Potential theory & dynamics
- lecture II -
The case of polynomials

$f : \mathbb{C} \to \mathbb{C}$ polynomial of degree $d \geq 2$.

<u>Aim:</u> build μ_f proba on $\mathbb{C}$ supported on J_f s.t.

(1) (J_f, f, μ_f) is ergodic,

(2) $\forall E$ countable, $\mu_f(E) = 0$,

(3) $\forall z_0 \notin \mathcal{E}_f$,

$$\frac{1}{d^n} \sum_{f^n(z) = z_0} \delta_z \longrightarrow \mu_f$$

(4) $$\frac{1}{d^n} \sum_{f^n(z) = z} \delta_z \longrightarrow \mu_f .$$

Tool : green function $g_f : \mathbb{C} \longrightarrow \mathbb{R}_+$ of the polynomial f.

$$g_f(z) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)| \geq 0$$

$$\{g_f = 0\} = \underbrace{K_f}_{\text{filled-in Julia set}} = \{z_0 \,/\, (f^n(z_0))_n \text{ bded}\}.$$

g_f is subharmonic on $\mathbb{C}$.

* Subharmonic functions

$u : \mathbb{C} \longrightarrow \mathbb{R} \cup \{-\infty\}$ is sh (subharmonic)

if (1) u u.s.c., $\boxed{u \not\equiv -\infty}$.

(2) u satisfies a submean inequality, $\forall z_0 \in \mathbb{C}$, $\forall \varepsilon > 0$ small enough,

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta})\, d\theta.$$

u harmonic if u & $-u$ are sh.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+\varepsilon e^{i\theta})\,d\theta$$

$\forall \varepsilon>0$ small eng.

Fact: u is sh $\Rightarrow u \in L^1_{loc}(\mathbb{C})$,

$\forall K \subset\subset \mathbb{C}$, $\int_K |u|\,d\lambda < \infty$

In particular, $\lambda(\{u=-\infty\}) = 0$.

Theorem u sh $\Longleftrightarrow$ u is usc and L^1_{loc} and $\Delta u \geq 0$ in the sens of distributions.

$$\langle \Delta u, \varphi\rangle := \int u\,\Delta\varphi$$

$\varphi \in \mathcal{C}^\infty_c(\mathbb{C})$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = i\partial\bar\partial = i\frac{\partial^2}{\partial z\partial\bar z}$.

$\Delta u \geq 0 \Leftrightarrow \forall \varphi \geq 0$, $\langle \Delta u, \varphi\rangle \geq 0$.

extends to $\mathcal{C}^0_c(\mathbb{C})$.

* potential of a measure μ

$$p_\mu(z) := \int_{\mathbb{C}} \log|z-w|\,d\mu(w).$$

Fact. p_μ is sh (when $\mu \geq 0$)

if μ has compact support,

$$p_\mu(z) = \log^+|z| + \underset{\substack{\uparrow \\ |z| \to \infty}}{o}(1)\,.$$

<u>Theorem</u> $\Delta p_\mu = 2\pi\mu$.

[u harmonic $\Leftrightarrow$ $\Delta u = 0$ in $\mathcal{D}'$

$\Leftrightarrow$ $\Delta u = 0$ and u smooth.]

<u>idea:</u> $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{C})$, $\langle \Delta p_\mu, \varphi \rangle \overset{?}{=} \int \varphi \mu$.

$$\langle \Delta p_\mu, \varphi \rangle = \int p_\mu \Delta\varphi$$

$$= \int \left(\int \log|z-w|\, d\mu(w) \right) \Delta\varphi(z)\, d\lambda(z)$$

$$\overset{\text{Fubini}}{=} \int \Big(\underbrace{\int \log|z-w|\, \Delta\varphi(z)\, d\lambda(z)}_{= 2\pi\varphi(w)} \Big)\, d\mu(w)$$

$$\overset{\text{Green's formula}}{=} \int \Big(\lim_{\varepsilon \to 0} \int_{|z-w| > \varepsilon} \log|z-w|\, \underbrace{\Delta\varphi(z)}_{\frac{\partial^2\varphi}{\partial x^2}(z) + \frac{\partial^2\varphi}{\partial y^2}(z)}\, d\lambda(z) \Big)\, d\mu(w)$$

$$= \int \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} \int_{r > \varepsilon} \log r \, \Delta\varphi(re^{i\theta}) \, r \, dr \, d\theta \right) d\mu(w)$$

$$= \int \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} \left(\varphi(w + re^{i\theta}) - \underbrace{r \log r \frac{\partial \varphi}{\partial r}(re^{i\theta})}_{\to 0} \right) \Big|_{r=\varepsilon} d\theta \right) d\mu(w)$$

use that φ has compact support.

$$= \int_{\mathbb{C}} \varphi(w) \, d\mu(w) \, . \qquad \blacksquare$$

Examples: 1 $\mu = \delta_0$ $\quad P_\mu = \log|z|$.

$$P_\mu(z) = \int \log|z - w| \, \delta_0(w) = \log|z|,$$

2 $\mu = \lambda_{S^1}$ $\qquad P_\mu(z) = \log^+|z|$.

idea: use the invariance by rotation of λ_{S^1}.

3 $h : \mathbb{C} \to \mathbb{C}$ holomorphic. $\log|h|$ is sh.

$$\Delta \log|h| = 2\pi \sum_{h(z) = 0} \delta_z$$

counted with multiplicity.

From measures supported on K to its equilibrium measure

$\mu \geq 0$ measure on $\mathbb{C}$, $I(\mu) = \int_{\mathbb{C}} p_\mu \, d\mu$.

Example: ① δ_0, $I(\delta_0) = \int \log|z| \delta_0 = -\infty$.

② $h : \mathbb{C} \to \mathbb{C}$ holomorphic, (non-constant)

$$\mu = \frac{\Delta}{2\pi} \log|h|, \quad I(\mu) = -\infty.$$

$$p_\mu = \underbrace{\int \log|z-w| \mu(w)}$$

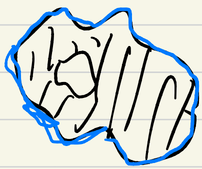
function with poles at points where $h=0$, $-\infty$

③ $\mu = \lambda_{S^1}$, $I(\mu) = \int_{\mathbb{C}} \log^+|z| \lambda_{S^1}(z) = 0$.

capacity of a compact set $K \subset\subset \mathbb{C}$:

$$\sup_{\substack{\nu \text{ supported on } K \\ \nu(\mathbb{C})=1}} I(\nu) =: -\log \operatorname{cap}(K).$$

(corrected)



$E \subset \mathbb{C}$ is called polar if $E \subset \{u = -\infty\}$, u sh.

$\boxed{\text{Theorem}}$ $\forall K \subset\subset \mathbb{C}$ $\exists$ a measure μ_K supported $\partial_e K$ s.t. $I(\mu_K) = e^{-\operatorname{cap}(K)}$

Moreover, if K non-polar, μ_K is unique & $\underline{I(\mu_K) > \infty}$

For K non-polar, define the Green function of K as $\boxed{g_K(z) := p_{\mu_K}(z) - \log \operatorname{cap}(K)}$.

<u>Proposition:</u> g_K green function of a non-polar $K \subset\subset \mathbb{C}$,

(1) $g_K(z) = \log^+|z| - \log \operatorname{cap}(K) + o(1)$, $|z| \to \infty$

(2) $g_K \geq 0$

(3) $g_K = 0$ exactly on $\widehat{K} \setminus E$ (E is a polar set). fill-in K.
E maybe $= \emptyset$

These characterize the Green function of K.

<u>Examples of polar sets:</u> F finite $\Rightarrow$ polar

$F = \{z_1, \ldots, z_k\}$ $\quad \log |(z-z_1) \cdots (z-z_k)| = u$

sh and $u \equiv -\infty$ on F.

(2) (z_n) sequence which converges to 0

$u = \sum_{n \geq 0} \log |z - z_n| \alpha_n$ $\quad$ if you choose $\alpha_n \to \infty$ fast enough

u is well-defined and sh.

$\{u = -\infty\} \supset \{z_n, n \geq 0\}$.

(3) If u is sh and locally bounded, $\forall K$, $\|u\|_{L^\infty(K)} < \infty$

$\Delta u(E) = 0$, $\forall E$ polar.

Another perspective, v sh s.t. $E \subset \{v = -\infty\}$,

$$v \in L^1_{loc}(\Delta u).$$

<u>Back to dynamics.</u> $f : \mathbb{C} \circlearrowleft$ polynomial

<u>Fact</u>

$$g_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|.$$

g_f is the Green function of K_f.

<u>proof</u>: $g_{K_f} \circ f = \log^+|f(z)| - \log \operatorname{cap}(K_f) + o(1)$

$\boxed{\text{si } f^{-1}\{\infty\} = \{\infty\}}$.

$d g_{K_f}$.

$$d(-\log \operatorname{cap}(K_f)) = -\log \operatorname{cap}(K_f) + \log|a_d|$$

$f(z) = a_d z^d + \cdots + a_0$.

$$\operatorname{cap}(K_f) = |a_d|^{-\frac{1}{d-1}}.$$

$$g_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left| a_d^{1+\cdots+d^{n-1}} z^{d^n} + \text{l.o.t.} \right|$$

$$= \log^+|z| + \frac{1}{d-1} \log|a_d| + o(1)$$

$g_f \geq 0$ and $g_f = 0$ on $K_f \implies g_f = g_{K_f}$ ■

$\mu_f := \Delta g_f =$ equilibrium measure of f.

$\to$ Brolin (1965)

idea of why preimage distribute towards μ_f.

Take $z_0 \in \mathbb{C} \setminus \mathcal{E}_f$. $\quad u_n = \frac{1}{d^n} \log |f^n(z) - z_0|$.

$$\frac{1}{2\pi} \Delta u_n = \frac{1}{d^n} \sum_{f^n(z) = z_0} \delta_z .$$

To prove $\frac{1}{2\pi} \Delta u_n \longrightarrow \mu_f$, you only need to prove that $u_n \to g_f$ in $L^1_{loc}(\mathbb{C})$.

indeed: $\varphi \in \mathcal{C}_c^\infty(\mathbb{C})$,

$$|\langle \Delta u_n - \Delta g_f, \varphi \rangle| = \left| \int (u_n - g_f) \Delta \varphi \right|$$

$$\leq C \int_{\mathrm{supp}(\varphi)} |u_n - g_f| \lambda \to 0$$

Hartogs: if (u_n) sequence of locally uniformly bdd from above sh fnts, then:

(1) either $u_n \to -\infty$ unif. on cpt sets,

(2) or $\exists (u_{n_k}) \xrightarrow[L^1_{loc}]{} u$ sh.

(u_k) locally unif bded from above:

$$\frac{1}{d^n}\log|f^n(z)-z_0| \le C \text{ on compact } K \Subset \mathbb{C}$$

if $f^n(z) \to \infty$ $\quad \frac{1}{d^n}\log|f^n(z)-z_0| = \frac{1}{d^n}\log^+|f^n(z)| + o(1)$ $\underset{n\to\infty}{\nearrow}$

so $u_n \to g_f$ on $\mathbb{C}\setminus K$.

* u_∞ a L^1_{loc}-limit of $(u_{n_k})_k$.

  . $u_\infty \le g_f$ and $u_\infty = g_f$ outside K_f.

  . Δu_∞ supported on $J_f = \partial K_f$.

  use that accumulation of $\{f^{-n}\{z_0\}\}$ is J_f.

$u_\infty \le 0$ on K_f ⊕ $u_\infty \equiv 0$ on $\partial K_f = J_f$

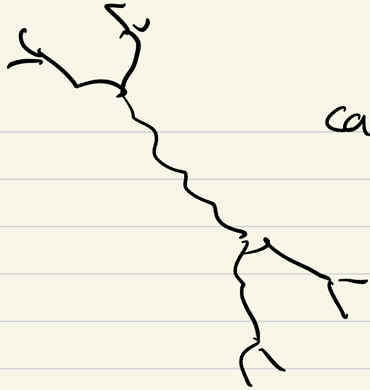
$\Rightarrow u_\infty = g_f$. ∎

$f : \mathbb{P}^1 \circlearrowleft$ with $J_f \Subset \mathbb{C}$ and if μ_f is cooked up later

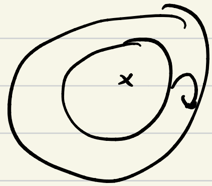
the equilibrium measure of J_f, then f is a polynomial

(Okuyama).

J_{z^2+i}



$\mathrm{cap}(J_f) = |1|^{-\frac{1}{1}} = 1.$



$\boxed{\mathrm{cap}(D(z,r)) = r}$

$K \subset J_f$

unif repeller $\rightsquigarrow$ $\nu(E) = 0$

$\boxed{E \text{ polar} \iff \text{for all } u \text{ sh and locally bdded } \Delta u(E) = 0.}$