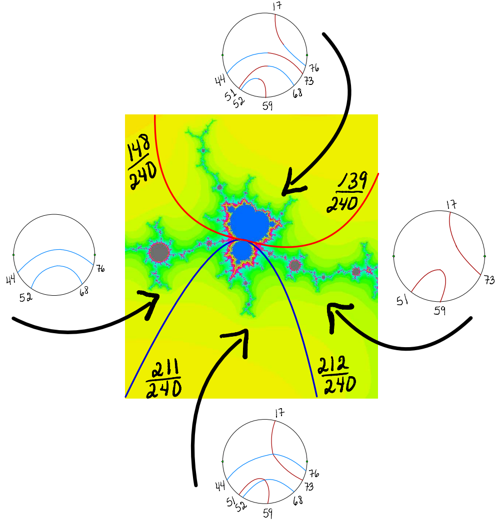
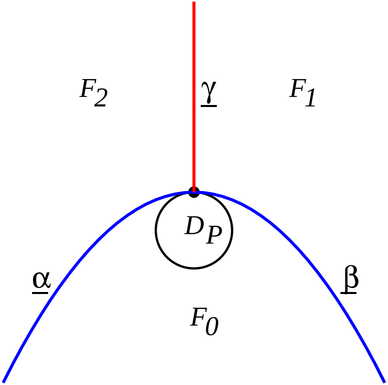


Mandelbrot copy across the boundary between 010− and the airplane region. Here $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}) = (211, 212, 139, 148)/240$ while $(\alpha, \beta, \gamma, \delta) = (51, 52, 59, 68)/80$.

3-Rays Landing Case



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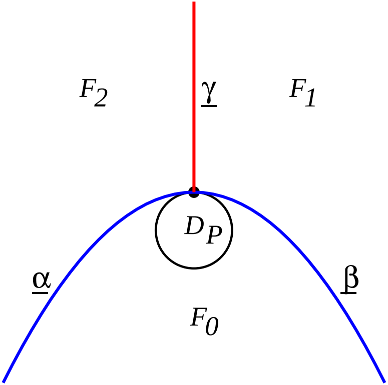

Consider three parameter rays landing at P, assuming the four Conjectures, and assuming that $\beta = \alpha + 1/(3^q - 1)$, the distinguishing relations are as follows.

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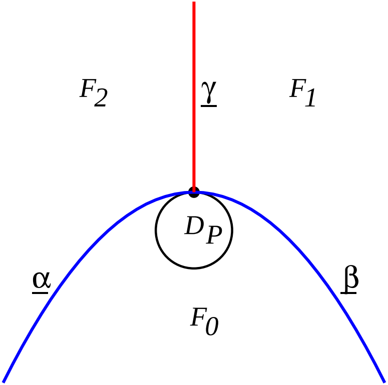

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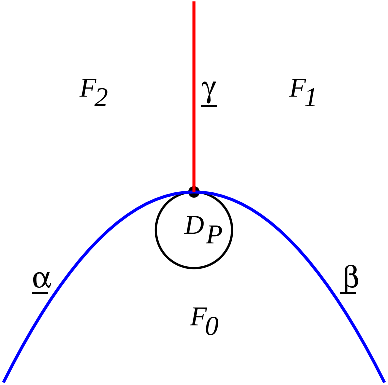

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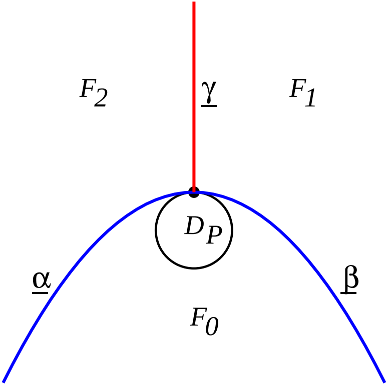

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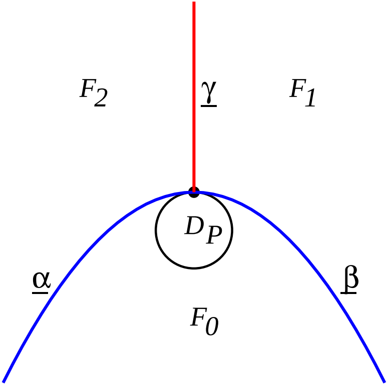

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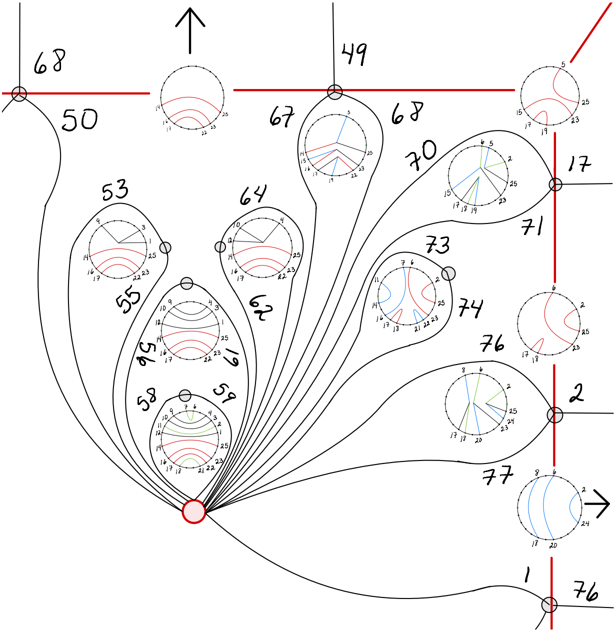
- For $\mathcal{F}_2$: $\quad 3^k\beta \sim 3^k\gamma \quad$ for $\quad k \geq 0$.

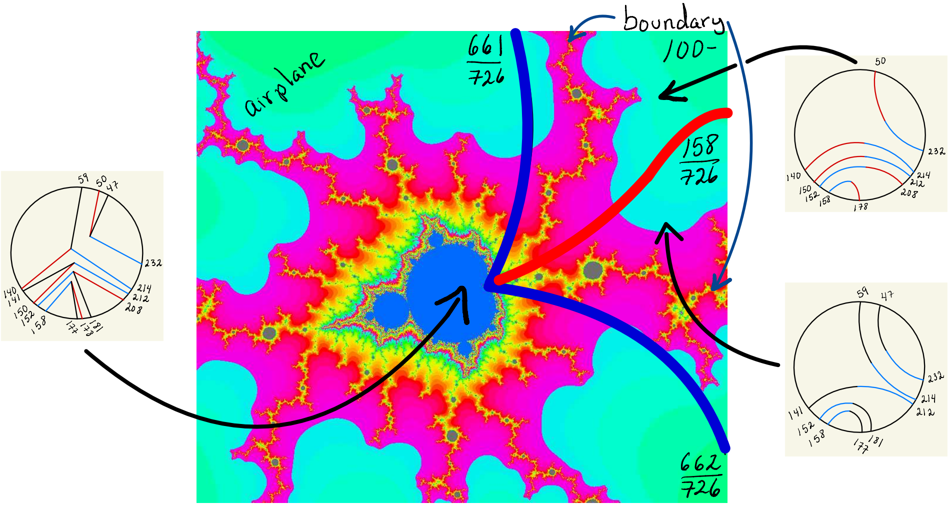
Thus the orbit portrait for $\mathcal{F}_0$ is the amalgamation of the orbit portraits for $\mathcal{F}_1$ and $\mathcal{F}_2$.

Example Orbit Portraits Airplane

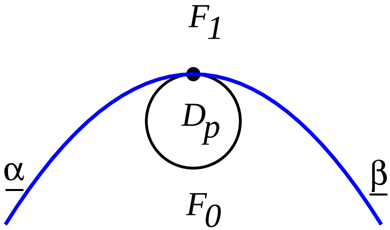
26.



Mandelbrot copy $M(5,3)$



The Two Ray Case



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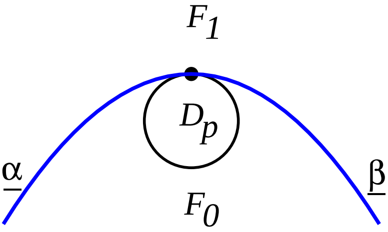
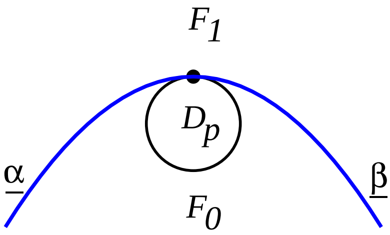


The distinguishing relations in the primary face $\mathcal{F}_0$ are

$$3^k\alpha \ \sim \ 3^k\beta \qquad \text{for} \quad k \geq 0 \ ;$$

and there are no distinguishing relations in the opposite face $\mathcal{F}_1$. The only obvious restriction is that $\alpha \neq \beta$.

The Two Ray Case



The distinguishing relations in the primary face $\mathcal{F}_0$ are

$$3^k\alpha \sim 3^k\beta \qquad \text{for} \quad k \geq 0\ ;$$

and there are no distinguishing relations in the opposite face $\mathcal{F}_1$. The only obvious restriction is that $\alpha \neq \beta$.

In some cases the two angles belong to the same grand orbit; while in other cases they belong to different grand orbits.

References

Cubic Polynomial Maps with Periodic Critical Orbit:

Part I, in “Complex Dynamics Families and Friends”, John Milnor, ed. D. Schleicher, A. K. Peters 2009, pp. 333-411.Also available in “Collected Papers of John Milnor VII” American Mathematical Society (2014) 409–476.

Part II: Escape Regions (Bonifant-Kiwi-Milnor), Conf. Geom. and Dyn. **14** (2010) 68–112. Also available in “Collected Papers of John Milnor VII”, American Mathematical Society (2014) 477–521.

Part III: External rays (Bonifant-Milnor), Work in Progress.