TRANSCENDENTAL ENTIRE FUNCTIONS WITH CANTOR BOUQUET JULIA SETS

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(joint work with L. Rempe)

MSRI, 4th February, 2022

The University of Manchester

Connections Workshop: Complex Dynamics

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map.

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In particular,

$$\boxed{J(f) = \partial I(f)}.$$

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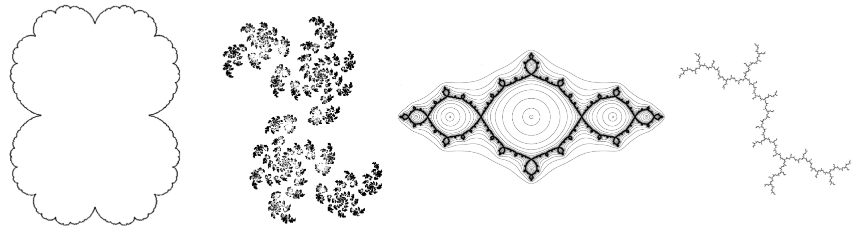

*Pictures from Wikimedia commons.

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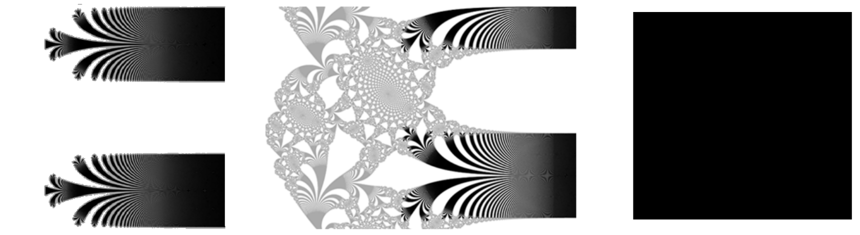

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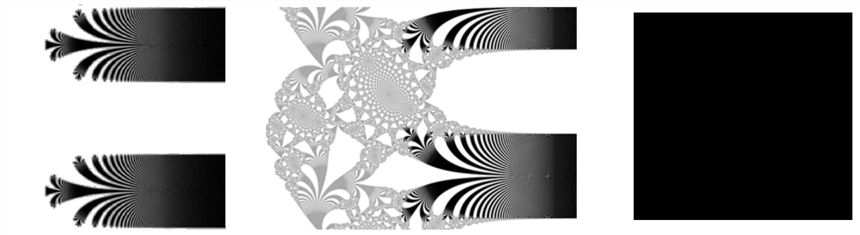

Goal: To describe (the dynamics of entire maps on their) Julia sets.

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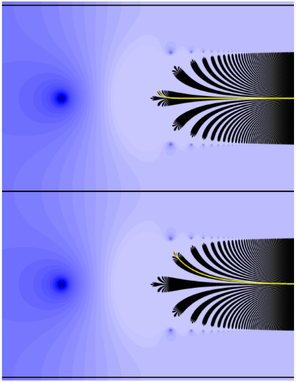

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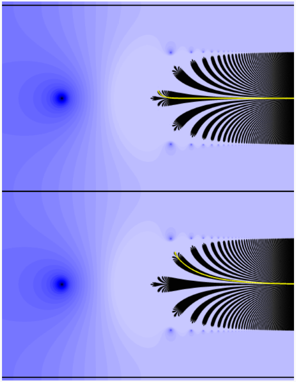
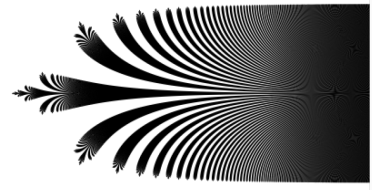


- These curves are known as (Devaney) hairs or dynamic rays.

Definition

$J(f)$ is a **Cantor bouquet** if

- Every conn. comp. of $J(f)$ is an arc to infinity, called hair;
- $J(f)$ is topologically straight, i.e., there is a homeo. $\varphi\colon \mathbb{C} \to \mathbb{C}$ such that the image of every hair is a straight horizontal line.



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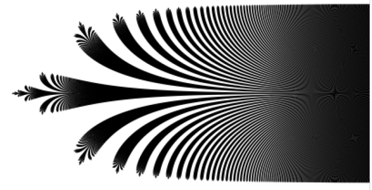
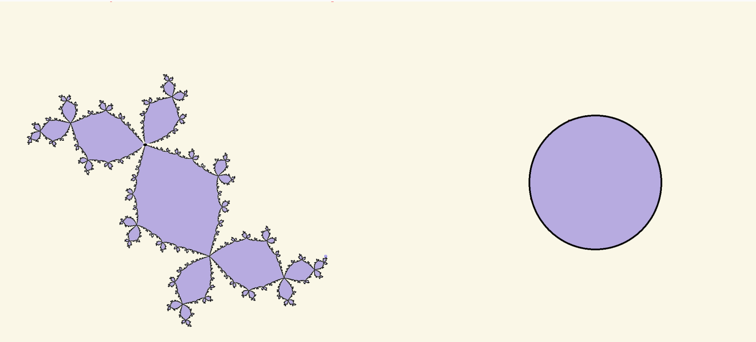


Theorem (Aarts–Oversteegen, ’93)

The Julia set of any $\lambda \sin(z)$ with $\lambda \in (0,1)$ and $\mu \exp(z)$ with $\mu \in (1, 1/e)$ is a Cantor bouquet.

⋆ Hairs of transcendental maps as analogues of external rays of polynomials.



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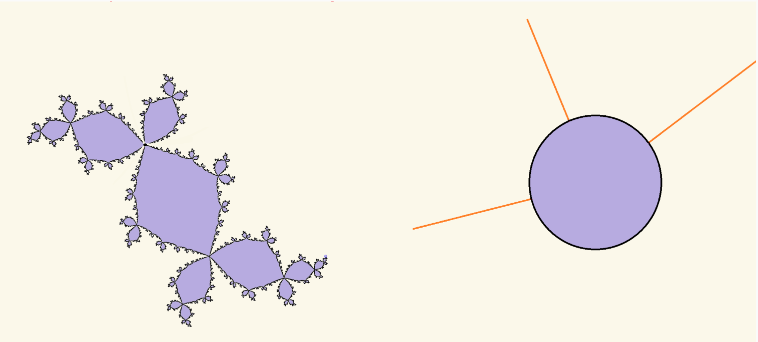

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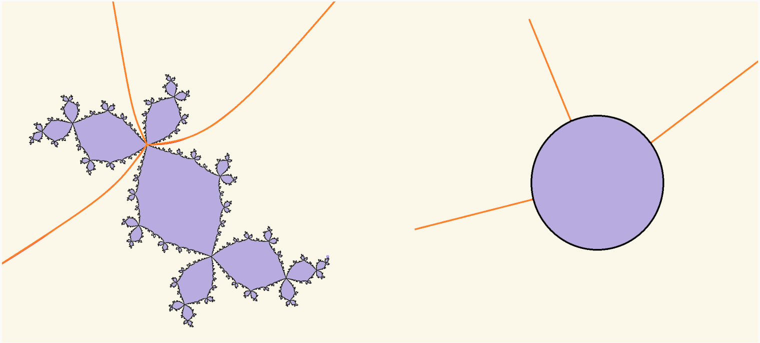

-**External rays** for p_d as preimages of straight lines under Bottcher's map.

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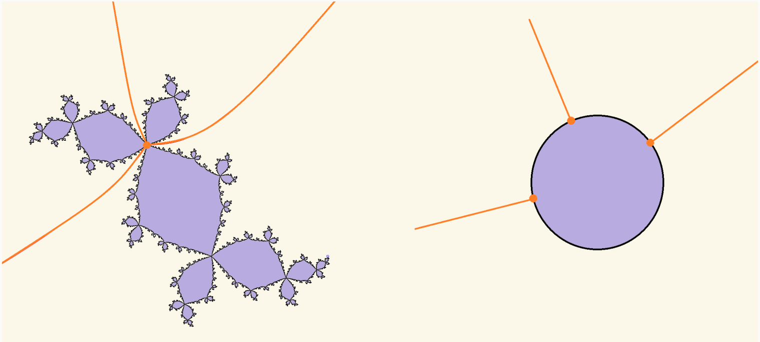

-When $J(p_d)$ is connected and locally connected, the conjugacy extends to Julia sets and dynamic rays **land.**

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Definition (Benini-Rempe '20)

An entire function f is **criniferous** if for every $z \in I(f)$ and for all sufficiently large n, there is an arc γ_n connecting $f^n(z)$ to ∞, such that

- f maps γ_n injectively onto γ_{n+1};
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Remark: If f is criniferous, then every $z \in I(f)$ can be connected to infinity by a curve of escaping points.

The set of **singular values** $S(f)$ is the smallest closed subset of $\mathbb{C}$ such that $f\colon \mathbb{C}\setminus f^{-1}(S(f)) \to \mathbb{C}\setminus S(f)$ is a covering map.

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- $\mathcal{B}_{RRRS}$: maps in $\mathcal{B}$ that satisfy a (uniform) head-start condition.
  [RRRS]. UHSC

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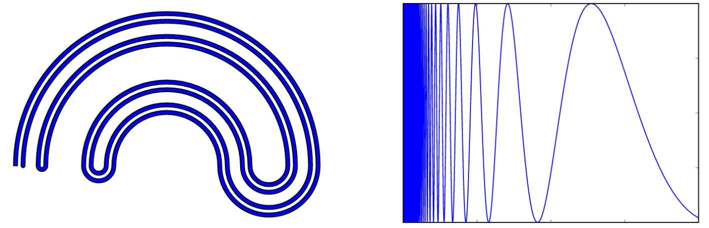

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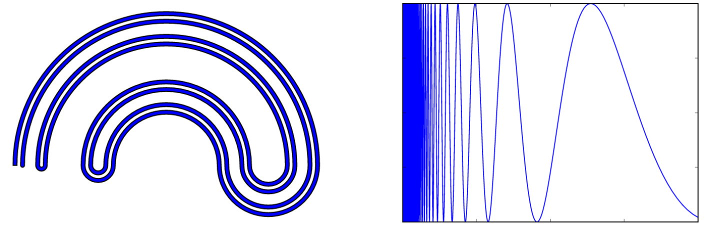
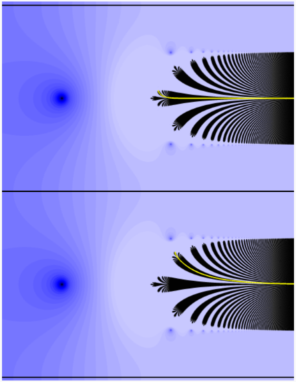


- Alternative to rays: *dreadlocks* [Benini-Rempe '20].

DISJOINT TYPE MAPS



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If $f \in \mathcal{B}$, then λf is of disjoint type for $|\lambda|$ sufficiently small.

- ⋆ λf is in the *parameter space* of f.
- ⋆ The dynamics of λf and f are related near infinity by some analogue of Böttcher's Theorem. [Rempe '09] Conjugacy

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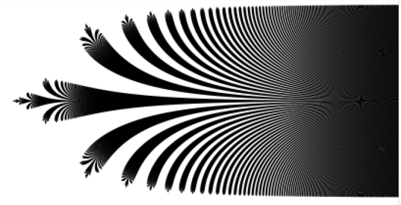
Theorem (Barański-Jarque-Rempe '12)

If $f \in \mathcal{B}$ is of finite order and of disjoint type, then $J(f)$ is a *Cantor bouquet.*



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Theorem A (P.- Rempe)

There is $f \in \mathcal{B}$ criniferous and of disjoint type such that $J(f)$ is <u>not</u> a Cantor bouquet.

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5. If $x \in X$ is accessible from $\mathbb{C} \setminus X$, then x is an endpoint of X. (Equivalently, **every hair** of X is **accumulated** on **by other hairs** from both sides.)

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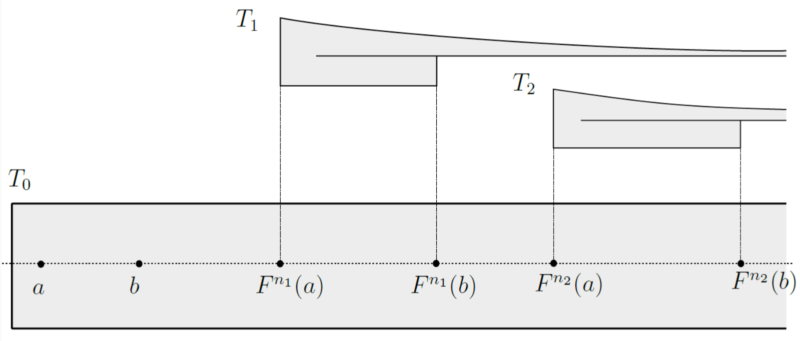
We *design* hooked thin tracts.



FURTHER RESULTS

Definition

We say that a subset $A \subset J(f)$ is **absorbing** if it is forward-invariant, every escaping point eventually enters A; i.e.

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Theorem B (P.-Rempe)

Let $f \in \mathcal{B}$ be of disjoint type. The following are equivalent.

(a) $J(f)$ is a Cantor bouquet.

(b) There is an absorbing Cantor bouquet $X \subset J(f)$.

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Theorem D (P.-Rempe)

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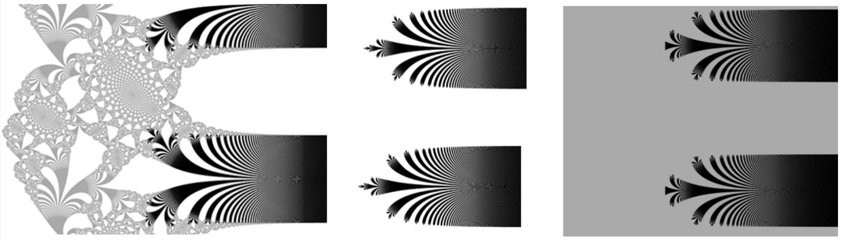

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THANKS FOR YOUR ATTENTION!

Theorem (Rempe '09)

Let $f \in \mathcal{B}$ and let $g := \lambda f$ be of disjoint type. Then there is $R > 0$ and a continuous map

$$\vartheta \colon \{z \in J(g) : |g^n(z)| \geq R \text{ for all } n \geq 1\} \to J(f)$$

such that $\vartheta \circ g = f \circ \vartheta$ and is a homeomorphism onto its image.

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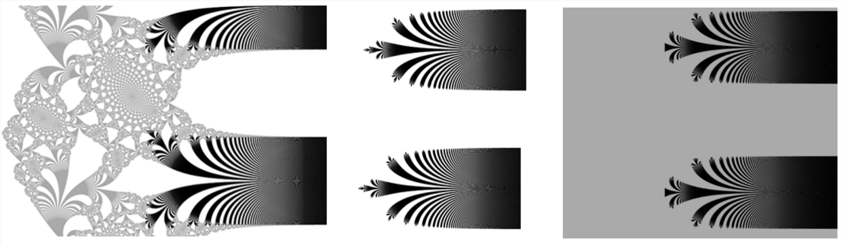

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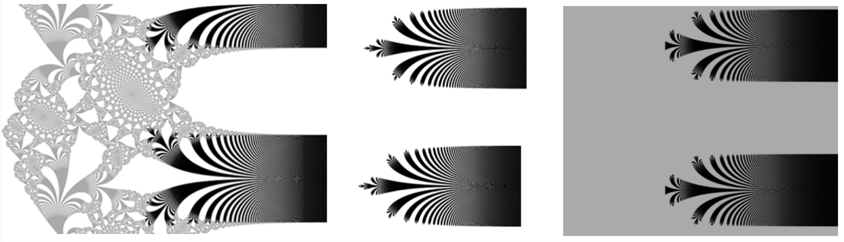

Disjoint type

Definition (Uniform head-start condition)

Let $f \in \mathcal{B}$. We say that f satisfies a **uniform head-start condition (with respect to $|z|$) on its Julia set** if there is an upper semicontinuous function $\varphi\colon [0,\infty) \to [0,\infty)$ with the following properties for all points z and w belonging to the same component of $J(f)$.

(i) If $|w| > \varphi(|z|)$, then $|f(w)| > \varphi(|f(z)|)$.

(ii) If $z \neq w$, then there is $n \geq 0$ such that either $|f^n(w)| > \varphi(|f^n(z)|)$ or $|f^n(z)| > \varphi(|f^n(w)|)$.

Note that the conditions imply, in particular, that $\varphi(t) > t$ for all t.

Criniferous