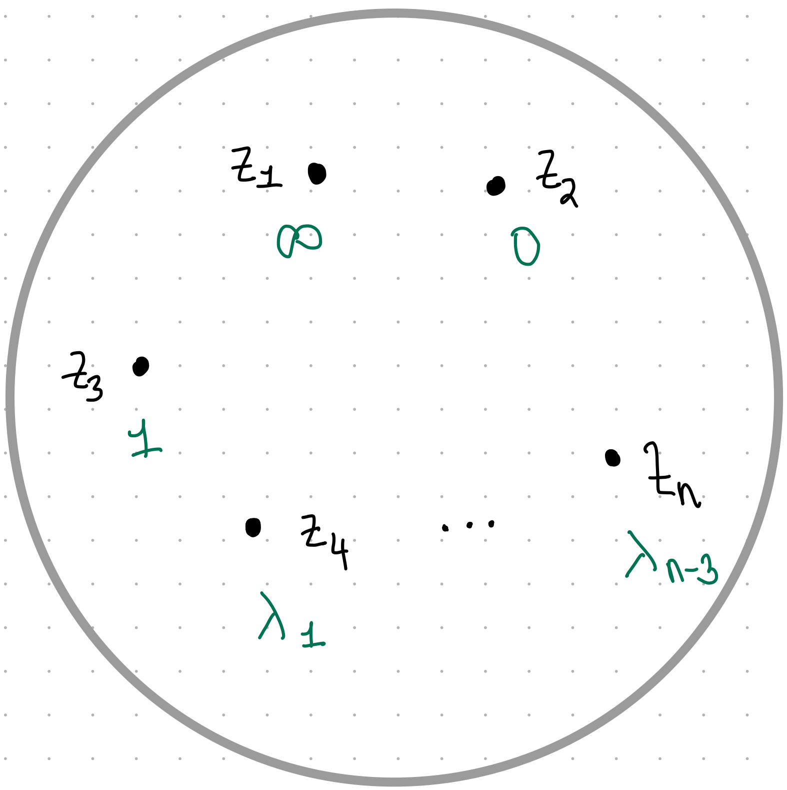
Moduli spaces and dynamics

Rohini Ramadas
University of Warwick / MSRI

Connections workshop

Configurations of n distinct points on $\mathbb{CP}^1$

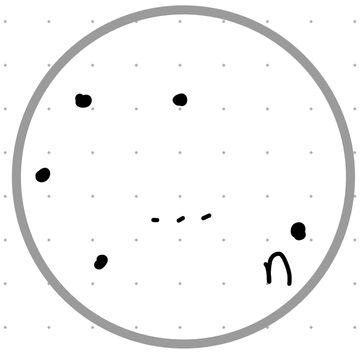


$\mathbb{CP}^1$

$$\rightsquigarrow (\lambda_1, \dots, \lambda_{n-3})$$

Classifies configuration up to
Möbius equivalence

Moduli space of configurations

$\mathcal{M}_{0,n} :=$ {  } / Möbius $\sim$

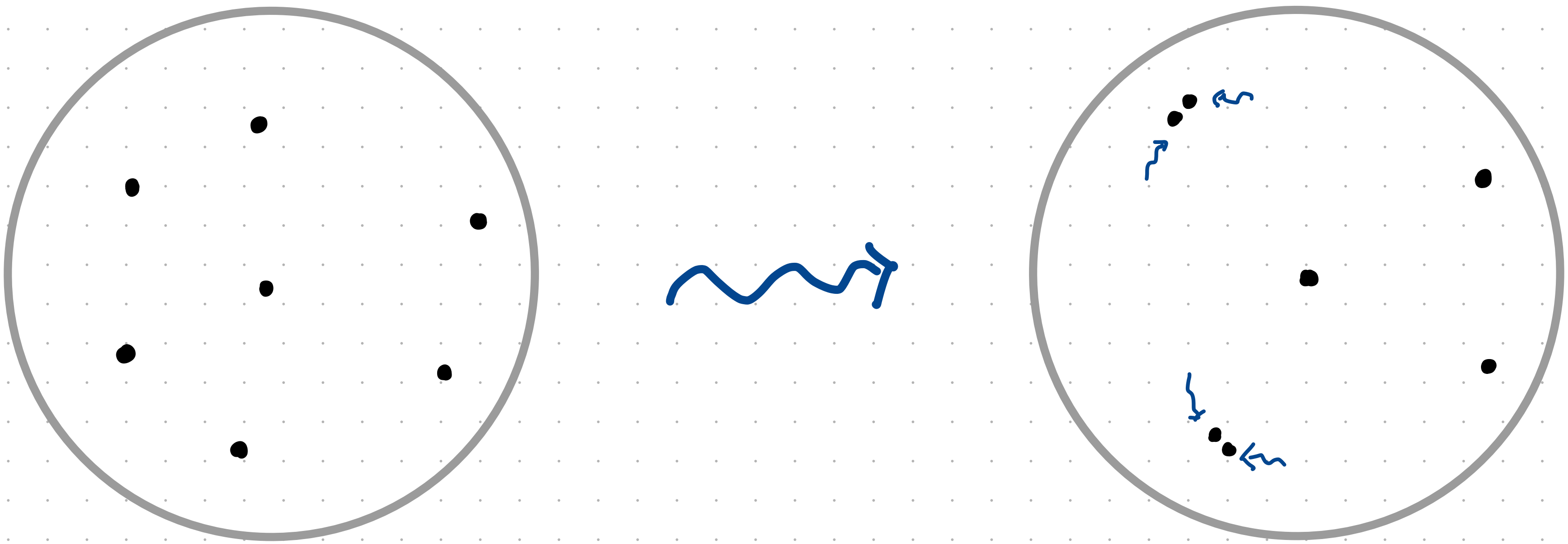
$$= \{ (\lambda_1, \dots, \lambda_{n-3}) \mid \lambda_i \neq 0, 1 \text{ etc.} \}$$

Smooth, (n-3)-dimensional, non-compact

Complex manifold / algebraic variety

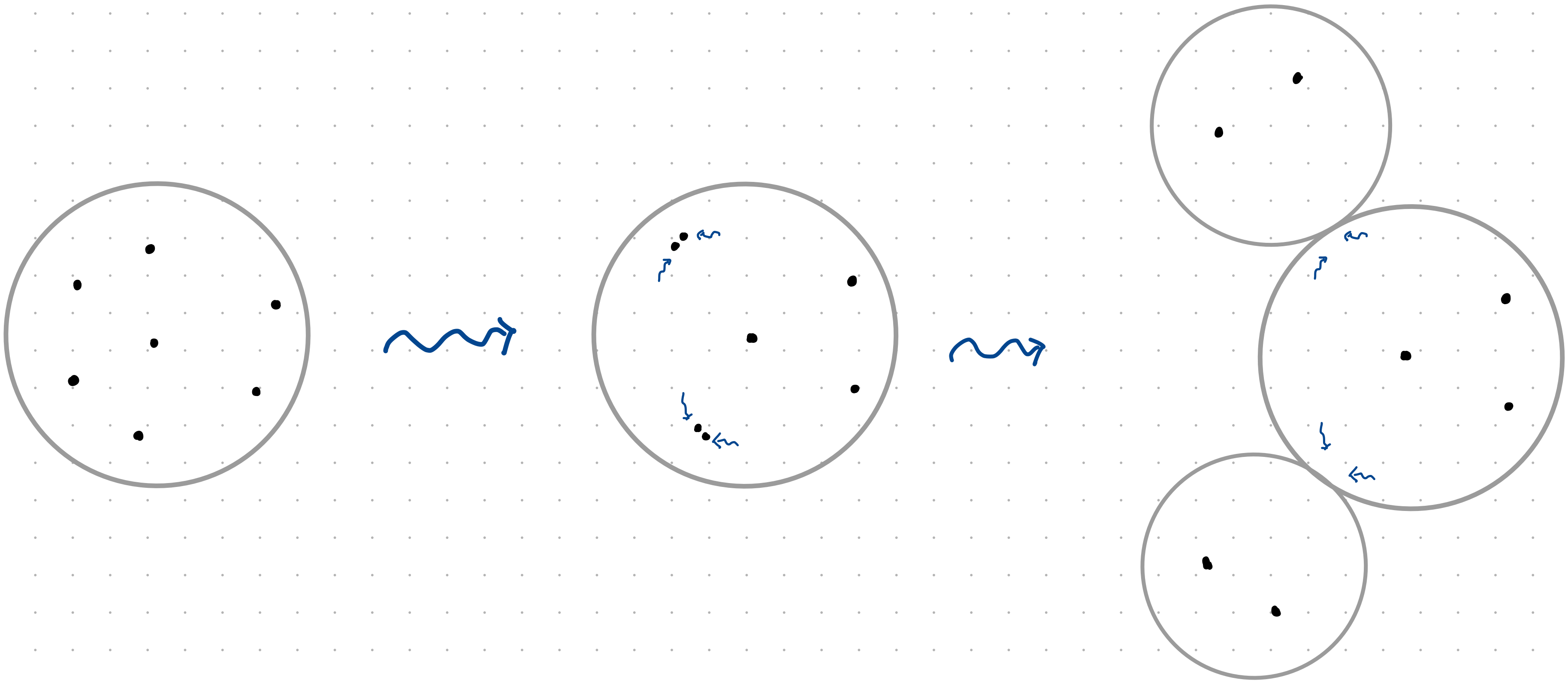
Universal cover is Teichmuller space

Going to infinity in $\mathcal{M}_{0,n}$



Compactify $\mathcal{M}_{0,n}$: interpret limits at infinity

Knudson-(Deligne-Mumford) compactification $\bar{\mathcal{M}}_{0,n}$

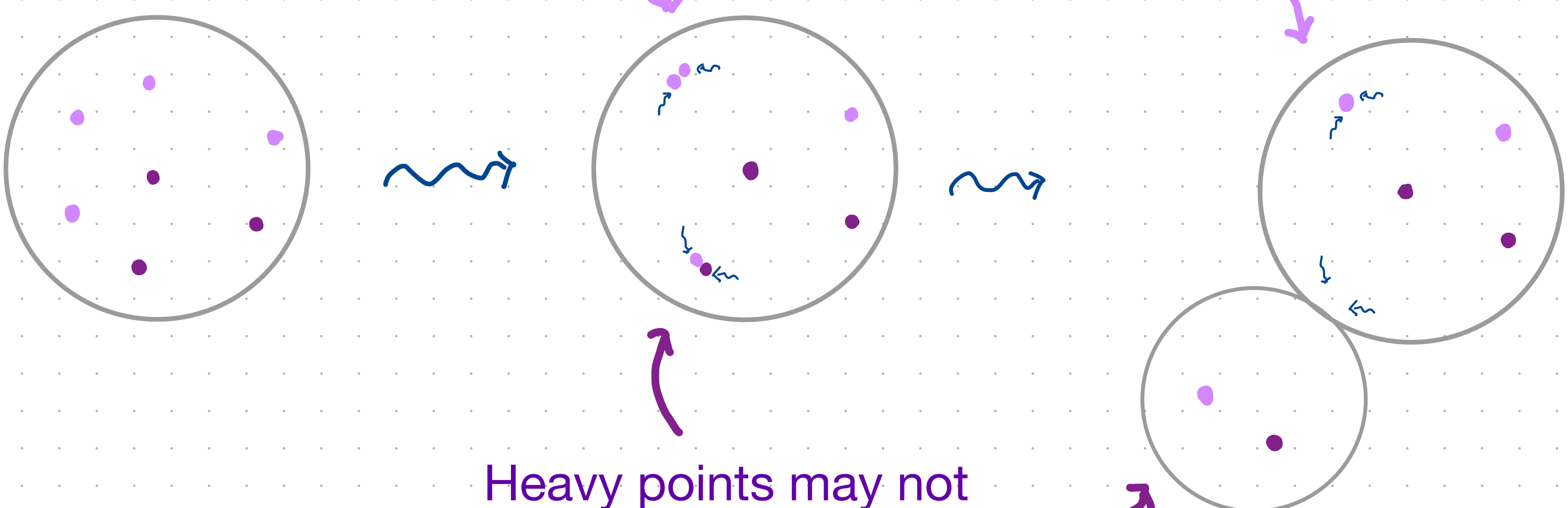


At infinity: singular surface with n distinct marked points

Hassett's compactifications * $X_{k,n-k}$:

(k) marks are "heavy"; (n-k) are "light"

Light points may coincide with each other



Heavy points may not coincide with other points

* Are more general

"Larger" compactifications: more heavy points, more at infinity, bigger homology groups

LARGER

$$\overline{\mathcal{M}}_{0,n} = X_{n,0}$$

$\downarrow$ birational, holomorphic

$$X_{n-1,1}$$

$$\downarrow$$

$$\vdots$$

$$X_{k,n-k}$$

$$\vdots$$

$$\downarrow$$

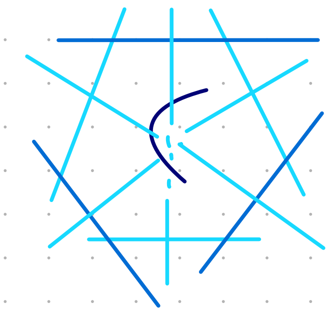
$$X_{2,n-2}$$

$$\downarrow$$

* $X_{1,n-1} \cong \mathbb{CP}^{n-3}$

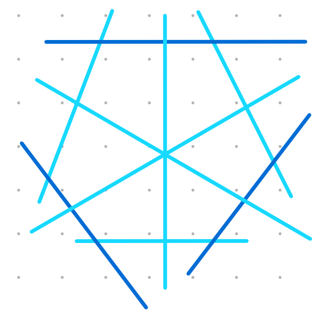
* caveat

$$\overline{\mathcal{M}}_{0,5} = X_{5,0}$$



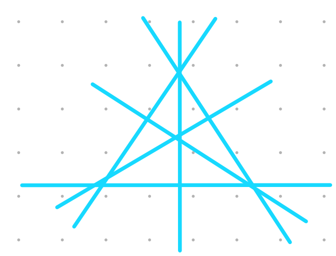
$$\downarrow$$

$$X_{2,3}$$



$$\downarrow$$

$$X_{1,4} = \mathbb{CP}^2$$



Configurations from rational maps

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$$

$$\rightsquigarrow$$

Fixed points/

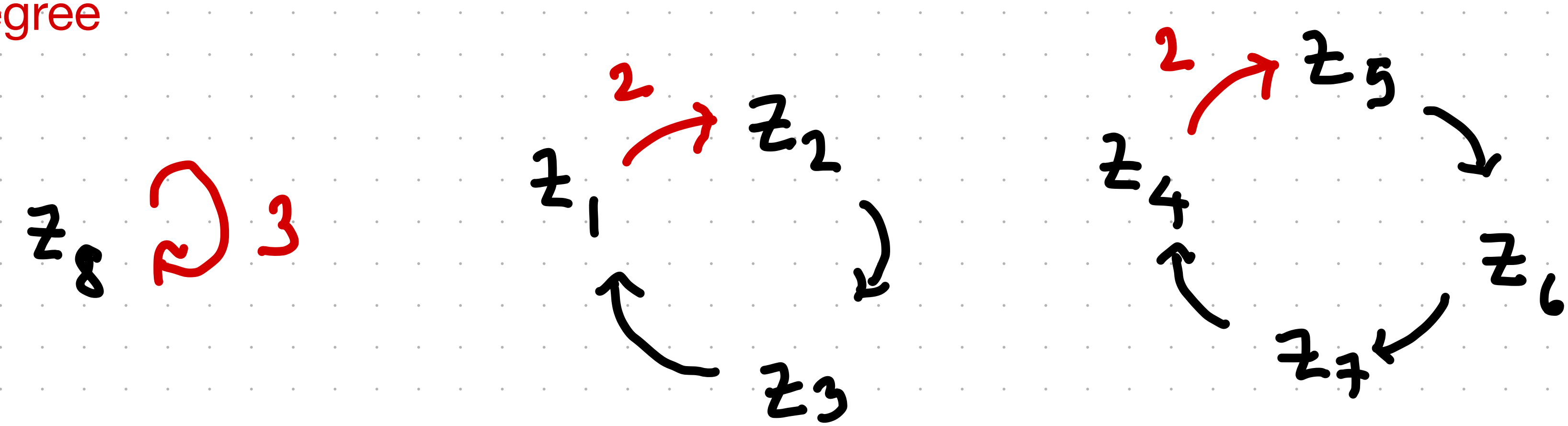
Critical points/

Critical values/

n-periodic points etc.

Post-critically finite/PCF: every critical point is (pre-)periodic

Local degree



"Portrait" of a PCF cubic polynomial

$$(z_1, \ldots, z_8) \in \mathcal{M}_{0,8}$$

PCF branched cover

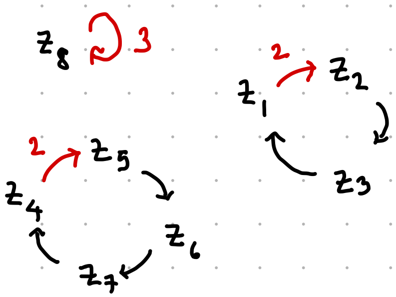
$S^2 \to S^2$

W. Thurston

$\longrightarrow$

Pullback map on Teichmuller space;

Fixed points are PCF rational maps



PCF portrait

S. Koch

$\longrightarrow$

Meromorphic map* on $\mathcal{M}_{0,n}$

Fixed points are PCF rational maps

* Caveat

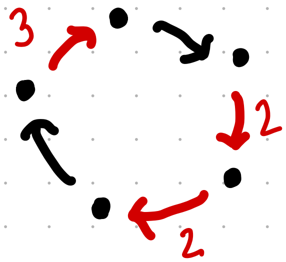
Portrait $\rightsquigarrow \mathcal{M}_{0,n} \dashrightarrow \mathcal{M}_{0,n}$

My Ph.D. + epsilon:

Describe dynamics by passing to compactifications

Eg. Can one extend holomorphically to a compactification?

“Algebraically stable map” = “Acts on homology groups as if it were holomorphic”

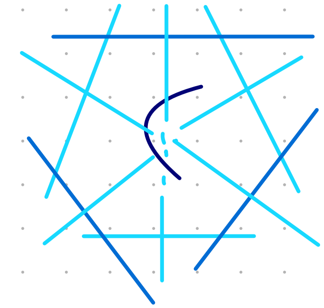


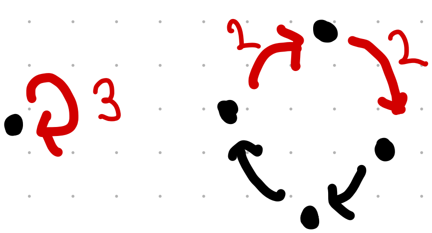
Koch-Roeder: certain portraits

Koch-Ramadas-Speyer: all portraits

$\Rightarrow$

Algebraically stable on $\overline{\mathcal{M}}_{0,n}$ $(= X_{n,0})$





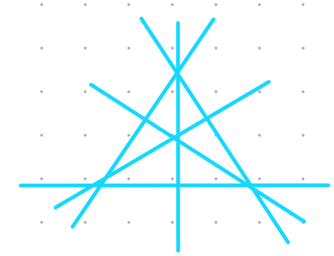
Koch: polynomial portrait*

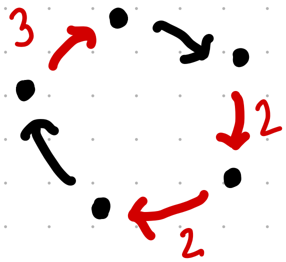
$\Rightarrow$

holomorphic/PCF on $\mathbb{C}\mathbb{P}^{n-3}$

$= X_{1,n-1}$

* + conditions





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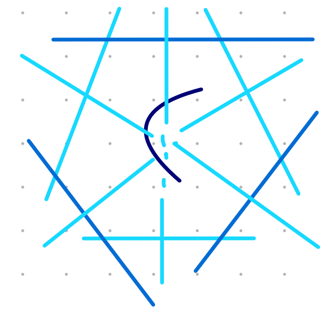
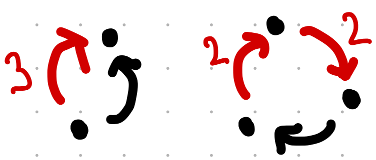
Algebraically stable on $\overline{\mathcal{M}}_{0,n}$ $(= X_{n,0})$



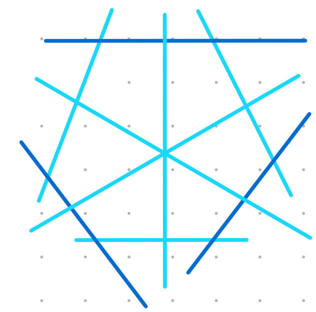


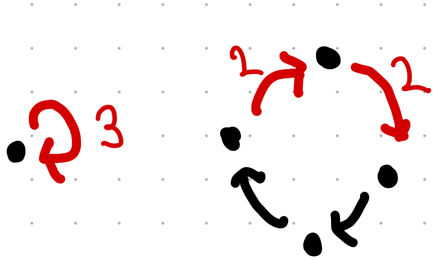
Ramadas: "polynomial-like"
(Fully ramified point in k-cycle)*

$\Rightarrow$

Algebraically stable on $X_{k,n-k}$

* $k \geq 2$



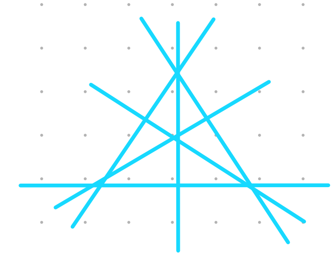


Koch: polynomial portrait*

$\Rightarrow$

holomorphic/PCF on $\mathbb{CP}^{n-3}$

$= X_{1,n-1}$



Dynamical moduli spaces, point-configurations, tropical geometry

Dynamical moduli spaces, point-configurations, tropical geometry

$$\mathcal{V}_{0,n} := \left\{ f : \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \quad \text{deg } 2 \right.$$

$$\left. \right\} = \mathrm{Per}_n(0)$$

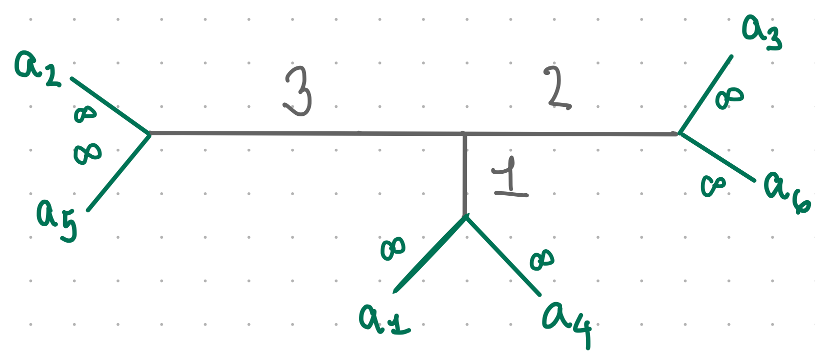
Algebraic curve/ punctured Riemann surface

$$\mathcal{V}_{0,n} \longrightarrow \mathcal{M}_{0,n} \qquad \qquad \overline{\mathcal{V}}_{0,n} \longrightarrow \overline{\mathcal{M}}_{0,n}$$

Puncture of $\mathcal{V}_{0,n}$ — infinitesimal 1-parameter degeneration in $\mathcal{M}_{0,n}$

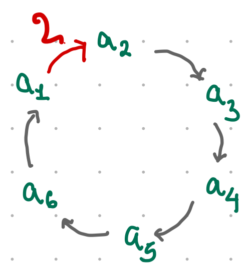
“Genus-0 tropical curve” = “Metric tree”

Eg.



A tropical curve represents an equivalence class of infinitesimal 1-parameter degenerations in $\mathcal{M}_{0,n}$

$\{ f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad \deg 2$



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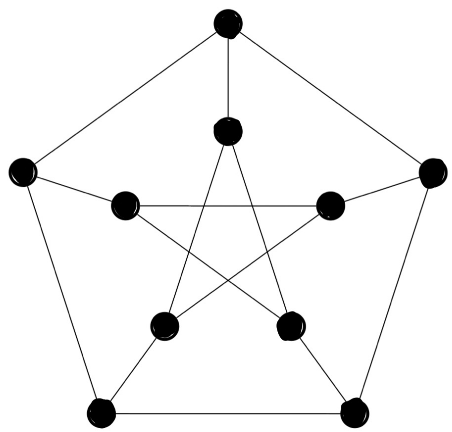
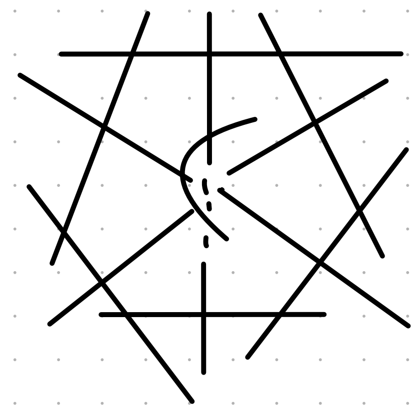
Ramadas-Silversmith: experimental paper for n=5

Tropical moduli spaces: Mikhalkin, Abramovich-Caporaso-Payne

$$\mathcal{M}_{0,n}^{trop} = \left\{ \begin{array}{l} \text{Tropical genus-0} \\ \text{curves on n leaves} \end{array} \right\} = \left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{infinitesimal 1-parameter} \\ \text{degenerations in } \mathcal{M}_{0,n} \end{array} \right\}$$



$\mathcal{M}_{0,5}^{trop}$: cone over



$\overline{\mathcal{M}}_{0,5}$

Dynamics on $\mathcal{M}_{0,n}^{trop}$?