Simply connected wandering domains

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MSRI

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Introduction

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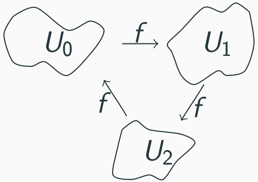
The connected components of the Fatou set, which map into each other, are called **Fatou components**.

Fatou components

Fatou components can be:



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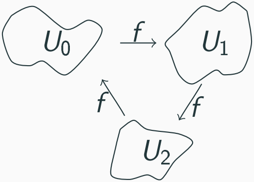
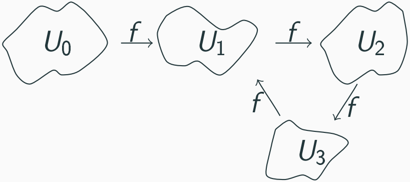


preperiodic



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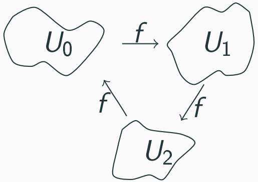

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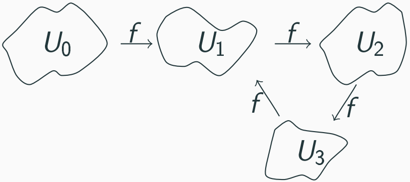
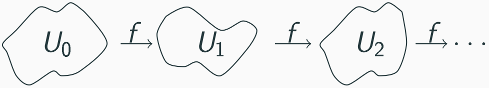


wandering domains



Definition
Let U be a Fatou component of f. If $f^n(U) \cap f^m(U) = \emptyset$, for all $m, n \in \mathbb{N}$, with $m \neq n$ then U is a **wandering domain**.

An example of a wandering domain

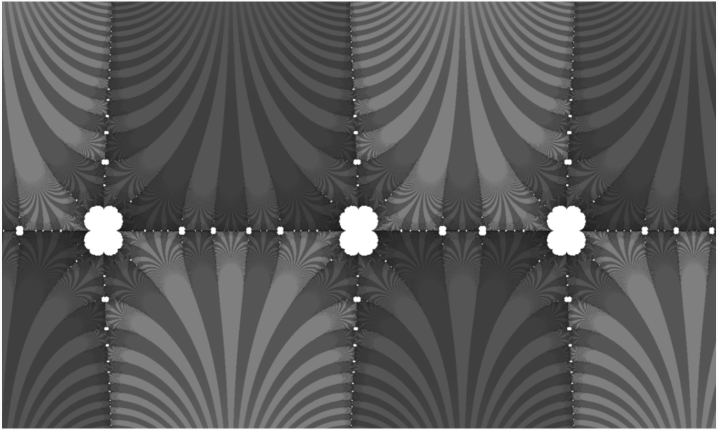


Figure 1: The dynamics of the function $f(z) = z + 2\pi \sin z$ (picture by Lasse Rempe).

Three types of wandering domains

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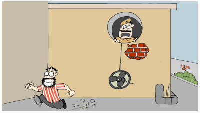

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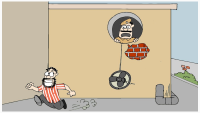

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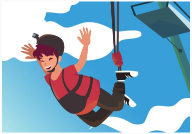

oscillating, if there exist $(n_k), (m_k)$ such that $f^{n_k}(z) \to \infty$ and $(f^{m_k}(z))$ stays bounded for all $z \in U$;

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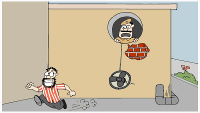

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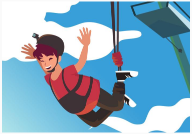

oscillating, if there exist $(n_k), (m_k)$ such that $f^{n_k}(z) \to \infty$ and $(f^{m_k}(z))$ stays bounded for all $z \in U$;

bounded (orbit) if $(f^n(z))$ stays bounded for all $z \in U$.

Early results

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Baker (1984) was the first to give an example of a transcendental entire function with a wandering domain. The wandering domain in Baker's example was multiply connected.

A detailed description of the dynamics in multiply connected wandering domains was given by Bergweiler, Rippon and Stallard in 2011.

Why wandering domains?

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- Wandering domains are the least understood of all Fatou components.
- There are several big open questions in Holomorphic Dynamics concerning wandering domains.

Our project

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Classifying simply connected wandering domains

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We obtained a nine-way classification of simply connected wandering domains

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- in terms of hyperbolic distances between orbits of points and
- in terms of converging to the boundary.

First classification theorem

Theorem. Let U be a simply connected wandering domain and suppose $z, w \in U$ have distinct orbits. Then there are three possibilities.

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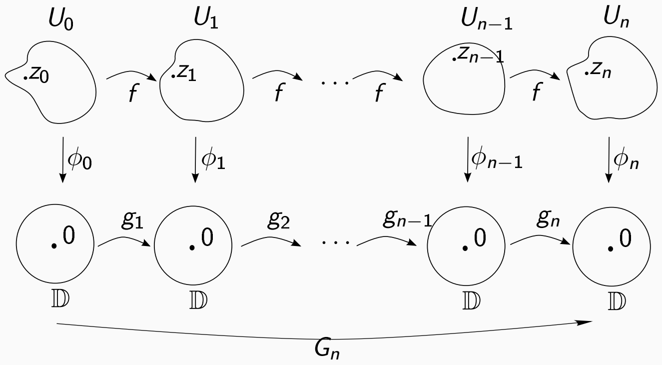
3. U is **eventually isometric**: for all such pairs $z, w \in U$, $d_{U_n}(f_n(z), f_n(w))$ is eventually constant.

Idea of the proof



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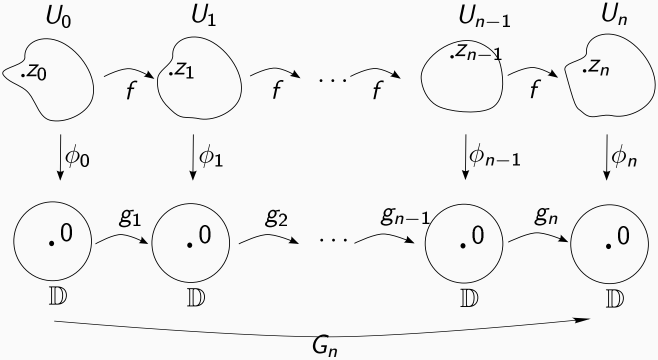

- $d_{U_n}(f_n(z), f_n(z_0)) = d_{\mathbb{D}}(G_n(w), 0)$
- $d_{\mathbb{D}}(G_n(w), 0) \to 0 \Rightarrow d_{\mathbb{D}}(G_n(w), G_n(w')) \to 0$

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(b) **bungee** For all $z \in U$, there is a subsequence $f_{n_k}(z)$ which converges to ∂U_{n_k} and a subsequence which stays away; or

(c) **converging** For all $z \in U$, $f_n(z)$ converges to ∂U_n.

All points together

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⚠ The behaviour of two points/one point in the wandering domain determines the type of the wandering domain with respect to the first/second classification.

For example, if the orbit of one point converges to the boundary of the wandering domain then all internal orbits do.

Possible types of wandering domains

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type	away	bungee	converging
contracting	x		x
semi-contracting			
eventually isometric	x		

A wandering domain coming from a lift

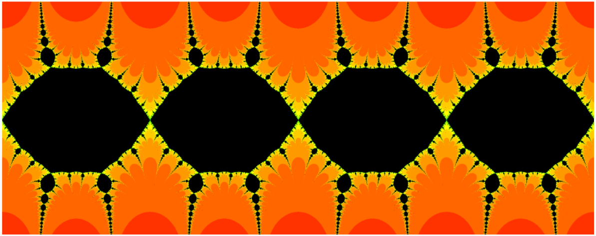
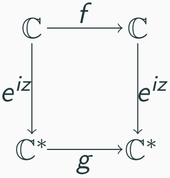


Figure 2: The dynamics of the function $f(z) = z + 2\pi + \sin z$ (picture by David Martí-Pete).



The function $g(z) = z\exp(\frac{1}{2}(\frac{1}{z} - z))$ has a superattracting basin, which lifts to a sequence of wandering domains.

All types are realisable!

type	away	bungee	converging
contracting	x	x	x
semi-contracting	x	x	x
eventually isometric	x	x	x

Examples of wandering domains - Advertisement 1

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More on this construction, as well as a recent exciting construction by Martí-Pete, Rempe and Waterman will be presented during the mini-course 'Approximation in Transcendental Dynamics'.

Boundary dynamics - Advertisement 2

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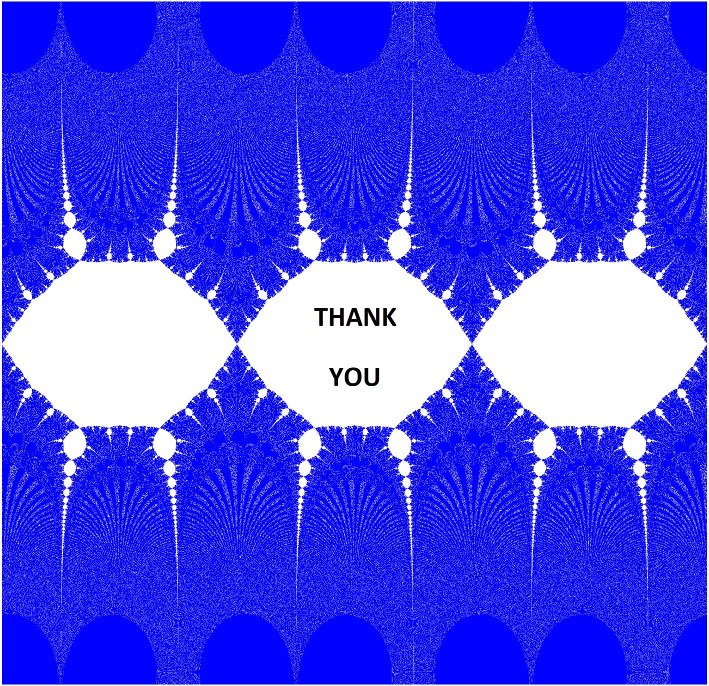
You will hear all about this work in Nuria's talk at the Introductory workshop.


THANK
YOU