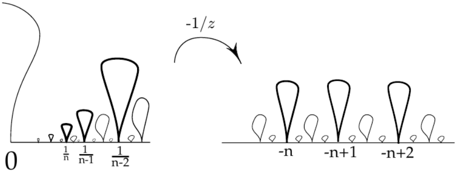
Complex rotation numbers and renormalization

Nataliya Goncharuk, University of Toronto
natalia.goncharuk@utoronto.ca

MSRI Connections Workshop: Complex Dynamics —
from special families to natural generalizations in one and several variables



February 3, 2022

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\operatorname{rot} f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\operatorname{rot} f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\operatorname{rot} f$ if f is C^2-smooth.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\operatorname{rot} f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\operatorname{rot} f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\operatorname{rot} f$ if f is C^2-smooth.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\operatorname{rot} f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\operatorname{rot} f \in \mathbb{R}\setminus\mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\operatorname{rot} f$ if f is C^2-smooth.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

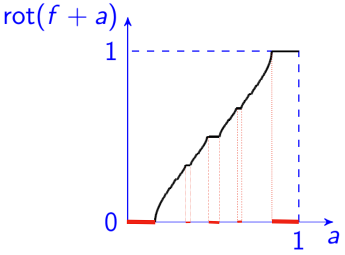
- $\mathrm{rot}\, f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\mathrm{rot}\, f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\mathrm{rot}\, f$ if f is C^2-smooth.

Rotation number of a circle diffeomorphism.

Let F be a lift of $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\operatorname{rot} f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\operatorname{rot} f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\operatorname{rot} f$ if f is C^2-smooth.

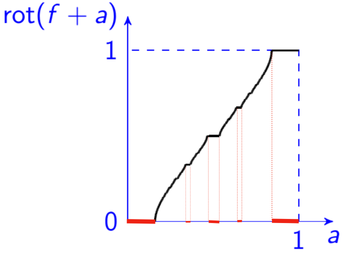


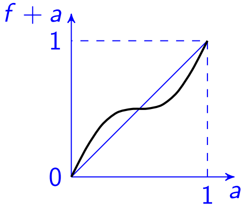
Rotation number of a circle diffeomorphism.

Let F be a lift of $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\text{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\text{rot}\, f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\text{rot}\, f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\text{rot}\, f$ if f is C^2-smooth.





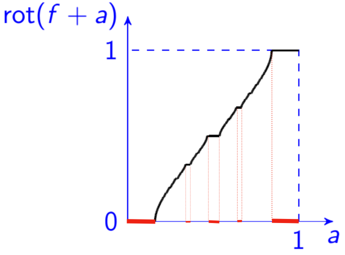
Hyperbolic: $(f + a)'(\text{f.p.}) \neq 1$

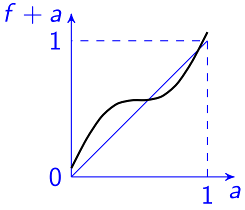
Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\mathrm{rot}\, f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\mathrm{rot}\, f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\mathrm{rot}\, f$ if f is C^2-smooth.





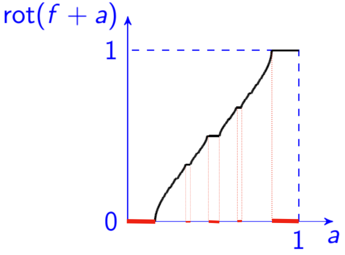
Hyperbolic: $(f+a)'(\text{f.p.}) \neq 1$

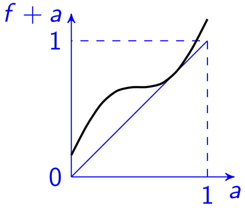
Rotation number of a circle diffeomorphism.

Let F be a lift of $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\operatorname{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\operatorname{rot} f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\operatorname{rot} f \in \mathbb{R} \setminus \mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\operatorname{rot} f$ if f is C^2-smooth.





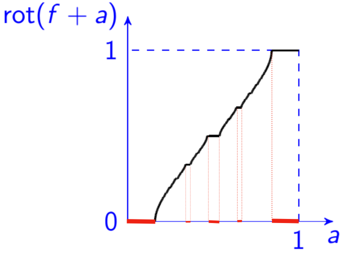
Parabolic: $(f + a)'(\text{f.p.}) = 1$

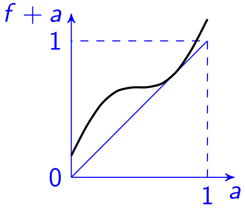
Rotation number of a circle diffeomorphism.

Let F be a lift of $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{\#\text{ turns around } \mathbb{R}/\mathbb{Z} \text{ under } n \text{ iterates}}{n}$$

- $\mathrm{rot}\, f \in \mathbb{Q} \Leftrightarrow f$ has a periodic orbit
- **[Denjoy]** $\mathrm{rot}\, f \in \mathbb{R}\setminus\mathbb{Q} \Leftrightarrow f$ is continuously conjugate to the rotation by $\mathrm{rot}\, f$ if f is C^2-smooth.



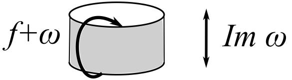


↑Stairs Parabolic: $(f+a)'(\text{f.p.}) = 1$

Arnold's construction (1978)

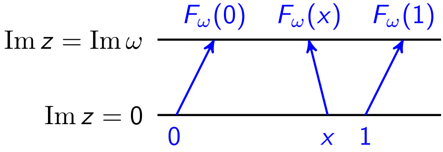
Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 \le \operatorname{Im} z \le \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

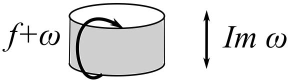


Arnold's construction (1978)

Let $f:\mathbb{R}/\mathbb{Z}\mapsto\mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

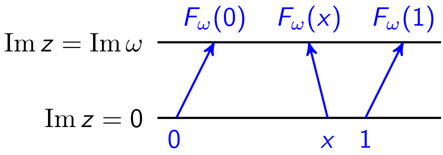


- Idea: let us add a *complex* shift to f, $f_\omega = f+\omega$.
- Take the quotient space of the annulus $0\le \mathrm{Im}\, z\le \mathrm{Im}\,\omega$ in $\mathbb{C}/\mathbb{Z}$ by $x\mapsto f(x)+\omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f+\omega$.

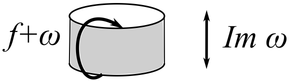


Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

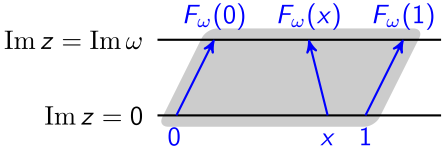


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 \le \operatorname{Im} z \le \operatorname{Im}\omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

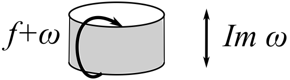


Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

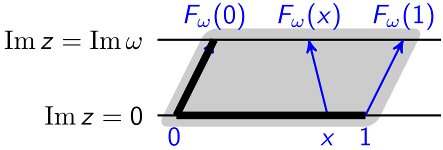


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 \le \operatorname{Im} z \le \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

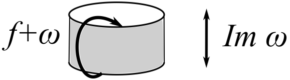


Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

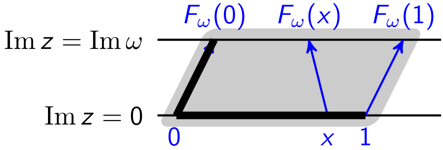


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 \le \operatorname{Im} z \le \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.

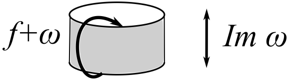


Arnold's construction (1978)

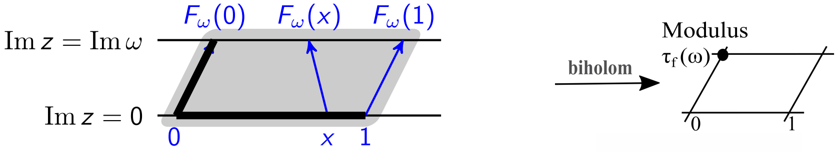
Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



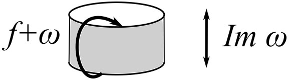
- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 \leq \operatorname{Im} z \leq \operatorname{Im} \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.



Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.

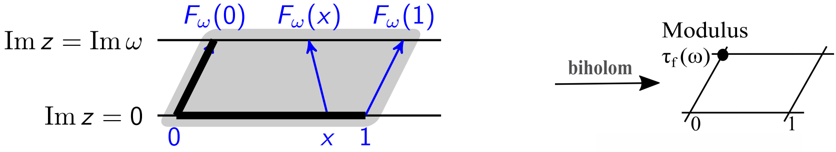


- Idea: let us add a *complex* shift to f, $f_\omega = f + \omega$.
- Take the quotient space of the annulus $0 \leq \mathrm{Im}\, z \leq \mathrm{Im}\, \omega$ in $\mathbb{C}/\mathbb{Z}$ by $x \mapsto f(x) + \omega$.
- We obtain a *complex torus* with marked generators.
- Consider its modulus $\tau_f(\omega)$ — the **complex rotation number** of $f + \omega$.



Arnold's construction (1978)

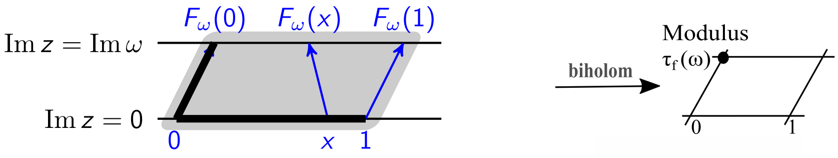
Let $f\colon \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's question, 1978:** what happens as $\omega \to a \in \mathbb{R}$?

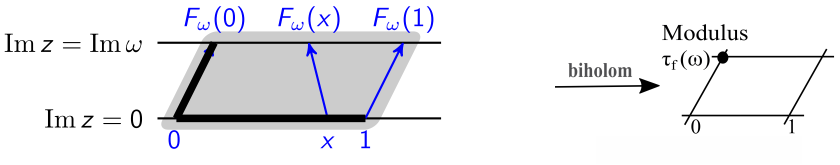
Arnold's construction (1978)

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's question, 1978:** what happens as $\omega \to a \in \mathbb{R}$?

Let $f : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ be an analytic circle diffeomorphism.



- **Example:** If $f(x)$ is a rotation by ϕ, then $\tau_f(\omega) = \omega + \phi$.
- **Remark:** τ_f is holomorphic.
- **Arnold's question, 1978:** what happens as $\omega \to a \in \mathbb{R}$?

Bubbles: overview of results

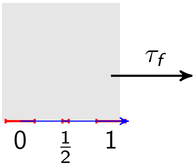
$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.

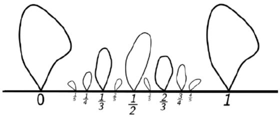
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



↑ Stairs

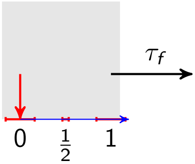


Limit values of τ_f on $\mathbb{R}$.

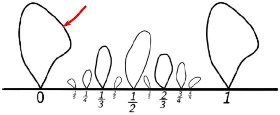
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



↑ Stairs

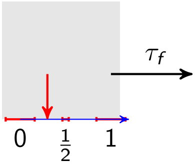


Limit values of τ_f on $\mathbb{R}$.

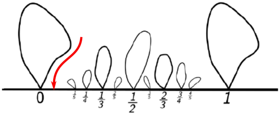
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



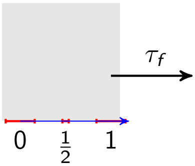
↑ Stairs



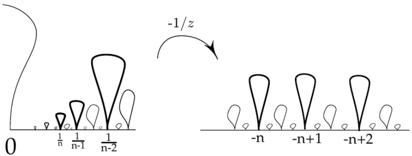
Limit values of τ_f on $\mathbb{R}$.

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \text{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



↑ Stairs

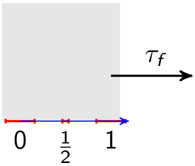


Limit values of τ_f on $\mathbb{R}$.

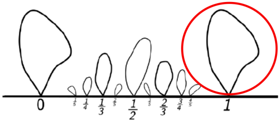
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \text{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



↑ Stairs

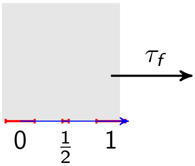


Limit values of τ_f on $\mathbb{R}$.

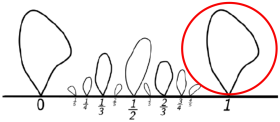
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



↑ Stairs

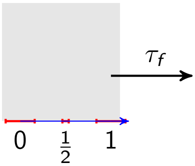


Limit values of τ_f on $\mathbb{R}$.

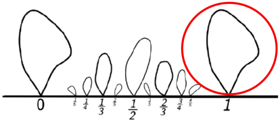
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. **(Stairs)**
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. **(Outside stairs)**
- Bubbles are (generically) self-similar near rational points.
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.
- Near Diophantine numbers, the bubbles are much smaller.



↑ Stairs

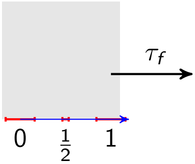


Limit values of τ_f on $\mathbb{R}$.

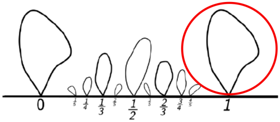
Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis; NG
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. Risler; Moldavskis; Lacroix; X.Buff, NG
- Bubbles are (generically) self-similar near rational points. NG
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$. X.Buff, NG
- Near Diophantine numbers, the bubbles are much smaller. NG, I.Gorbovickis



↑ Stairs

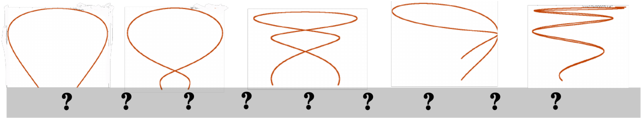


Limit values of τ_f on $\mathbb{R}$.

Bubbles: overview of results

$\tau_f : \mathbb{H} \to \mathbb{H}$ extends continuously to $\mathbb{R}$. X.Buff, NG Let $\hat{\tau}_f(a) := \lim_{\omega \to a} \tau_f(\omega)$.

- $f + a$ is hyperbolic $\Rightarrow \hat{\tau}_f(a) \in \mathbb{H}$. Ilyashenko, Moldavskis; NG
- Otherwise, $\hat{\tau}_f(a) = \mathrm{rot}(f + a)$. Risler; Moldavskis; Lacroix; X.Buff, NG
- Bubbles are (generically) self-similar near rational points. NG
- Size of the $\frac{p}{q}$-bubble is at most $\frac{C}{q^2}$.X.Buff, NG
- Near Diophantine numbers, the bubbles are much smaller. NG, I.Gorbovickis



Zero bubbles for perturbations of $z \mapsto \frac{az+b}{cz+d}$, approximation.

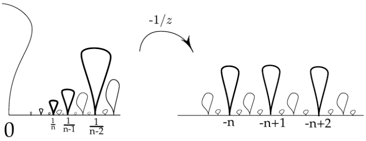
Self-similarity of bubbles

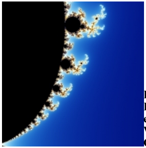
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

$\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

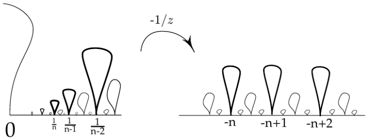
Self-similarity of bubbles

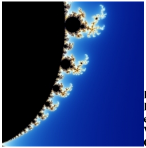
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$

Lavaurs maps — through the eggbeater

$\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed
CC-BY-SA 3.0

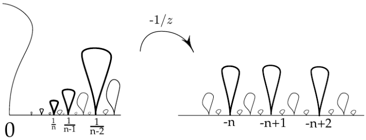
Self-similarity of bubbles

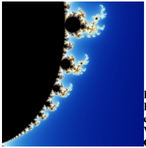
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

$\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed
CC-BY-SA 3.0

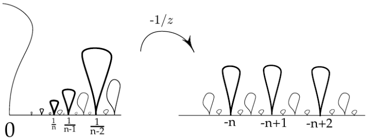
Self-similarity of bubbles

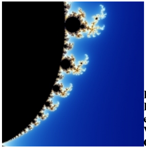
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

$\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





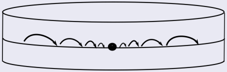
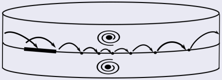
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

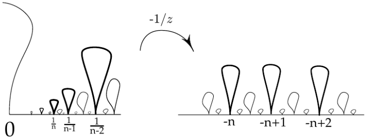
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

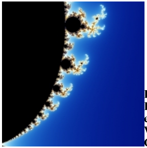
$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

$\mathcal{R}(f+a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





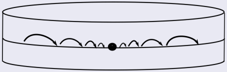
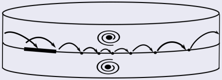
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed
CC-BY-SA 3.0

Self-similarity of bubbles

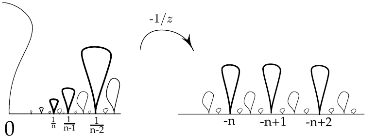
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

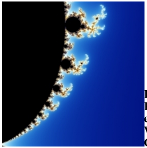
$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

$\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





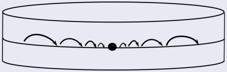
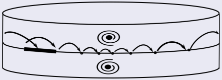
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

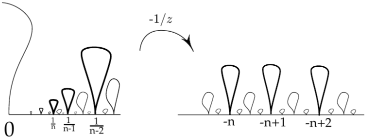
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

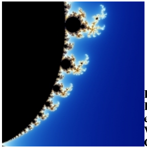
$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

$\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.





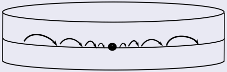
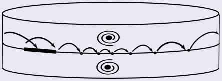
Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Self-similarity of bubbles

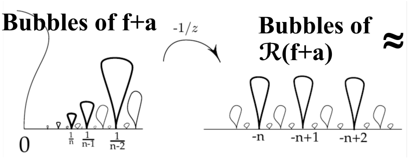
Renormalization $\mathcal{R}f$ is the first-return map under f to the circle $[0, f(0)]/f$.

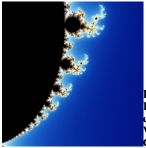
$$\hat{\tau}(\mathcal{R}f) = -\frac{1}{\hat{\tau}(f)} \mod 1.$$

Lavaurs maps — through the eggbeater

Dynamics of f  and $f + a$, $a > 0$: 

$\mathcal{R}(f + a) \to L_c$ as $a \to 0$ where L_c are Lavaurs maps, $c \in \mathbb{R}/\mathbb{Z}$.

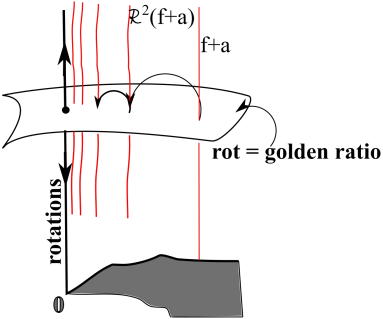




Elephant Valley
Produced by Ultra Fractal 3
en.wikipedia.org/wiki/Mandelbrot set,
Wolfgang Beyer, zoomed

Renormalization and bubbles

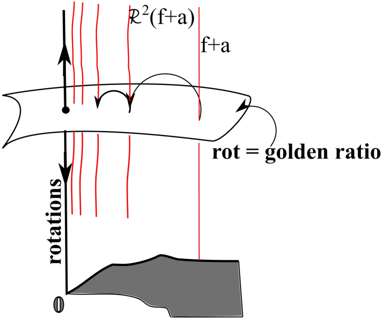
- Golden ratio rotation is a hyperbolic fixed point for $\mathcal{R}^2$
- $\Rightarrow$ bubbles are small near the golden ratio (Gorbovickis, NG; in progress).
- Do critical maps have bubbles? Are they self-similar?



Thank you for your attention!

Renormalization and bubbles

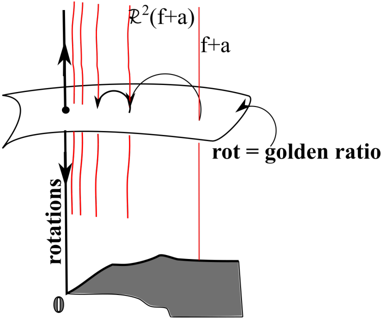
- Golden ratio rotation is a hyperbolic fixed point for $\mathcal{R}^2$
- $\Rightarrow$ bubbles are small near the golden ratio (Gorbovickis, NG; in progress).
- Do critical maps have bubbles? Are they self-similar?



Thank you for your attention!

Renormalization and bubbles

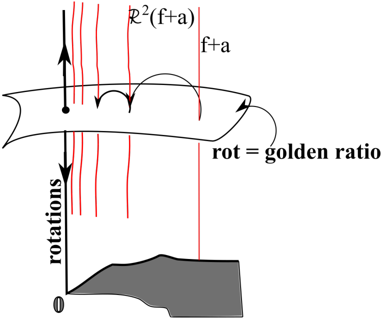
- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Brjuno numbers (Gorbovickis, NG; in progress).
- Do critical maps have bubbles? Are they self-similar?



Thank you for your attention!

Renormalization and bubbles

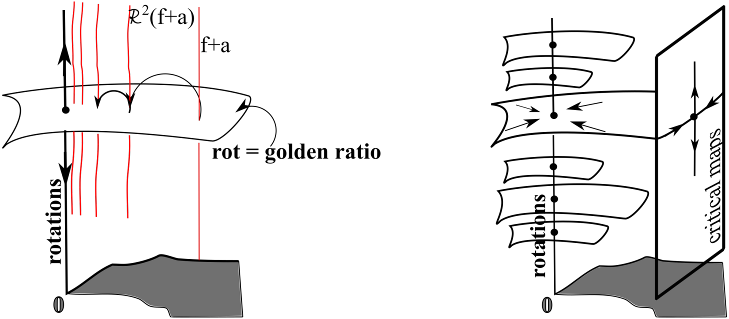
- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Brjuno numbers (Gorbovickis, NG; in progress).
- Do critical maps have bubbles? Are they self-similar?



Thank you for your attention!

Renormalization and bubbles

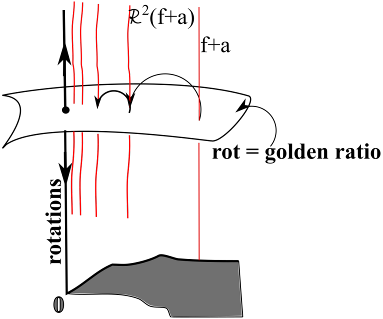
- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Brjuno numbers (Gorbovickis, NG; in progress).
- Do critical maps have bubbles? Are they self-similar?

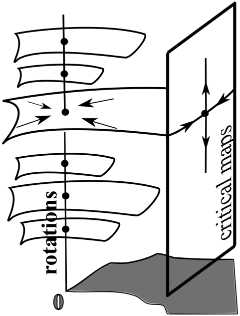


Thank you for your attention!

Renormalization and bubbles

- Brjuno rotations are a hyperbolic set for $\mathcal{R}$ (joint with M. Yampolsky).
- $\Rightarrow$ bubbles are small near Brjuno numbers (Gorbovickis, NG; in progress).
- Do critical maps have bubbles? Are they self-similar?





Thank you for your attention!