Dynamics of complex Hénon maps

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Motivation: Real Analysis

In 1963 Lorenz proposed a system of three coupled first order ODEs with a strange attractor to model atmospheric convection:

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz.\end{aligned}$$

Based on numerical experiments by Pomeau, Lanford, and Ruelle, in 1976 Michel Hénon proposed a model map for a Poincaré map of the Lorenz system:

$$H_{c,a}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x^2+c-ay\\ x\end{pmatrix}.$$

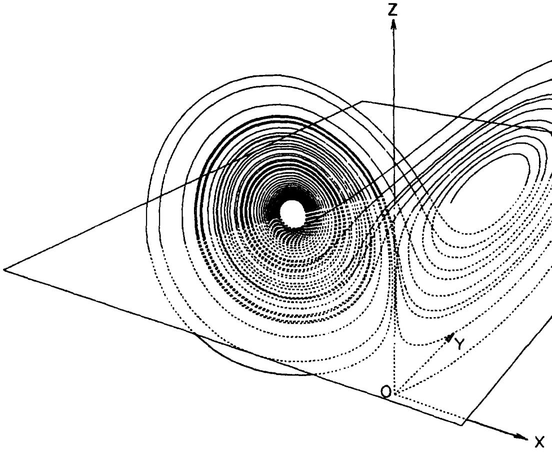


Figure: Poincaré section of the Lorenz system (O. Lanford 1977)

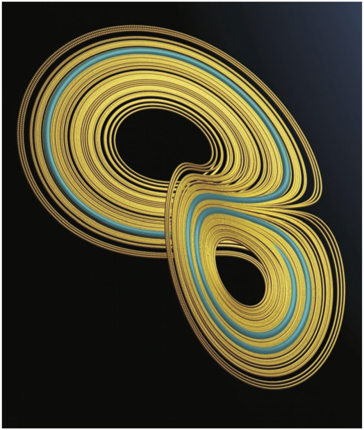


Figure: The Lorenz attractor in yellow and a particular trajectory in blue (E. Ghys 2013)

Motivation: Complex Analysis

Theorem (Friedland, Milnor 1989)

Every polynomial automorphism of $\mathbb{C}^2$ is conjugate by a polynomial automorphism to one of the following maps:

(a) *affine maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k \\ k' \end{pmatrix}$, $ad - bc \neq 0$

(b) *elementary maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax + p(y) \\ by + c \end{pmatrix}$, $ab \neq 0$

(c) *compositions of Hénon maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} p(x) - ay \\ x \end{pmatrix}$, $a \neq 0$

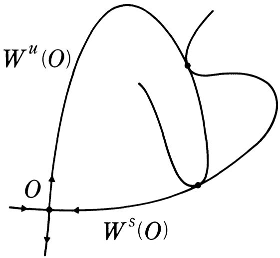
Motivation: Universality

Berger, Palis, Takens: Hénon-like maps

$$f_{c,a}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x^2+c-ay\\x\end{pmatrix}+g(x,y).$$

where g has small norm, appear in unfoldings of homoclinic tangencies between the stable and unstable manifold of a saddle periodic point in dissipative systems with one unstable Lyapunov exponent.

The study of part of the local dynamics in these unfoldings is reduced to the study of the dynamics of Hénon-like maps.



Dynamical sets of a Hénon map

For the standard complex Hénon map $H_{c,a} : \mathbb{C}^2 \to \mathbb{C}^2$

$$H_{c,a} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}, \quad a \neq 0.$$

we define the dynamical objects:

$$K^{\pm} = \text{points in } \mathbb{C}^2 \text{ with bounded forward/backward orbit}$$

$$J^{\pm} = \partial K^{\pm}, \quad K = K^- \cap K^+ \text{ and } J = J^- \cap J^+$$

$$U^{\pm} = \mathbb{C}^2 - K^{\pm} \text{ (escaping sets)}$$

$$J^* = \text{closure of saddle periodic points}$$

The sets J, J^+ and J^- are the Julia sets of the Hénon map.

$J^* \subset J$ is the *small Julia set*. It is open problem whether $J = J^*$ (of J. Hubbard).

Differences between 1D and 2D

Different methods: Many tools from complex analysis and complex dynamics in one variable do not extend to higher dimensions.
Different phenomenons: For example, unlike for 1D polynomials, the number of attracting periodic points (sinks) of a Hénon map is not bounded by the degree of the map!

Theorem (Newhouse Phenomenon)

There exist Hénon maps with infinitely many sinks, accumulating on a Smale's horseshoe.

Theorem (Coexistence Phenomenons - Benedicks, Palmisano)

For every $k \geq 1$ there exists a set of parameters E_k in the parameter space of real Hénon maps such that for every $(a, b) \in E_k$ the map $f_{a,b}$ has at least k attractive periodic orbits and a strange attractor.

Differences between 1D and 2D

1D: Critical points play a fundamental role for the dynamics of polynomials.

- The Julia set of a polynomial is connected if and only if all critical points have bounded forward orbits.
- Easy to plot in the parameter space.
- A polynomial of degree d has at most $d-1$ non-repelling cycles.

2D: The Hénon map is a biholomorphism of $\mathbb{C}^2$, hence it has no critical points in the usual sense.

Böttcher coordinates

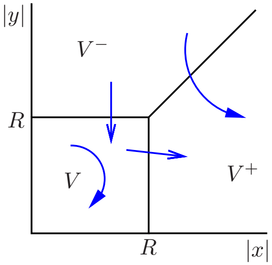


Figure: Dynamical filtration of $\mathbb{C}^2$

The (forward) escaping set is given by $U^+ = \bigcup_{k \geq 0} H^{-\circ k}(V^+)$.

There exists a unique holomorphic function $\varphi^+ : V^+ \to \mathbb{C} - \overline{\mathbb{D}}$,

- $\varphi^+ \circ H = (\varphi^+)^2$
- $\varphi^+(x, y) \sim x$, when $(x, y) \to \infty$ in V^+.

Böttcher coordinates

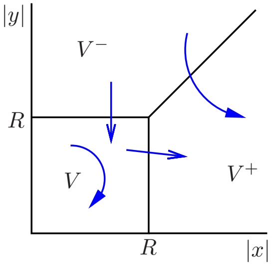


Figure: Dynamical filtration of $\mathbb{C}^2$

The (backward) escaping set is given by $U^- = \bigcup_{k \geq 0} H^{\circ k}(V^-)$.

There exists a unique holomorphic function $\varphi^- : V^- \to \mathbb{C} - \overline{\mathbb{D}}$,

- $(\varphi^-/a) \circ H^{-1} = (\varphi^-/a)^2$
- $\varphi^-(x, y) \sim y$, when $(x, y) \to \infty$ in V^-.

The foliation of U^+

U^+ is foliated by copies of $\mathbb{C}$, which have a natural affine structure.

- φ^+ defines a holomorphic foliation on V^+.
- $(\varphi^+)^{2^k} = \varphi^+ \circ H^{\circ k}$ defines a holomorphic foliation on $H^{-\circ k}(V^+)$.

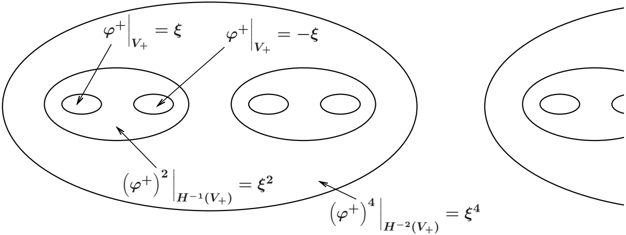


Figure: A fiber $\mathcal{F}_\xi$ of the foliation of U^+

Critical locus

The critical locus $\mathcal{C}$ is the set of tangencies between the foliation of the escaping set U^+ and the foliation of U^-.

The set $\mathcal{C}$ is a nonempty, closed analytic subvariety of $U^+ \cap U^-$ and is invariant under the Hénon map.

Stable/unstable critical loci

Unstable Critical Locus $\mathcal{C}^u$: the set of tangencies between the foliation of U^+ and the “lamination” of J^-.

Stable Critical Locus $\mathcal{C}^s$: set of tangencies between the foliation of the escaping set U^- and the “lamination” of J^+.
J^+ and J^- are not always laminar.

Theorem (Bedford, Smillie)

$$\overline{\mathcal{C}} \cap J^+ \cap U^- \neq \emptyset \text{ and } \overline{\mathcal{C}} \cap J^- \cap U^+ \neq \emptyset.$$

It is not true that $\mathcal{C}^s = \partial\mathcal{C} \cap (J^+ \cap U^-)$ and $\mathcal{C}^u = \partial\mathcal{C} \cap (J^- \cap U^+)$.

The relation between $\mathcal{C}$, $\mathcal{C}^s$ and $\mathcal{C}^u$ is "rather mysterious is general".

General Questions

- General properties of the critical locus $\mathcal{C}$.
  Is the critical locus smooth or can it have singularities?
- Topological models of the critical locus $\mathcal{C}$.
- Relations between the critical loci $\mathcal{C}$, $\mathcal{C}^s$, $\mathcal{C}^u$.
- Connections between the properties of the critical locus and the dynamical properties of the map, and the connectivity of the Julia set J.

Unstable connectivity

Theorem (Bedford, Smillie)

For a dissipative complex Hénon map the following are equivalent:

- J is connected.
- K is connected.
- $W^u(p) \cap K^+$ is connected for some saddle periodic point p.
- $W^u(p) \cap K^+$ is connected for any saddle periodic point p.

Remark The unstable manifold of any saddle periodic point is dense in J^-.

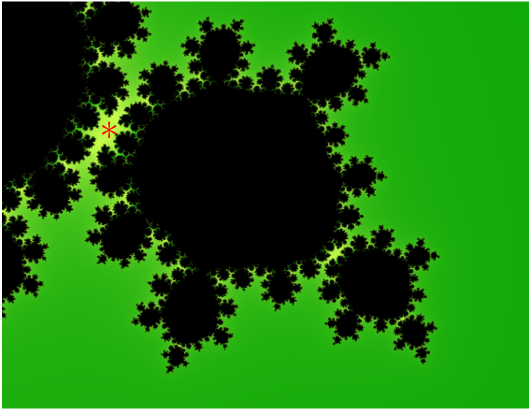


Figure: The unstable manifold of the hyperbolic fixed point $\mathbf{q}$ of a complex Hénon map $H_{c,a}(x,y) = (x^2 + x + ay, ax)$ with a semi-Siegel fixed point with an eigenvalue $\lambda = e^{2\pi i\alpha}$, $\alpha = (\sqrt{5}-1)/2$ and small Jacobian. The black region represents K^+ restricted to the unstable manifold. We notice an unstable critical point (in red) of the Green function $G^+|_{W^u(\mathbf{q})}$ and that $W^u(\mathbf{q}) \cap K^+$ is disconnected.

Models for the Critical locus I

Theorem (Lyubich, Robertson)

Let H be a Hénon map that is a small perturbation of a hyperbolic quadratic polynomial p with connected Julia set.

- There exists a unique primary component $\mathcal{C}_0$ of the *critical locus*, which is asymptotic to the x-axis.
  $\mathcal{C}_0$ is biholomorphic to $\mathbb{C} - \overline{\mathbb{D}}$ and it is everywhere transverse to the foliations of U^+ and U^-. Its boundary is homeomorphic to the Julia set J_p.
- All other components of $\mathcal{C}$ are iterates of $\mathcal{C}_0$ under H.

So the Julia set is connected & the critical locus in disconnected.

Models for the Critical locus II

Theorem (Firsova)

Let H be a Hénon map that is a small perturbation of a hyperbolic quadratic polynomial with disconnected Julia set. The critical locus is connected, smooth and homeomorphic to a Riemann surface which is a countable collection of truncated spheres with countably many handles, glued by dynamics.

Remark: this model was conjectured by J. Hubbard.

So the Julia set is disconnected & the critical locus in connected.

General Question: The Julia set J of the Hénon map is connected if and only if the critical locus $\mathcal{C}$ is disconnected.

From each hemisphere we remove a countable collection of disks and a Cantor set. We further remove a point from the equator. The resulting topological object is a *truncated sphere*.

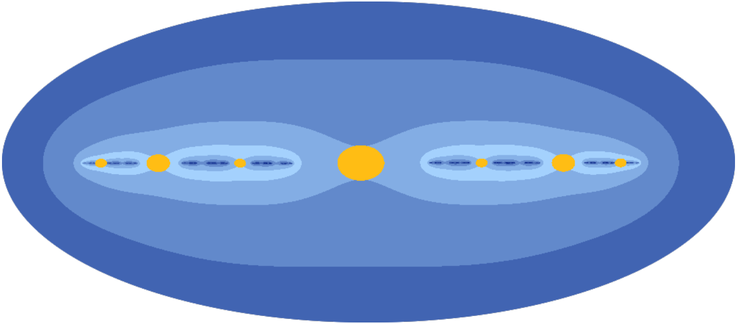


Figure: A hemisphere of a truncated sphere.

Application: description of J

We used the Lyubich-Robertson critical locus as a common transverse to the foliation of U^+ and the lamination of J^+ to extend the Hubbard-Oberste-Vorth analytic structure of the escaping set U^+ to the boundary and describe the Julia set J^+:

Theorem (T.)

Consider complex Hénon maps that are singular perturbations of a hyperbolic polynomial with connected Julia set. The Julia set J^+ is homeomorphic to the quotient of $\mathbb{S}^1 \times \mathbb{C}$ by a discrete group of automorphisms isomorphic to $\mathbb{Z}[1/2]/\mathbb{Z}$ and an equivalence relation.

Critical locus in the Horseshoe Region

The Hénon map is a complex horseshoe in the HOV region

$$\mathsf{HOV} = \left\{|c| > 2(1+|a|)^2\right\}$$

For $\beta = (5+2\sqrt{5})/4$ this was proved by Oberste-Vorth in his thesis.
In particular, in the Horseshoe Region, the Hénon map is hyperbolic, J is a Cantor set and $J = J^*$.

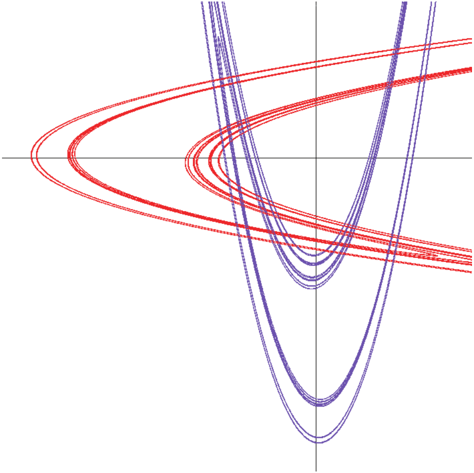


Figure: A Hénon horseshoe (by H. Dullin)

For $\beta \geq 2$ we look at HOV-like regions from the Horseshoe Region:

$$\mathsf{HOV}_\beta = \left\{(c,a) \in \mathbb{C}^2 : |c| > \beta(1+|a|)^2\right\}.$$

Theorem (Firsova, Radu, T.)

There exists $\beta > 2$ (for example, $\beta = 18.75$) such that in the HOV_β region, the critical locus is connected, smooth and homeomorphic to a Riemann surface which is a countable collection of truncated spheres with countably many handles, glued by dynamics.

Remark: As expected, since HOV_β contains small perturbations, the model is the same as the Firsova model for small perturbations, but in this theorem there are no restrictions on the Jacobian. The proof is based on non-perturbative, non-numerical techniques. The bound on β is not optimal and can be improved.

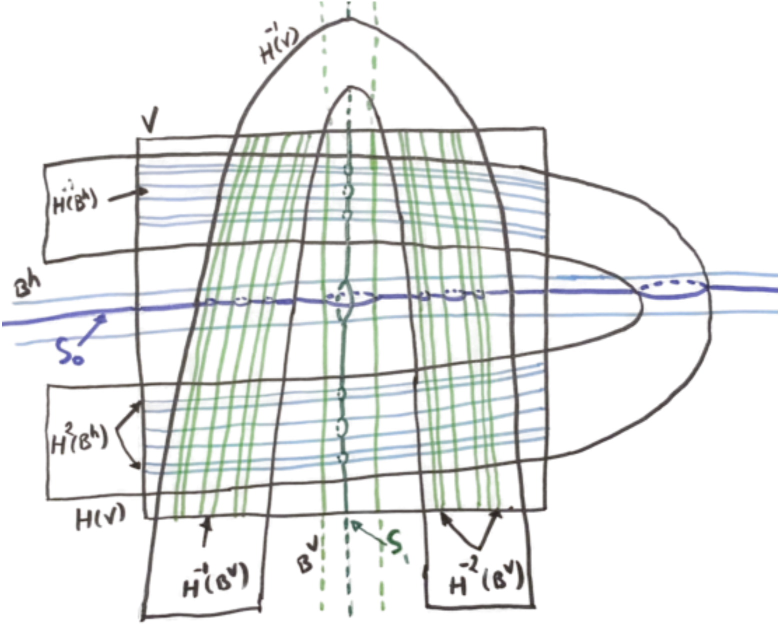


Figure: The punctured disks S_0 and S_1 are part of the critical locus

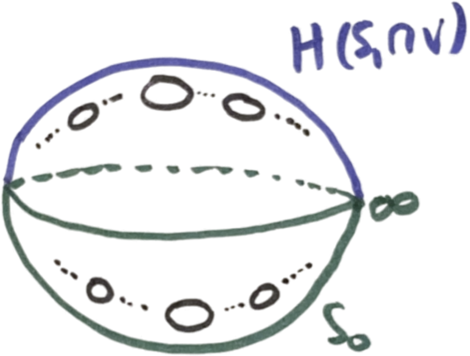


Figure: Truncated sphere

Parabolic implosion

Let $\mathcal{P}_\lambda$ denote the set of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map has a fixed point with an eigenvalue λ. In particular, $\mathcal{P}_1 : c = (1+a)^2/4$ is the curve of Hénon maps with a semi-parabolic fixed point with an eigenvalue 1. It passes through the tip of the Mandelbrot set at $c = 1/4$.

The parametric region

$$\mathcal{H}_{\delta,\delta'} = \left\{(c, a) \in \mathcal{P}_{1+it} \,:\, 0 < |a| < \delta \;\text{ and }\; -\delta' < t < \delta',\; t \neq 0\right\}$$

sits at the right of the semi-parabolic curve $\mathcal{P}_1$.

Theorem (Bedford, Smillie, Ueda)

The sets J, J^+, K and K^+ vary discontinuously with the parameters as $t \to 0$, while J^- and K^- vary continuously.

The phenomenon described is a 2D analogue of parabolic implosion in 1D (Lavaurs, Douady, Hubbard, etc.).

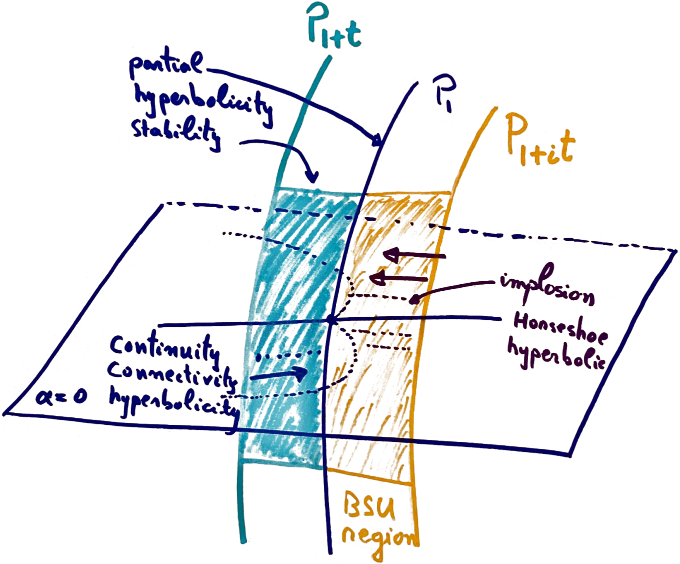


Figure: Semi-parabolic implosion from the right (BSU), continuity of J & J^+ from the left (Radu-T.)

Using the Radu-T. characterization of semi-parabolic Hénon maps and their nearby perturbations, the Lyubich-Robertson description of the critical locus holds for semi-parabolic Hénon maps from $\mathcal{P}_1$ with small Jacobian and throughout the corresponding parametric region to the left of $\mathcal{P}_1$.

Question: Does the critical locus varies discontinuously with the parameters as we approach the semi-parabolic curve $\mathcal{P}_1$ from the horseshoe region?

Theorem (Firsova, Radu, T. – work in progress)

The model of the critical locus holds throughout the region $\mathcal{H}_{\delta,\delta'}$.

Thank you!