# Potential theory & complex dynamics.

## – lecture III –

$$\left[ -\log \operatorname{cap}(K) = \sup_{\substack{\mu \text{ supported} \\ \text{on } K}} I(\mu) \right]$$

* Action of $f$ by pushforward & pullback

$$f_* : \mathcal{C}^0(\mathbb{P}^1) \longrightarrow \mathcal{C}^0(\mathbb{P}^1)$$

$f : \mathbb{P}^1 \circlearrowleft$ rational map

$$(f_*\varphi)(x) := \sum_{f(y)=x} \varphi(y).$$

Well-defined : $f$ is proper and has finite degree.

By duality define $f^*\nu$, $\nu(\mathbb{P}^1) < \infty$, $\nu \geq 0$ :

$$\langle f^*\nu, \varphi \rangle := \langle \nu, f_*\varphi \rangle, \quad \forall \varphi \in \mathcal{C}^0.$$

If $\nu$ is a smooth measure (locally $\nu = \Delta \underset{\text{smooth}}{\psi}$ )

then $f^*\nu = \Delta(\psi \circ f)$ locally

so $f^*\nu$ is again a smooth measure.

$$\int_{\mathbb{P}^1} f^*\nu = \langle f^*\nu, 1 \rangle = \langle \nu, \underbrace{f_* 1}_{} \rangle = d \int_{\mathbb{P}^1} \nu .$$

$$(f_* 1)(x) = \sum_{f(y)=x} 1 = d \cdot 1$$

$\frac{1}{d} f^* : \{\text{probability measures on } \mathbb{P}^1\} \circlearrowleft$ .

<u>Reminder</u> : We want to use $\frac{1}{d} f^*$ to build a good measure to study the dynamics of $f$.

$\ast$ $\nu$ smooth measure $\longleftrightarrow$ 2-form on $\mathbb{P}^1$

If $\nu$ and $\mu$ are 2 smooth proba on $\mathbb{P}^1$,

$\exists\, u \in \mathcal{C}^\infty$ s.t. $\boxed{\nu = \mu + \Delta u} \Leftarrow \boxed{\Delta u \geq -\mu}$

$$\{\nu\} = \{\mu\} \text{ in } H^2_{dR}(\mathbb{P}^1, \mathbb{R}) = \frac{\{2\text{-forms}\}}{\operatorname{im}(\Delta)} .$$

$\Delta : \{\text{functions}\} \to \{2\text{ forms}\}$.

* $\omega_{FS}$ on $\mathbb{C}$ coincides with $\Delta \frac{1}{2}\log(1+|z|^2)$.

⚠ We can't write $\omega_{FS} = \Delta u$, $u : \mathbb{P}^1 \to \mathbb{R}$.

because $\mathbb{P}^1$ is compact. Max principle

$\Rightarrow$ sh f° are <u>constant</u>.

* <u>quasish green function</u>:

apply $\frac{1}{d^n}(f^n)^*$

$$\frac{1}{d} f^* \omega_{FS} = \omega_{FS} + \Delta g_0$$

$\left( f = \frac{P}{Q} \right.$ you take $g_0 = \frac{1}{2\pi}\left[\frac{1}{2d}\log(|P(z)|^2 + |Q(z)|^2) - \frac{1}{2}\log(1+|z|^2)\right]\Big)$

$$\frac{1}{d^{n+1}}(f^{n+1})^* \omega_{FS} = \frac{1}{d^n}(f^n)^* \omega_{FS} + \underbrace{\frac{1}{d^n}\Delta g_0 \circ f^n}_{\Delta\left(\frac{g_0 \circ f^n}{d^n}\right)}$$

$$\underbrace{\frac{g_0 \circ f^n}{d^n}}_{\|\cdot\| \leq \frac{C}{d^n}}$$

$$\boxed{g_f := \sum_{n=0}^{\infty} \frac{1}{d^n} g_0 \circ f^n \text{ is a uniform limit}}$$

The equilibrium measure is $\mu_f = \omega_{FS} + \Delta g_f$

Again $\langle \Delta g_f, \varphi \rangle := \langle g_f, \Delta\varphi \rangle$.

$$\underline{\underline{\Delta g_f}} \geq -\omega_{FS}.$$

* Back to polynomials: $f : \mathbb{C} \to \mathbb{C}$ degree $d$ polynomial

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|.$$

$\boxed{\text{Claim}}$ $G_f(z) = \boxed{\frac{1}{2}\log(1+|z|^2)} + g_f(z)$.

$$\Rightarrow \Delta G_f = \Delta \frac{1}{2}\log(1+|z|^2) + \Delta g_f$$
$$= \omega_{FS} + \Delta g_f.$$

<u>proof</u>:

$$G_n = \frac{1}{2}\log(1+|z|^2) + \sum_{j=0}^{n-1} \frac{g_0 \circ f^j}{d^j}$$

$$G_n = \frac{1}{2}\log(1+|z|^2) + \sum_{j=0}^{n-1} \frac{1}{2d^{j+1}} \log\left(1+|f^{j+1}(z)|^2\right) - \frac{1}{2d^j}\log\left(1+|f^j(z)|^2\right)$$

$$= \frac{1}{2d^n}\log\left(1+|f^n(z)|^2\right) = \frac{1}{d^n}\log^+|f^n(z)| + O\left(\frac{1}{d^n}\right).$$

$\boxed{\text{Proposition}}$ $\operatorname{Supp}(\mu_f) = J_f$.

Lemma: $\forall \nu$ smooth measure, $\nu(\mathbb{P}^1) = 1$

$\frac{1}{d^n}(f^n)^* \nu \longrightarrow \mu_f$ in the sense of proba.

proof: $\nu = \omega_{FS} + dd^c u$ $\quad \left[ dd^c = \frac{\Delta}{2\pi} \right]$

$u$ is $\mathcal{C}^\infty$

$$\frac{1}{d^n}(f^n)^* \nu = \underbrace{\frac{1}{d^n}(f^n)^* \omega_{FS}}_{\to \mu_f} + dd^c \Big( \underbrace{\frac{1}{d^n} u \circ f^n}_{\to 0 \text{ unif.}} \Big)$$

$\blacksquare$

proof (Prop): Take $U \subset \mathcal{F}_f$ small $\boxed{\mu_f(U) = 0 \, ?}$

$(f^n)$ is normal on $U$. Up to extracting, $\exists V \Subset \mathcal{F}$ open set st. $f^n(U) \subset V$ $\forall n$.

Take $\nu$ smooth but supported outside $V$.

$\mu_f = \lim_{n \to \infty} \frac{1}{d^n}(f^n)^* \nu$ on $U$ $\Rightarrow$ $\mu_f \equiv 0$ on $U$.

$\Longrightarrow \operatorname{supp}(\mu_f) \subset J_f$.

Take $U \subset (\operatorname{supp}(\mu_f))^c$. On $U$,

$$\|(f^n)'\|^2_{FS}\,\omega_{FS} = (f^n)^*\omega_{FS} = d^n\Big[\frac{1}{d^n}(f^n)^*\omega_{FS} - \underline{\underline{\mu_f}}\Big] \text{ on } U$$

$\mu_f \equiv 0$ on $U$.

By construction,

$$\frac{1}{d^n}(f^n)^*\omega_{FS} - \mu_f = dd^c g_n, \quad g_n = O\Big(\frac{1}{d^n}\Big).$$

On $U$

$$\|(f^n)'\|^2_{FS}\,\omega_{FS} = d^n\, dd^c g_n$$

Take $U' \subset\subset U$, $\varphi$ smooth $\equiv 1$ on $U'$

$\equiv 0$ outside $U$

$\varphi \geq 0.$

$$\int_{U'} \|(f^n)'\|^2_{FS}\,\omega_{FS} \leq \int_{\mathbb{P}^1} \varphi\,(f^n)^*\omega_{FS} = \int_{\mathbb{P}^1} d^n g_n\, dd^c\varphi \leq C(U)$$

$C(U) \to 0$ when $U$ shrinks.

$\Rightarrow (f^n)$ equicontinuous. ∎

$\ast$ Invariance and mixing of $\mu_f$:

By construction, $\frac{1}{d} f^*\mu_f = \mu_f$: $f^*\mu_f = d\mu_f$. | constant jacobian

$\Rightarrow$ invariance.

Fact: $\forall \mu$ $f_* f^*\mu = d\mu$.

proof: $\langle f_* f^* \mu, \varphi \rangle = \langle f^* \mu, \varphi \circ f \rangle$

$= \langle \mu, f_*(\varphi \circ f) \rangle$

$$f_*(\varphi \circ f)(x) = \sum_{f(y)=x} \varphi(f(y)) = d\varphi(x). \quad \blacksquare$$

$$f_*(d\mu_f) = f_*(f^* \mu_f) = d\mu_f.$$

$\Rightarrow f_* \mu_f = \mu_f.$

Reminder: $\mu_f$ is mixing if for all Borel sets $A, B \subset \mathbb{P}^1$

$$\mu_f(f^{-n}(A) \cap B) \xrightarrow[n \to \infty]{} \mu_f(A) \mu_f(B).$$

Theorem: $\exists C > 0$, s.t. $\forall \varphi, \psi \in \mathcal{C}^2$,

$$\left| \int \varphi \circ f^n \cdot \psi \, d\mu_f - \int \varphi \, d\mu_f \int \psi \mu_f \right| \leq \frac{C}{d^n} \|\varphi\|_{L^\infty} \|\psi\|_{\mathcal{C}^2}.$$

proof: Assume $\varphi \geq 0$, $\int \varphi \mu_f = 1$.

$\varphi \mu_f = \mu_f + dd^c u \qquad u \in \mathcal{C}^0.$

$\left[ \sup_{\mathbb{P}^1} u = 0 \right].$

$$\left|\int_{\mathbb{P}^1} \varphi\circ f^n \cdot \psi \, d\mu_f - \int_{\mathbb{P}^1} \psi \, d\mu\right| = \left|\langle (\varphi\circ f^n)\cdot \overset{\downarrow\ \mu_f = \frac{1}{d^n}(f^n)^*\mu_f}{\mu_f} - \mu_f, \psi \rangle\right|$$

$$= \left|\langle \frac{1}{d^n}(f^n)^*(\varphi\mu_f - \mu_f), \psi\rangle\right|$$

$$= \left|\langle \frac{1}{d^n} dd^c(u\circ f^n), \psi\rangle\right|$$

$$= \left|\int_{\mathbb{P}^1} \frac{1}{d^n} u\circ f^n \cdot dd^c\psi\right|$$

$$\leq \frac{1}{d^n} \underbrace{\|u\|_{L^\infty}}_{\substack{\Delta \text{ is an elliptic operator} \\ \Rightarrow \leq C\|\varphi\|_{L^\infty}}} \cdot \|\psi\|_{C^2}.$$

$\boxed{\text{Theorem}}$ $\nu$ probability measure on $\mathbb{P}^1$,

$$\frac{1}{d^n}(f^n)^*\nu \longrightarrow \mu_f \iff \nu(\mathcal{E}_f) = 0.$$

In particular $\displaystyle \frac{1}{d^n}\sum_{f^n(z)=z_0} \delta_z \longrightarrow \mu_f$ iff $z_0 \notin \mathcal{E}_f$.

## Distribution of periodic pts

$$\mu_n := \frac{1}{d^n} \sum_{\substack{f^n(z)=z \\ z \text{ repelling}}} \delta_z$$

Thm (Lyubich, F-L-M)

$$\mu_n \longrightarrow \mu_f.$$

proof: $\underbrace{\mu_n}(\mathbb{P}^1) \le 1 \implies$ Up to extraction $(\mu_n) \to \mu_\infty$.

Left to prove: $\mu_\infty = \mu_f$.

We prove: $\forall D$ small enough, $\mu_f(D) \le \mu_\infty(D)$

$$\implies \mu_\infty(\mathbb{P}^1) \ge 1 \;\&\; \mu_\infty \ge \mu_f \implies \underline{\mu_\infty = \mu_f}$$

**Lemma** $\forall \varepsilon > 0$, $\exists \ell \ge 1$, $\exists C > 0$, $\forall n \ge \ell$ and all disks $D$ avoiding $\bigcup_{j=0}^{\ell-1} f^j(\mathcal{C}_f)$, $\exists$ at least $(1-\varepsilon)d^n$ inverse branches of $f^n$, $(f_i^{-n})_i$ s.t.

$$\mathrm{diam}(f_i^{-n}(D)) \le C d^{-\frac{n}{2}}.$$

proof: Koebe $\frac{1}{4}$-theorem! ∎

Take $D$ small enough avoiding $\bigcup_{n=0}^{\ell-1} f^k(\mathcal{C}_f)$ $\left(\begin{smallmatrix}\varepsilon>0\\ \text{fixed}\end{smallmatrix}\right)$

Let $D_i^{-n} = f_i^{-n}(D)$, $n \ge \ell$.

Take $D' \subset D$ a little smaller


$\mu_f(D) \simeq \mu_f(D')$

possible because $g_f$ has Hölder regularity.

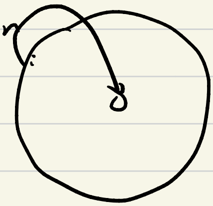
$\mu_f$ mixing $\Rightarrow \Big[\, \mu_f(D)^2 \simeq \mu_f(D)\,\mu_f(D') \simeq \mu_f\big(D' \cap f^{-n}(D)\big)$

$$\simeq \sum_{i=1}^{(1-\varepsilon)d^n} \mu_f\big(D' \cap D_i^{-n}\big)$$

Two situations: * $D' \cap D_i^{-n} = \emptyset$.

* $D_i^{-n} \Subset D$

$f_i^{-n}$



$f_i^{-n}$ inverse of $f^n$ : $\exists\, z \in D' \cap D_i^{-n}$

repelling & $f^n(z) = z$.

$$\mu_f(D)^2 \simeq \sum_{i=1}^{(1-\varepsilon)d^n} \mu_f\big(D_i^{-n} \cap D'\big) \leq \sum_i \underbrace{\mu_f\big(D_i^{-n}\big)}_{\text{cst jacobian}}\, d^n \mu_n\big(D_i^{-n}\big)$$

$$= \frac{1}{d^n}\mu_f(D)$$

$$\leq \mu_f(D) \times \mu_n(D)$$

$$\Rightarrow \quad \mu_f(D) \leq \mu_n(D) \qquad \blacksquare$$

$\boxed{\text{Question}}$ What the rate of convergence?

$$\frac{n}{d^n}$$