Bootstrapping to optimal exponential decay

Gaiotto-Moore-Neitzke's Conjecture (Schematically)

Pick a point $u \in \mathcal{B} \simeq \mathbb{C}$ and let $|u| = r$.

$$g_{L^2} - g_{\mathrm{sf}} = \sum_{\gamma \in H_1(\Sigma_u;\mathbb{Z})} \Omega(\gamma, u) \mathrm{e}^{-\ell(\gamma,u)\sqrt{r}}.$$

The first correction is $\Omega(\gamma_0, u) = 8$ and $\ell(\gamma_0, u) = 2\sqrt{\frac{2}{\mathrm{Im}\tau}}$.

Theorem [F-Mazzeo-Swoboda-Weiss]

Let $\mathcal{M}$ be a (strongly-parabolic) $SU(2)$ Hitchin moduli space for the four-punctured sphere. The rate of exponential decay for the Hitchin moduli space is as Gaiotto-Moore-Neitzke conjecture:

$$g_{L^2} - g_{\mathrm{sf}} = O(\mathrm{e}^{-2\sqrt{\frac{2}{\mathrm{Im}\tau}}\sqrt{r}}).$$

$(\mathcal{M}, g_{L^2})$ is an ALG metric asymptotic to the model metric g_{sf}.

Bootstrapping to optimal exponential decay

LeBrun gave a framework to describe all Ricci-flat Kähler metrics of complex-dimension two with a holomorphic circle action in terms of two functions u, w.

Generalized Gibbons-Hawking Ansatz specialized to our case:

Consider a hyperkähler metric on $T^2_{x,y} \times \mathbb{R}^+_r \times S^1_\theta$ with holomorphic circle action. The hyperkähler metric is

$$g_{L^2} = \mathrm{e}^u u_r(\mathrm{d}x^2 + \mathrm{d}y^2) + u_r \mathrm{d}r^2 + u_r^{-1}\mathrm{d}\theta^2$$

where $u : T^2_{x,y} \times \mathbb{R}^+_r \to \mathbb{R}$ solves

$$\Delta_{T^2} u + \partial_r^2 \mathrm{e}^u = 0.$$

The semiflat metric g_{sf} corresponds to $u_{\mathrm{sf}} = \log r$.

Goal

Show that $u - u_{\mathrm{sf}}$ has conjectured rate of exponential decay.

Bootstrapping to optimal exponential decay

Let $v = u - u_{\mathrm{sf}}$. Then,

$$\underbrace{\Delta_T v + r\partial_r^2 v + 2\partial_r v}_{Lv} = \underbrace{-e^v r(\partial_r v)^2 - (e^v - 1)\left(r\partial_r^2 v + 2\partial_r v\right)}_{Q(v,\partial_r v,\partial_{rr} v)},$$

Observation #1: The first exponentially-decaying function in $\ker L$ decays like $\mathrm{e}^{-2\lambda_T\sqrt{r}}$, where λ_T^2 is the first positive eigenvalue of $-\Delta_{T^2}$. In the torus T_τ^2 with its semiflat metric $\lambda_T^2 = \frac{2}{\mathrm{Im}\ \tau}$.

Observation #2: If $v \sim \mathrm{e}^{-\varepsilon\sqrt{r}}$, then $Q(v, \partial_r v, \partial_{rr} v) \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$.

Solving the non-homogeneous problem $Lv = f$ for $f \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$, we find

$$v \sim \mathrm{e}^{-2\min(\varepsilon,\lambda_T)\sqrt{r}}.$$

Conclusion: $v \sim \mathrm{e}^{-2\lambda_T\sqrt{r}}$ where $\lambda_T = \sqrt{\frac{2}{\mathrm{Im}\ \tau}}$

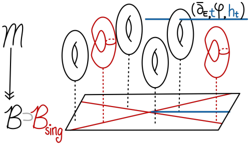
Two hyperkähler metrics on the regular locus $\mathcal{M}'$

- g_{L^2} Hitchin's L^2 hyperkähler metric—uses h
- $g_{\rm sf}$ semiflat metric—from integrable system structure

Gaiotto-Moore-Neitzke's Conjecture

Fix $(\bar\partial_E, \varphi) \in \mathcal{M}'$. Along the ray $T_{(\bar\partial_E, t\varphi, h_t)}\mathcal{M}'$,

$$g_{L^2} - g_{\rm sf} = \Omega e^{-\ell t} + \text{faster decaying}$$



Progress:

- Mazzeo-Swoboda-Weiss-Witt proved polynomial decay for $SU(2)$-Hitchin moduli space. ['17]
- Dumas-Neitzke proved exponential* decay in $SU(2)$-Hitchin section with its tangent space. ['18]
- **F** proved exponential* decay for $SU(n)$-Hitchin moduli space. ['18]
- **F**-Mazzeo-Swoboda-Weiss proved exponential* decay for $SU(2)$ parabolic Hitchin moduli space. (Higgs field has simple poles along divisor $D \subset C$.) ['20]

$*$: Rate of exponential decay is not optimal.

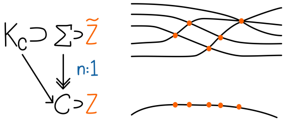
Main Theorem

Theorem [F, F-Mazzeo-Swoboda-Weiss]

Fix $(\bar\partial_E, \varphi) \in \mathcal{M}'$ and a Higgs bundle variation $(\dot\eta, \dot\varphi) \in T_{(\bar\partial_E, \varphi)}\mathcal{M}$.
Along the ray $T_{(\bar\partial_E, t\varphi, h_t)}\mathcal{M}'$, as $t \to \infty$,

$$\|(\dot\eta, t\dot\varphi)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

As $t \to \infty$, $F_{D(\bar\partial_E, h_t)}$ concentrates along branch divisor $Z \subset C$.
The limiting metric h_∞ is flat with singularities along Z.



The main difficulty is dealing with the contributions to the integral $\|\cdot\|_{g_{L^2}} = \int_C \cdots$ from infinitesimal neighborhoods around Z.

Idea #1: Semiflat metric is an L^2-metric

Hitchin's hyperkähler metric g_{L^2} on $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$ is

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} = 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t\right|^2_{h_t} + t^2 \left|\dot\varphi + [\dot\nu_t, \varphi]\right|^2_{h_t}$$

where the metric variation $\dot\nu_t$ of h_t is the unique solution of

$$\partial_E^{h_t} \bar\partial_E \dot\nu_t - \partial_E^h \dot\eta - t^2 \left[\varphi^{*_{h_t}}, \dot\varphi + [\dot\nu_t, \varphi]\right] = 0.$$

The semiflat metric, from the integrable system structure, on $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$ is an L^2-metric defined using h_∞.

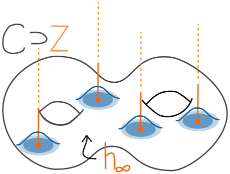
$$\|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_\infty\right|^2_{h_\infty} + t^2 \left|\dot\varphi + [\dot\nu_\infty, \varphi]\right|^2_{h_\infty},$$

where the metric variation $\dot\nu_\infty$ of h_∞ is independent of t and solves

$$\partial_E^{h_t} \bar\partial_E \dot\nu_\infty - \partial_E^h \dot\eta = 0 \qquad \left[\varphi^{*_{h_\infty}}, \dot\varphi + [\dot\nu_\infty, \varphi]\right] = 0.$$

Idea #2: Approximate solutions

Desingularize h_∞ (singular at Z) by gluing in solutions h_t^{model} of Hitchin's equations on neighborhoods of $p \in Z$. $\rightsquigarrow$ h_t^{approx}.



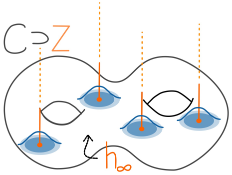
$$\bar\partial_E = \bar\partial \qquad t\varphi = t \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \mathrm{d}z$$

$$h_\infty = \begin{pmatrix} |z|^{\frac{1}{2}} & 0 \\ 0 & |z|^{-\frac{1}{2}} \end{pmatrix} \qquad h_t^{\mathrm{model}} = \begin{pmatrix} |z|^{\frac{1}{2}} \mathrm{e}^{u(t^{2/3}|z|)} & 0 \\ 0 & |z|^{-\frac{1}{2}} \mathrm{e}^{-u(t^{2/3}|z|)} \end{pmatrix}$$

$$h_t^{\mathrm{approx}} = \begin{pmatrix} |z|^{\frac{1}{2}} \mathrm{e}^{\chi u(t^{2/3}|z|)} & 0 \\ 0 & |z|^{-\frac{1}{2}} \mathrm{e}^{-\chi u(t^{2/3}|z|)} \end{pmatrix}$$

Idea #2: Approximate solutions

Desingularize h_∞ (singular at Z) by gluing in solutions h_t^{model} of Hitchin's equations on neighborhoods of $p \in Z$. $\rightsquigarrow$ h_t^{approx}.



Perturb h_t^{approx} to an actual solution h_t using a contracting mapping argument.

(Difficulty: Showing the first eigenvalue of $L_t : H^2 \to L^2$ is $\geq Ct^{-2}$)

Theorem

$$h_t(v,w) = h_t^{\mathrm{app}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w) \qquad \text{for } \|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}.$$

Idea #2: Approximate solutions

Define an *non-hyperkähler L^2-metric* g_{app} on $\mathcal{M}'$ using variations of the metric h_t^{app}.

$$\begin{aligned}
\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t\right|^2_{h_t} + t^2 \left|\dot\varphi + [\dot\nu_t, \varphi]\right|^2_{h_t} \\
\|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_\infty\right|^2_{h_\infty} + t^2 \left|\dot\varphi + [\dot\nu_\infty, \varphi]\right|^2_{h_\infty} \\
\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t^{\mathrm{app}}\right|^2_{h_t^{\mathrm{app}}} + t^2 \left|\dot\varphi + [\dot\nu_t^{\mathrm{app}}, \varphi]\right|^2_{h_t^{\mathrm{app}}} .
\end{aligned}$$

Then, break the $g_{L^2} - g_{\mathrm{sf}}$ into two pieces:

$$\left(\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}}\right) + \left(\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}}\right)$$

Corollary

Since $h_t(v,w) = h_t^{\mathrm{app}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w)$ for $\|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}$, as $t \to \infty$ along the ray $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$,

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} = O(\mathrm{e}^{-\varepsilon t}).$$

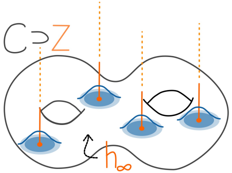
Idea #2: Approximate solutions

Our goal is to show that the following sum is $O(\mathrm{e}^{-\varepsilon t})$:

$$\underbrace{\left(\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}}\right)}_{O(\mathrm{e}^{-\varepsilon t})} + \left(\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}}\right)$$

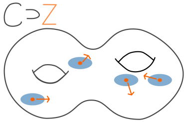
It remains to show that $\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$.

Since h_t^{app} differs from h_∞ only on disks around $p \in Z$, the difference $g_{\mathrm{app}} - g_{\mathrm{sf}}$ localizes (up to exponentially-decaying errors) to disks around $p \in Z$.



Idea #3: Holomorphic variations

When Mazzeo-Swoboda-Weiss-Witt proved that $g_{L^2} - g_{\mathrm{sf}}$ was at least polynomially-decaying in t, all of their possible polynomial terms came from infinitesimal variations in which the branch points move.



Dumas-Neitzke used a family of biholomorphic maps on local disks (originally defined by Hubbard-Masur) to match the changing location of the branch points. This uses subtle geometry of Hitchin moduli space. E.g. for $SU(2)$, conformal invariance.

Remarkably, this can be generalized off of the Hitchin section and from $SU(2)$ to $SU(n)$.

Theorem [F, F-Mazzeo-Swoboda-Weiss]

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

Main Theorem

Gaiotto-Moore-Neitzke's Conjecture

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$,

$$\|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{L^2}} - \|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{\mathrm{sf}}} = \Omega \mathrm{e}^{-\ell t} + \text{faster decaying.}$$

Theorem [**F**, **F**-Mazzeo-Swoboda-Weiss]

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$ and a Higgs bundle variation $(\dot{\eta}, \dot{\varphi}) \in T_{(\bar{\partial}_E, \varphi)}\mathcal{M}$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$, as $t \to \infty$,

$$\|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{L^2}} - \|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t}).$$

Ideas:

#1 Semiflat metric is an L^2-metric for h_∞

#2 Build approximate solutions h_t^{approx} that are exponentially close to h_t

#3 Use local biholomorphic flow to match the changing location of the branch points.

Thank you!