– Lecture 1 –

Potential theory & cplx dynamics

* motivations & examples
* applications to the dynamics of $f : \mathbb{P}^1 \circlearrowleft$
* applications to parameter spaces.

- $f : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$ rational map of degree $d \geq 2$.

  $z_0 \in \mathbb{P}^1(\mathbb{C}) \rightsquigarrow$ understand the longterm behavior of

  $$(f^n(z_0))_n, \qquad f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

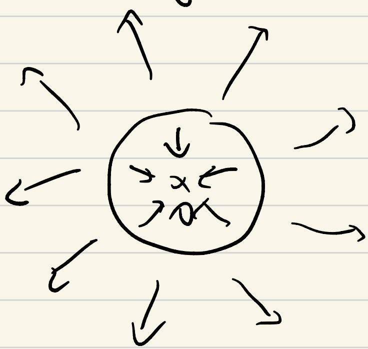
describe what happens to $(f^n(z_0))_n$ and its variation when z_0 moves.

$\mathcal{F}_f$ = Fatou set = $\{z_0 \,/\, (f^n)_n \text{ equicont. at } z_0\}$

$J_f = \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{F}_f$ = Julia set.

J_f & $\mathcal{F}_f$ are totally invariant $f^{-1}(J_f) = f(J_f) = J_f$.

Ex: $z \overset{f}{\longmapsto} z^d$



$J_f = \mathbb{S}^1$ unit circle of $\mathbb{C}$.

Ex: $z \overset{f}{\longmapsto} z^2 - 2$. Chebychev polynomial

$$f\left(z + \frac{1}{z}\right) = z^2 + \frac{1}{z^2}.$$

$\varphi : z \mapsto z + \frac{1}{z}$

$$f \circ \varphi = \varphi(z^2)$$

$$J_f = \varphi(\mathbb{S}^1) = [-2, 2].$$

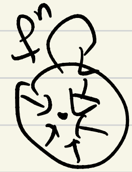
Other examples? Not easy to draw!!

$\rightarrow$ Fractal sets.

* Other description of J_f

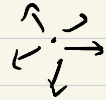
$z \in \mathbb{P}^1(\mathbb{C})$ is *periodic* if $\exists\, n \geq 1$ s.t. $f^n(z) = z$.

of period n if $\forall\, k < n$ $f^k(z) \neq z$.



z is attracting if $|(f^n)'(z)| < 1$

repelling if $|(f^n)'(z)| > 1$



neutral otherwise.

. repelling periodic points belong to J_f.

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Theorem (Fatou, Julia)

(1) J_f is the closure of repelling periodic pts.

(2) $\exists\, \mathcal{E}_f \subset \mathbb{P}^1(\mathbb{C})$, $\#\mathcal{E}_f \leq 2$ s.t. $\forall z_0 \notin \mathcal{E}_f$,

$$J_f \subset \overline{\bigcup_{n \geq 0} f^{-n}\{z_0\}}$$

$\mathcal{E}_f$ is the exceptional set of f:

$$\mathcal{E}_f = \{z \,/\, \bigcup_n f^{-n}\{z\} \text{ is finite}\}$$

Fact: if $\#\mathcal{E}_f = 2$, then f conjugate to $z^{\pm d}$.

if $\#\mathcal{E} = 1$, f is conjugate to a polynomial $\neq z^d$.

Theorem (Chéritat-Buff) $\exists f$ polynomials s.t. $J(f)$ has >0 lebesgue measure.

$f|_{J_f} : J_f \circlearrowleft$ is "complicated"

$\boxed{\text{idea}}$ Use invariant measures to describe this dynamics more precisely.

→ Find μ_f probability measure supported on J_f s.t. (J_f, f, μ_f) is an ergodic dynamical system & this measure describes the theorem of Fatou & Julia from above.

Done by $\begin{cases} \text{Brolin (65) for polynomials.} \\ \left.\begin{array}{l}\text{Freire Lopes Mañé} \\ \text{Lyubich}\end{array}\right\} \text{(83) rational maps.}\end{cases}$

Example: $z \xrightarrow{f} z^d$. Natural candidate λ_{S^1} lebesgue measure on S^1.

- $f_*(\lambda_{\mathbb{S}^1}) = \lambda_{\mathbb{S}^1}$, i.e for all B Borel subset of $\mathbb{S}^1$

$$\underbrace{\lambda_{\mathbb{S}^1}(f^{-1}(B))}_{\text{definition of } f_*(\lambda_{\mathbb{S}^1})(B).} = \lambda_{\mathbb{S}^1}(B)$$

Check it for $(-\varepsilon,\varepsilon) \subset \mathbb{S}^1 \longleftarrow$ identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$

$$f : x \in \mathbb{R}/\mathbb{Z} \longmapsto dx \pmod 1 \in \mathbb{R}/\mathbb{Z}$$

$$f^{-1}(-\varepsilon,\varepsilon) = \bigcup_{k=0}^{d-1} \left(\frac{k}{d} - \frac{\varepsilon}{d}, \frac{k}{d} + \frac{\varepsilon}{d}\right) \bmod 1$$

$$\lambda_{\mathbb{S}^1}(f^{-1}(-\varepsilon,\varepsilon)) = \sum_k \lambda_{\mathbb{S}^1}\left(\frac{k-\varepsilon}{d}, \frac{k+\varepsilon}{d}\right) = 2\varepsilon.$$

$$\lambda_{\mathbb{S}^1}(-\varepsilon,\varepsilon) = 2\varepsilon.$$

* $\lambda_{\mathbb{S}^1}$ is <u>ergodic</u> : if B is a Borel set with $f^{-1}(B) = B$, $\lambda_{\mathbb{S}^1}(B) \in \{0,1\}$.

* $\lambda_{\mathbb{S}^1}$ is <u>mixing</u> : $\forall A, B$ Borel sets,

$$\lim_{n\to\infty} \lambda_{\mathbb{S}^1}(f^{-n}(A) \cap B) = \lambda_{\mathbb{S}^1}(A)\,\lambda_{\mathbb{S}^1}(B).$$

<u>mixing $\Rightarrow$ ergodic</u> : if $A = f^{-1}(A)$, take $A = B$ in the definition of mixing: $\lambda_{\mathbb{S}^1}(A) = \lambda_{\mathbb{S}^1}(A)^2$.

Why do we care?

Theorem (Birkhoff) (X, f, μ) an ergodic dyn. system. Then μ-a.e $x \in X$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \longrightarrow \mu \quad \text{in the weak sense of measures:}$$

$$\forall \varphi \in \mathcal{C}^0_c(X), \qquad \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) \xrightarrow[n\to\infty]{} \int_X \varphi \, d\mu .$$

If μ_f is ergodic, it describes the longterm behavior of μ_f- a.e. orbits.

Example: T_d Chebychev polynomial of degree d.

$$T_d \circ \varphi = \varphi(z^d), \quad \varphi : z \mapsto z + \frac{1}{z}.$$

- $\mu_{T_d} = \varphi_*(\lambda_{S^1})$ is an ergodic measure for T_d.

  μ_{T_d} is absolutely continuous w.r.t. $\lambda_{\mathbb{R}}|_{[-2,2]}$.

Aim 2: Have a "quantified" version of the theorem by Fatou & Julia:

(1) $J_f = \overline{\{\text{repelling periodic pts}\}}$

(2) $\forall z \notin \mathcal{E}_f$, $J_f \subset \overline{\bigcup_{n\geq 1} f^{-n}\{z\}}$.

* ideally: (1) $\forall \varphi \in \mathcal{C}^0(\mathbb{P}^1)$,

$$\frac{1}{d^n}\sum_{f^n(z)=z} \varphi(z) \longrightarrow \int_{\mathbb{P}^1} \varphi \, d\mu_f$$

(2) $\forall z_0 \notin \mathcal{E}_f$, $\forall \varphi \in \mathcal{C}^0(\mathbb{P}^1)$,

$$\frac{1}{d^n}\sum_{f^n(z)=z_0} \varphi(z) \longrightarrow \int_{\mathbb{P}^1} \varphi \, d\mu_f$$

Motivations: To count points (periodic or preimages) in a given subset A with $\mu_f(A) > 0$.

Examples: for $f : z \mapsto z^d$. $\quad \overbrace{f^n(z)}^{z^{d^n}} = z \iff \begin{cases} z = \infty \\ z = 0 \\ z^{d^n - 1} = 1 \end{cases}$.

$$\frac{1}{d^n}\sum_{f^n(z)=z} \delta_z = \frac{1}{d^n}\left(\delta_0 + \delta_\infty + \sum_{z \in \Phi_{d^n-1}} \delta_z\right)$$

- $\forall U$ small interval of $\mathbb{S}^1$. $\quad \dfrac{\#\{z \in U, z^{d^n-1} = 1\}}{d^n} \sim \#\{z \,/\, z^{d^n-1} = 1\}$

$$\Big\downarrow_{n\to\infty}$$

$$\lambda_{\mathbb{S}^1}(U).$$

- $z_0 \in \mathbb{S}^1$, $\quad \dfrac{\#\{z \in U, f^n(z) = z_0\}}{d^n} \sim \lambda_{\mathbb{S}^1}(U)$.

Theorem (Brolin, Lyubich, Freire-Lopes-Mañé) $\forall f : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$

$\exists !\ \mu_f$ probability measure supported on J_f s.t.

(1) (J_f, f, μ_f) is mixing (with exponential rate of cv)

(2) $\forall E \subset \mathbb{P}^1$ countable, $\mu_f(E) = 0$,

* (3) $\forall z_0 \in \mathcal{E}_f$, $\frac{1}{d^n} \sum_{f^n(z) = z_0} \delta_z \longrightarrow \mu_f$

↳ (4) $\frac{1}{d^n} \sum_{f^n(z) = z} \delta_z \longrightarrow \mu_f$

Many additional : [μ_f is the unique measure of maximal entropy of f.

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* What about parameter spaces ?

Focus on the family $f_\lambda(z) = z^2 + \lambda$, $\lambda \in \mathbb{C}$.

Mandelbrot set $M = \{ \lambda \in \mathbb{C} \,/\, (f_\lambda^n(0)) \text{ bounded} \}$.

Why 0 ? $f_\lambda'(z) = 2z$: 0 is the only (finite) critical point of f_λ !

Question: What is the regularity of the map $\lambda \mapsto J_\lambda$?

Is it continuous (for the Hausdorff top. at the target)?

→ **No**. It is discontinuous along a dense subset of ∂M.

∂M is the closure of discont$(\lambda \mapsto J_\lambda)$.

Discontinuities come from parabolic implosion.

There are "chaotic" parameters: ∂M.

→ Can we adapt the use of measure theory to this setting → Yes!

Theorem (Levin, Przytycki, Graczyk-Świątek, G. Vigny)

$\exists\, \mu_M$ proba supported on ∂M s.t.

(1) $\forall E$ countable, $\mu_M(E) = 0$,

(2) for μ_M-a.e. $\lambda \in \mathbb{C}$, $0 \in J_{f_\lambda}$ and

$$\lim_{n\to\infty} \frac{1}{n} \log |(f_\lambda^n)'(\lambda)| = \log 2,$$

(3) $\forall \varphi \in \mathcal{C}^2_c(\mathbb{C})$,

$$\left| \frac{1}{2^{n-1}} \sum_{f_\lambda^n(0)=0} \varphi(\lambda) - \int_{\mathbb{C}} \varphi \, d\mu_M \right| \leq C \frac{n}{2^n} \|\varphi\|_{\mathcal{C}^2}$$

universal and computable.

Theorem (Zdunik) $f : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$ & if

μ_f is not orthogonal to some $\mathcal{H}^\alpha$, $\alpha > 0$.

then $\alpha = 1$ or 2 and

- if $\alpha = 1$, f is monomial or Chebychev
- if $\alpha = 2$, f is Lattès.

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft L"] & \mathbb{C}/\Lambda \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f"'] & \mathbb{P}^1 \end{tikzcd} \Rightarrow J_f = \mathbb{P}^1.$$