Ergodic methods in complex dynamics

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Rational maps in one complex variable

Let $\hat{\mathbb{C}}$ be the Riemann sphere. A *rational map* $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a map of $\hat{\mathbb{C}}$ of the form

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where $p(z)$ and $q(z)$ are polynomials in one complex variable.

The *degree* of a rational map $h(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ have no common factors, is the maximum of the degrees of $p(z)$ and $q(z)$.

Degree 1 rational maps are Mobius transformations.

Therefore, we consider rational maps of degree at least 2.

Hyperbolic rational maps

A rational map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *hyperbolic* if there exists a constant $C > 1$ and a smooth conformal metric ρ on $\hat{\mathbb{C}}$ such that

$$||h'(z)||_\rho > C > 1$$

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The dynamical system $h : J(h) \to J(h)$ is uniformly hyperbolic.

Markov partitions and Thermodynamic Formalism

For any $\varepsilon > 0$, there is a Markov partition $P_1, ..., P_k$ for $J = J(h)$ where P_j are compact subsets of J of diameter at most ε such that

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- $int_J P_j \cap int_J P_i = \emptyset$ if $i \neq j$;
- for each j, $f(P_j)$ is a union of P_i.

Subshift of finite type

Let A be a $k \times k$ matrix with

$$A_{ij} = \begin{cases} 1 \text{ if } P_j \subset f(P_i) \\ 0 \text{ otherwise} \end{cases} .$$

Note that A is aperiodic (topologically mixing), i.e. there exists $n \in \mathbb{N}$ such that $(A^n)_{ij} > 0$ for all $i, j = 1, .., k$.

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The matrix A defines a (one-sided) *subshift of finite type (SFT)*

$$\Sigma_A = \{(i_0, i_1, \cdots) \mid i_j \in \{1, \cdots, k\}, A_{i_j, i_{j+1}} = 1\}.$$

Define the shift $\sigma : \Sigma_A \to \Sigma_A$ by $\sigma(i_0, i_1, \cdots) = (i_1, i_2, \cdots)$

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Metric on Σ_A: $d(\underline{i}, \underline{j}) = 2^{-N}$ where $N = \min\{n \mid i_n \neq j_n\}$.

Part 1: Pressure

Let $\phi : \Sigma \to \mathbb{R}$ be a continuous function. The *pressure* $P(\phi)$ of ϕ is defined by

$$P(\phi) = \sup_{m \in \mathcal{M}_\sigma} \left(h_m(\sigma) + \int_\Sigma \phi dm \right)$$

where $h_m(\sigma)$ is the measure-theoretic entropy of σ with respect to the measure m and $\mathcal{M}_\sigma$ be the set of σ-invariant probability measures on Σ.

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A measure $m(\phi) \in \mathcal{M}_\sigma$ is called an *equilibrium state* of ϕ if $P(\phi) = h_{m(\phi)}(\sigma) + \int_\Sigma \phi dm(\phi)$.

Pressure metric

We say that a function $\phi : \Sigma \to \mathbb{R}$ is *Hölder continuous* if it is α-Hölder continuous for some $\alpha \in (0, 1]$.
(There exists a constant $C > 0$ and $\alpha \in (0, 1]$ such that for all $\underline{i}, \underline{j} \in \Sigma$, we have

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Recall that two continuous functions ϕ_1 and ϕ_2 are *cohomologous* if there exists a continuous function $h : \Sigma \to \mathbb{R}$ such that $\phi_1 - \phi_2 = h \circ \sigma - h$. A continuous function $\phi : \Sigma \to \mathbb{R}$ is a coboundary if it is cohomologous to the zero function.

Pressure metric cont'd

If $[f] \in \mathcal{Z}(\Sigma)$ and f has an equilibrium state m, then the tangent space at $[f]$ is given by

$$T_{[f]}\mathcal{Z}(\Sigma) = \{g \mid \int g dm = 0\}/\text{coboundaries}.$$

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The *pressure metric* $||\cdot||_P$ on $T_{[f]}\mathcal{Z}(\Sigma)$ is defined by

$$||[g]||_P^2 = \frac{var(g,m)}{-\int fdm}.$$

where $var(g,m) = \frac{d^2}{dt^2}\Big|_{t=0} P(f+tg)$.

Thermodynamic mapping

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Consider the *thermodynamic mapping*

$$\begin{aligned} \mathcal{E} : \ \mathcal{H} &\to \mathcal{Z}(\Sigma) \\ c &\mapsto [-\delta(c) \log |f_c'|] \end{aligned}$$

where $\delta : \mathcal{H} \to \mathbb{R}$ is the Hausdorff dimension function, $\delta(c) =$ Hausdorff dimension of the Julia set J_c.

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We can pull-back the pressure metric on $T_{[-\delta(c) \log |f_c'|]}\mathcal{Z}(\Sigma)$ by $\mathcal{E}$ to $T_c\mathcal{H}$. Namely, if $c(t)$ is a path in $\mathcal{H}$ with $c(0) = c$, then $v = \frac{d}{dt}|_{t=0} c(t) \in T_c\mathcal{H}$. Then

$$||v||_{\mathcal{P}} := ||\frac{d}{dt}|_{t=0} \mathcal{E}(c(t))||_P$$

Theorem 1

Theorem (H.-Nie)

We construct a Riemannian metric on a hyperbolic component $\mathcal{H}$ of the Mandelbrot set which is conformal equivalent to the (pull-back of the) pressure metric on $\mathcal{H}$.

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Related work: Bridgeman-Taylor, McMullen, Pollicott-Sharp, Bridgeman-Canary-Labourie-Sambarino.

Part 2: Dynamical zeta functions

Consider $\tau(z) = \log |h'(z)|$. If $z \in J$ is a periodic point of period n, write $\tau_n(z) = \sum_{j=1}^{n-1} \tau(h^j(z))$.

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The Ruelle/dynamical zeta function is defined as

$$\zeta(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{h^n(z)=z} e^{-s\tau_n(z)} \right), s \in \mathbb{C}.$$

Multipliers

If $z \in \hat{\mathbb{C}}$ satisfies the condition $h^n(z) = z$ for some $n \in \mathbb{N}$, then the set $\hat{z} := \{z, h(z), h^2(z), ..., h^{n-1}(z)\}$ is a *periodic orbit of period n*.

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A periodic orbit $\hat{z} = \{z, h(z), h^2(z), ..., h^{n-1}(z)\}$ is called *primitive* if $f^n(z) = z$ and $f^m(z) \neq z$ for any $1 \leq m < n$.

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Let $\mathcal{P}$ be the set of primitive periodic orbit.

If $\hat{z} \in \mathcal{P}$, then the quantity

$$\lambda(\hat{z}) := (h^n)'(z)$$

is called the *multiplier* of $\hat{z}$.

The Oh-Winter theorem: I

Theorem (Oh-Winter, 2017)

Let h be a hyperbolic rational map of degree $d \geq 2$ which is not conjugate to a monomial $z \mapsto z^{\pm d}$. Then there exists $\eta > 0$ such that

$$N_t := \#\{\hat{z} \in \mathcal{P} : |\lambda(\hat{z})| < t\} \\ = Li(t^\delta) + O(t^{\delta-\eta})$$

where δ is the Hausdorff dimension of the Julia set of h.

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The Oh-Winter theorem: II

Theorem (Oh-Winter, 2017)

Let h be a hyperbolic rational map of degree at least 2 whose Julia set is not contained in a circle in $\hat{\mathbb{C}}$. Then there exists $\eta > 0$ such that for any $\varphi \in C^4(\mathbb{S}^1)$,

$$\sum_{\hat{z} \in \mathcal{P} : |\lambda(\hat{z})| < t} \varphi(\arg(\lambda(\hat{z}))) = \int_0^1 \varphi(e^{2\pi i \theta}) d\theta \cdot Li(t^\delta) + O(t^{\delta - \eta})$$

where the implied constant depends only on the C^4-norm of φ.

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where the implied constant depends only on the C^4-norm of φ.

In particular, if $I \subset (-\pi, \pi]$, then

$$\frac{\#\{\hat{z} \in \mathcal{P} : |\lambda(\hat{z})| < t, \arg(\lambda(\hat{z})) \in I\}}{N_t} \sim \frac{|I|}{2\pi}$$

as $t \to \infty$, where $|I|$ is the length of the interval I.

Setup

Given $K \gg 1$, we divide the interval $(-\pi, \pi]$ into K disjoint intervals of equal length. Such intervals are of the form $[\theta - \frac{\pi}{K}, \theta + \frac{\pi}{K}], \theta \in (-\pi, \pi]$.

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Study the existence of multiplier angles $\mathrm{Arg}(\lambda(\hat{z}))$ falling into each such interval subject to the constraint $|\lambda(\hat{z})| < t$ for some fixed $t \gg 1$.

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Theorem 2

Theorem (H.-Nie)

Let $h \in \mathbb{C}(z)$ *be a hyperbolic rational map of degree at least 2. Suppose that* $J(h)$ *is not contained in a circle in* $\widehat{\mathbb{C}}$. *For any given* $K \gg 1$, *nearly every interval* $\left[\theta - \frac{\pi}{K}, \theta + \frac{\pi}{K}\right]$ *contains at least one multiplier angle* $\mathrm{Arg}(\lambda(\hat{z}))$ *with*

$$|\lambda(\hat{z})| \le K^{\frac{7}{2\alpha}},$$

where $\alpha = \min\left\{\frac{\delta}{2}, 2\eta\right\}$, *and* δ *and* η *are as in Oh-Winter's theorem.*

Thank you!