Potential theory & dynamics
– lecture IV –

* $f : \Lambda \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ holomorphic

Λ cplx manifold = parameter space

$f_\lambda := f(\lambda, \cdot)$ rational of degree d independent of λ.

Mimick Fatou/Julia description:

Def: $(f_\lambda)_{\lambda \in \Lambda}$ is <u>stable</u> if $\forall \lambda_0 \in \Lambda$, $\exists U \ni \lambda_0$ neigh.

and $h : U \times J_{\lambda_0} \longrightarrow \mathbb{P}^1$ $\mathcal{C}^0$ and

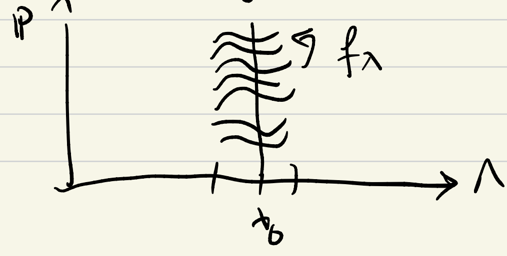
$(\lambda, z) \longmapsto h_\lambda(z)$

(1) $\forall \lambda \in U$, $h_\lambda : J_{\lambda_0} \longrightarrow J_\lambda$ homeo s.t.

$$f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}.$$

(2) $\forall z \in J_{\lambda_0}$, $\lambda \longmapsto h_\lambda(z)$ is holomorphic

(3) $h_{\lambda_0} = id_{J_{\lambda_0}}$.



<u>Reduction:</u> Assume $\exists\, c_1, \dots, c_{2d-2} : \Lambda \to \mathbb{P}^1$ holo

s.t. $\mathcal{C}_{f_\lambda} = \{c_1(\lambda), \dots, c_{2d-2}(\lambda)\}$.

Theorem (Mañé–Sad–Sullivan, Lyubich, 82)

$f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$ family. $U \subset \Lambda$ open connected set TFAE:

(1) $(f_\lambda)_{\lambda \in U}$ is stable,

(2) Repelling cycles can be followed holomorphically on U.

(3) $\forall\, 1 \leq i \leq 2d-2$, the sequence $\{c_{n,i} : \lambda \longmapsto f_\lambda^n(c_i(\lambda))\}_n$ is equicontinuous /normal on U.

Moreover, The set $\mathrm{Stab} = \{\lambda_0 \,/\, (f_\lambda) \text{ is stable near } \lambda_0\}$ is a dense open subset of Λ. $\mathrm{Bif} : \Lambda \setminus \mathrm{Stab}$.

Conjecture (Fatou) If $\Lambda = \{f : \mathbb{P}^1 \circlearrowleft \; \deg(f) = d\}$

Then Stab contains only hyperbolic parameters.

[* Case of families of polynomials with $\dim \Lambda = 1$]

$f_\lambda(z) := a_d(\lambda) z^d + \cdots + a_0(\lambda)$, $a_d, \ldots, a_0 : \Lambda \to \mathbb{C}$
a_d does not vanish.

$G_{f_\lambda}(z) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f_\lambda^n(z)|$.
plug-in $c_i(\lambda)$.

$G_{f,c_i}(\lambda) := G_{f_\lambda}(c_i(\lambda))$ It defines a $\mathcal{C}^0$ sh function on Λ.

<u>Theorem</u> (DeMarco, 2000) $\mu_{bif} := \sum_{i=1}^{2d-2} dd^c G_{f,c_i}$ (bif measure)
| is supported exactly on Bif.

<u>Rk</u>: the proof is valid for families of rational maps & when $\dim \Lambda \geq 1$.

<u>proof</u>: Prove $dd^c G_{f,c_i}$ is supported on the locus where

$\{c_{n,i}\}_n$ is not equicontinuous $\quad c_{n,i} : \lambda \mapsto f_\lambda^n(c_i(\lambda))$

If $\{c_{n,i}\}$ equicontinuous on U, $\exists\, c_{n_k,i} \longrightarrow \underline{c_{\infty,i}}$ unif loc.
on U: either $c_{\infty,i} \equiv \infty \Rightarrow |c_{n_k,i}(\lambda)| \gg 1$, $\forall k \gg 1$.

$$\frac{1}{d^{n_k}} \log^+ |c_{n_k,i}(\lambda)| = \frac{1}{d^{n_k}} \log |c_{n_k,i}(\lambda)| = \text{harmonic}.$$

$\Rightarrow dd^c G_{f,c_i} \equiv 0$ on U.

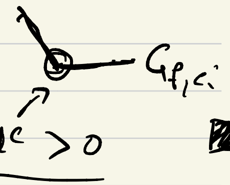
or $c_{\infty,i} : U \to \mathbb{C}$ holomorphic $\Rightarrow |c_{n,i}(\lambda)| \leq C$, $\forall \lambda \in U$, $\lambda n \gg 1$.

$\Rightarrow \quad \frac{1}{d^n}\log^+|c_{n,i}(\lambda)| = O\left(\frac{1}{d^n}\right) \longrightarrow 0 \quad : \quad dd^c G_{f,c_i} \equiv 0$ on U

The other implication? If $\{c_{n,i}\}_n$ not equicontinuous at λ_0,
& $dd^c G_{f,c_i} \equiv 0$ near λ_0

- $\lambda_n \to \lambda_0$: c_i escapes at λ_n
- $\lambda_n' \to \lambda_0$: c_i bounded orbit at λ_n'

$\Rightarrow$



∎

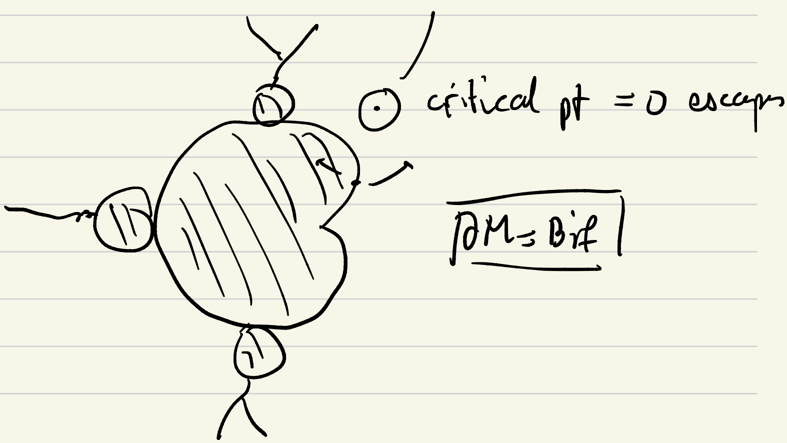
The family $(z^d+\lambda)_{\lambda\in\mathbb{C}}$ quadratic

Theorem (Douady-Hubbard)

The measure $d\mu_{bif}$ is the equilibrium measure of the degree d Mandelbrot set:

$$M := \{\lambda\in\mathbb{C} \,/\, (f_\lambda^n(0))_{n\geq 1} \text{ bounded}\}$$

$f_\lambda(z) = z^d+\lambda$.



$\boxed{\partial M = \mathrm{Bif}}$

$G_M := dG_{f_\lambda}(0) \left(= G_{f_\lambda}(\lambda)\right)$ $\qquad f_\lambda(0) = \lambda.$

$$(*)\begin{cases} \cdot\ G_M \geq 0 \text{ and harmonic on } \mathbb{C}\setminus\partial M \\ \cdot\ G_M = 0 \text{ exactly on } M, \\ \cdot\ G_M(\lambda) = \log^+|\lambda| + o(1) \end{cases}$$

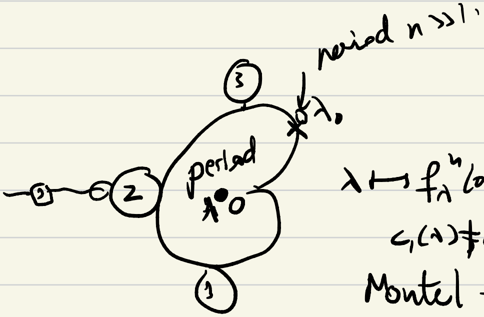
$(*) \Longrightarrow G_M$ is the Green function of $M \Longrightarrow d\mu_{bif} = dd^c G_M$ is its equilibrium measure.

induction: $f_\lambda^{n+1}(0) = \lambda^{d^n} + d^n \lambda^{d^n-d+1} + \text{l.o.t.}$

$$\frac{1}{d^n}\log^+|f_\lambda^{n+1}(0)| = \log^+|\lambda| + \frac{1}{d^n}\log^+\left|1 + \frac{d^n}{\lambda^{d-d+1}} + \cdots\right|$$
$$= \log^+|\lambda| + o(1). \qquad \blacksquare$$

$*$ $\partial M \subset \overline{\{\text{centers of hyperbolic components}\}}$.

Hyperbolic component of period n = component of $\mathring{M}$ that contains param. with a n-periodic attr. point.



$\lambda \mapsto f_\lambda^n(0)$ not normal near λ_0

$c_1(\lambda) \neq c_2(\lambda) \neq 0$

Montel $\Rightarrow$ $\{\lambda \mapsto f_\lambda^n(0)\}$ can not avoid $\{c_1(\lambda), c_2(\lambda), 0\}$

* The distribution of centers:

Theorem $\exists C \geq 1$ computable explicitly s.t. $\forall \varphi \in \mathcal{C}^2_c(\mathbb{C})$,

$$\left| \frac{1}{d^n} \sum_{f_\lambda^n(0)=0} \varphi(\lambda) - \int \varphi \, \mu_{bif} \right| \leq C \frac{n}{d^n} \|\varphi\|_{\mathcal{C}^2}.$$

$\frac{1}{d^n}$

(G-Vigny) previous Levin (no speed), Favre-Rivera-Letelier ($\mathcal{C}^1$-version)

adapts Brolin's proof

use Arakelov theory of height functions.

Generalizations [Favre-G., Ghioca-Ye] G. DeMarco-Baker

idea: $\nu_n = \frac{1}{d^n} \sum_{f_\lambda^n(0)=0} \delta_\lambda = dd^c \left(\underbrace{\frac{1}{d^n} \log|f_\lambda^n(0)|}_{=g_n} \right)$

$\downarrow$ G_M in $L^1_{loc}(\mathbb{C})$.

$\limsup_{n\to\infty} g_n \leq G_M$.

$\lim_{n\to\infty} g_n = G_M$ on $\mathbb{C} \setminus M$.

Up to extraction $g_n \xrightarrow{L^1_{loc}(\mathbb{C})} u$ (Hartogs)

Use $\underbrace{f_\lambda^n(0)}_{Q_n(\lambda)} = 0$ has simple roots. $\left(\mathrm{Disc}(Q_n, Q_n') = 1 \, [d] \right)$

$\Delta u = 0$ on $\mathring{U}$ $\Rightarrow$ $u = G_M$. ∎

$*$ Use argument of Przytycki: $c \in J_f$

$\kappa = \frac{1}{16}$, $M = \sup_J \|f'\|_{FS}$

$$\forall\, d(c, f^n(c)) \geq \frac{\kappa}{M^n}$$

• $\Rightarrow \exists C_1, \forall \lambda \in \mathbb{C} \setminus M,$

$$\left| \frac{1}{d^{n-1}} \log |f_\lambda^n(0)| - G_M(\lambda) \right| \leq \frac{C_1 n}{d^n}.$$

• Poincaré-Wirtinger inequality

$$\| \log |f_\lambda^n(0)| \|_{L^1(M)} \leq C n \operatorname{Area}(M).$$

Remarks: ① You can generalize all these statements in quite general families.

⚠ Problem: you get to study higher order bifurcations.

② these statements allow to count the number of hyperbolic components with attr. cycles of given periods.

$\exists n_0 \geq 1$ s.t. $\forall n_1 \geq \dots \geq n_{2d-2} \geq n_0$,

the # hyp. components having attr. cycles of rep. periods $n_1, \dots, n_{2d-2}$ is $\simeq d^{n_1 + \dots + n_{2d-2}} \left(\int \mu_{bif} + O\left(\frac{1}{d^{n_{2d-2}}}\right) \right)$

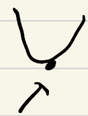
$\mathcal{M}_d = \mathrm{Rat}_d/\sim$ $(2d-2)$-dim variety.

$d=2$

Euler f°

$$\left[\ \int_{\mathcal{M}_2} \mu_{bif} = \frac{1}{3} - \frac{1}{8}\sum_{n\geq 1} \frac{\phi(n)}{(2^n-1)^2}.\right.$$

$$\int_{\mathcal{M}_d} \mu_{bif} > 0.$$



degree d polynomials $=$ $\int_{\mathcal{P}_d} \mu_{bif} = (d-1)!^{-1}$

<u>Polynomials</u>: $\boxed{d^{n(d-1)}}$ $\quad d^{n(d-1)}\left(1 + O\left(\frac{1}{d^n}\right)\right)$

of disjoint type

$\Rightarrow$ $\underline{\leq C d^{n(d-2)}}$